Generalized interval exchanges and the $2-3$ conjecture

SHMUEL FRIEDLAND
Department of Mathematics, Statistics and Computer Science,
University of Illinois at Chicago
Chicago, Illinois 60607-7045, USA
email: friedlan@uic.edu

BENJAMIN WEISS
Institute of Mathematics
Hebrew University
Jerusalem 91904, Israel
email: weiss@math.huji.ac.il

January 20, 2005

Abstract

We introduce the notion of a generalized interval exchange $\phi_{\mathcal{A}}$ induced by a measurable k-partition $\mathcal{A} = \{A_1, ..., A_k\}$ of $[0, 1)$. $\phi_{\mathcal{A}}$ can be viewed as the corresponding restriction of a nondecreasing function $f_{\mathcal{A}}$ on $\mathbb{R}$ with $f_{\mathcal{A}}(0) = 0, f_{\mathcal{A}}(k) = 1$. $\mathcal{A}$ is called λ-dense if $\lambda(A_i \cap (a, b)) > 0$ for each i and any $0 \le a < b \le 1$. We show that the $2-3$ Furstenberg conjecture is invalid if and only if there are 2 and 3 λ-dense partitions $\mathcal{A}$ and $\mathcal{B}$ of $[0, 1)$, such that $f_{\mathcal{A}} \circ f_{\mathcal{B}} = f_{\mathcal{B}} \circ f_{\mathcal{A}}$. We give necessary and sufficient conditions for this equality to hold. We show that for each integer $m \ge 2$, such that $3 \nmid 2m+1$, there exist 2 and 3 non λ-dense partitions $\mathcal{A}$ and $\mathcal{B}$ of $[0, 1)$, corresponding to the interval exchanges on $2m$ intervals, for which $f_{\mathcal{A}}$ and $f_{\mathcal{B}}$ commute.

2000 Mathematical Subject Classification: 37A05, 37A35
Keywords: Generalized interval exchange, entropy, 2-3 conjecture.

1 Introduction

Let Σ the σ-algebra of measurable sets in $\mathbb{R}$ with respect to the Lebesgue measure λ. Let $k \in \mathbb{N}$ and $J \in \Sigma$. $\mathcal{A} := \{A_1, ..., A_k\}$ is called a partition (or k-partition) of J if $A_1, ..., A_k$ are pairwise disjoint measurable sets whose union is J. Let $I = [0,1)$. Then a k-partition $\mathcal{A}$ of I induces the following partition $\{I_1, ..., I_k\}$ of I to k intervals:

$$I_j = [\beta_{j-1}, \beta_j), \quad j = 1, ..., k, \quad \beta_0 = 0, \ \beta_j = \sum_{i=1}^{j} \lambda(A_j), \ j = 1, ..., k. \tag{1.1}$$

$\mathcal{A}$ is called regular if $\lambda(A_j) > 0$ for $j = 1, ..., k$. For $A \subset \mathbb{R}$ let $\chi_A(x)$ be the characteristic function of A. Then the partition $\mathcal{A}$ induces the following generalized k-interval exchange $\phi_{\mathcal{A}} : I \to I$:

$$\phi_{\mathcal{A}} : A_j \to \overline{I}_j, \quad \phi_{\mathcal{A}}(x) = \beta_{j-1} + \int_0^x \chi_{A_j} d\lambda, \quad x \in A_j, \quad j = 1, ..., k. \tag{1.2}$$

$\phi_{\mathcal{A}} : I \to I$ is a measure preserving transformation of $(I, \Sigma(I), \lambda)$. If each A_j is a finite union of intervals then $\phi_{\mathcal{A}}$ is an orientation preserving interval exchange. See [1] for other generalizations of interval exchange maps.

Let $A \subset \mathbb{R}$ be the following measurable set induced by $\mathcal{A}$:

$$A \cap [m-1, m) = A_i + m - 1 \quad \text{for } m \in \mathbb{Z} \text{ with } m \equiv i \bmod k. \tag{1.3}$$

Define

$$f_{\mathcal{A}}(x) := \int_0^x \chi_A d\lambda, \quad x \in \mathbb{R}. \tag{1.4}$$

Clearly $f_{\mathcal{A}}$ is a continuous nondecreasing function on $\mathbb{R}$ with the properties

$$f_{\mathcal{A}}(0) = 0, \quad f_{\mathcal{A}}(x+k) = f_{\mathcal{A}}(x) + 1, \ x \in \mathbb{R}. \tag{1.5}$$

A measurable set $T \subset [s,t]$ is called λ-dense if

$$\lambda(T \cap (a,b)) > 0 \quad \text{for all } s \le a < b \le t.$$

$\mathcal{A}$ is called λ-dense if each A_j is λ-dense in I. Then $f_{\mathcal{A}}$ is increasing on $\mathbb{R}$ if and only if $\mathcal{A}$ is λ-dense. Assume that $f_{\mathcal{A}}$ is increasing on $\mathbb{R}$. Let $F_{\mathcal{A}}$ be the inverse function of $f_{\mathcal{A}}$. Then $F_{\mathcal{A}}(0) = 0$ and $F_{\mathcal{A}}(1) = k$. Furthermore $F_{\mathcal{A}} = F$ is expansive:

$$y - x < F(y) - F(x), \quad \text{for all } x < y. \tag{1.6}$$

Let $S^1 = \mathbb{R}/\mathbb{Z}$. Then $F_{\mathcal{A}}$ induces an expansive orientation preserving k-covering map $\tilde{F}_{\mathcal{A}} : S^1 \to S^1$, which fixes 0 and preserves λ. Furthermore $\tilde{F}_{\mathcal{A}}$ is λ-invertible. The λ-inverse of $F_{\mathcal{A}}$ is $\phi_{\mathcal{A}}$. Hence the entropy $h_\lambda(\phi_{\mathcal{A}})$ is 0 if $\mathcal{A}$ is λ-dense. (We prove that $h_\lambda(\phi_{\mathcal{A}}) = 0$ for any partition $\mathcal{A}$ of I.)

We show that $\tilde{F}_{\mathcal{A}}$ is conjugate to the standard k-covering map $\tilde{G}_k$, where $G_k(x) = kx,\ x \in \mathbb{R}$. λ is conjugate to a nonatomic probability measure ω on I whose support is S^1. $\tilde{G}_k$ preserves ω and $\tilde{G}_k$ is ω invertible. Vice versa, a nonatomic $\tilde{G}_k$-invariant probability measure, ω whose support is S^1 and which is invertible with respect to ω, is conjugate to $\tilde{F}_{\mathcal{A}}$ for some λ-dense k-partition $\mathcal{A}$.

Recall the $2-3$ conjecture of Furstenberg [2]. Let ω be a nonatomic probability measure on S^1 which is invariant for $\tilde{G}_2, \tilde{G}_3$. Then $\omega = \lambda$. Furstenberg showed that the support of ω is S^1. Rudolph [4] proved the $2-3$ conjecture if either $h_\omega(\tilde{G}_2)$ or $h_\omega(\tilde{G}_3)$ are positive. Thus it is left to consider the $2-3$ conjecture in the case $h_\omega(\tilde{G}_2) = h_\omega(\tilde{G}_3) = 0$. This is equivalent to the ω invertibility of $\tilde{G}_2$ and $\tilde{G}_3$. We show

Theorem 1.1 *The $2-3$ conjecture is false if and only there exist 2 and 3 λ-dense partitions $\mathcal{A}$ and $\mathcal{B}$ of I respectively such that*

$$F_{\mathcal{A}} \circ F_{\mathcal{B}} = F_{\mathcal{B}} \circ F_{\mathcal{A}}. \tag{1.7}$$

Clearly the condition (1.7) yields that condition

$$f_{\mathcal{A}} \circ f_{\mathcal{B}} = f_{\mathcal{B}} \circ f_{\mathcal{A}}, \tag{1.8}$$

which in turn implies

$$\phi_{\mathcal{A}} \circ \phi_{\mathcal{B}} = \phi_{\mathcal{B}} \circ \phi_{\mathcal{A}}. \tag{1.9}$$

We give necessary and sufficient conditions for the equality (1.8) for any 2 and 3-partitions $\mathcal{A}$ and $\mathcal{B}$ respectively. A k-partition $\mathcal{C}$ is called a k-n-partition if it is induced by the partition of I to n equal length intervals. ($\mathcal{C}$ is not λ-dense.) Assume that $\mathcal{A}$ and $\mathcal{B}$ are 2-n and 3-n-partitions of I respectively. Then $\phi_{\mathcal{A}}, \phi_{\mathcal{B}}$ induce permutation σ, η respectively on the set $< n >:= \{1, ..., n\}$. Assume that (1.8) holds. Then σ and η are two commuting permutations. The equality (1.8) gives the precise structure of σ and η. We show that for $n \le 3$ there are no regular 2-n and 3-n-partitions for which (1.8) holds. For $n = 4$ there are unique regular 2-4 and 3-4-partitions which satisfy (1.8)

$$\mathcal{A} = \{\{[\frac{1}{4}, \frac{1}{2}), [\frac{3}{4}, 1)\}, \{[0, \frac{1}{4}), [\frac{1}{2}, \frac{3}{4})\}\},\ \mathcal{B} = \{\{[\frac{1}{2}, \frac{3}{4})\}, \{[0, \frac{1}{4}), [\frac{3}{4}, 1)\}, \{[\frac{1}{4}, \frac{1}{2})\}\}. \tag{1.10}$$

It is possible to extend this example in a trivial way to any $n \ge 5$, by letting σ and η to fix a few first and last integers in the interval $[1, n]$. For each integer $m \ge 2$, where $3 \nmid 2m + 1$, the maps G_2, G_3 induce regular $2 - 2m$ and $3 - 2m$ partitions which satisfy (1.8). It seems that the non-validity of the $2-3$ conjecture is closely related to the existence of other type 2-n and 3-n-partitions which satisfy (1.8).

We now summarize briefly the contents of the paper. Section 2 is devoted to the discussion of the connection between k-λ dense partitions and a nonatomic invariant measure of $\tilde{G}_k$ whose support is S^1. In Section 3 we discuss the map $\phi_{\mathcal{A}}$ for any k-partition of I. In particular we show that the λ entropy of $\phi_{\mathcal{A}}$ is zero. In Section 4 we discuss the conditions on 2 and 3 partitions $\mathcal{A}$ and $\mathcal{B}$ of I which satisfy the condition (1.8). In the last section we discuss the combinatorial conditions on 2-n and 3-n-partitions of I which satisfy (1.8). In particular we show that the example (1.10) is the first nontrivial example of 2-4 and 3-4-partitions of I satisfying (1.8). This example is a particular case of the examples of $2-2m$ and $3-2m$ partitions $(3 \nmid 2m+1)$ satisfying (1.8), induced by the maps G_2, G_3.

2 Covering maps of S^1

Let $F : \bar{I} \to \mathbb{R}$ be a continuous function such that $F(0) = 0,\ F(1) = k$ for some $1 \le k \in \mathbb{Z}$. We then extend F to $\mathbb{R}$

$$F(0) = 0, \quad F(x+1) = F(x) + k \text{ for all } x \in \mathbb{R}. \tag{2.1}$$

Then F induces the map $\tilde{F} : S^1 \to S^1$ where the degree of $\tilde{F}$ is k. $\tilde{F}$ is a k-covering map if and only if F is increasing on $\mathbb{R}$. We call F *expansive* if (1.6) holds.

Theorem 2.1 *Let $\mathbb{F} : \mathbb{R} \to \mathbb{R}$ be a continuous increasing function on $\mathbb{R}$ satisfying* (2.1) *for an integer $k \ge 2$. Assume that F is expansive. Then there exist a unique continuous increasing function $H : \mathbb{R} \to \mathbb{R}$ satisfying* (2.1) *with $k = 1$ such that*

$$F \circ H = H \circ G_k, \tag{2.2}$$

where $G_k(x) = kx$. In particular $\tilde{F}$ is conjugate to $\tilde{G}_k$ on S^1.

Proof. Observe that (2.1) implies that $F(j) = jk$ for $j \in \mathbb{Z}$. Let $1 \le m \in \mathbb{Z}$ and define $F^{\circ m} = \underbrace{F \circ \ldots \circ F}_{m}$. Then $F^{\circ m}(1) = k^m$. Observe that $F^{\circ m}$ is also expansive. For $i \in [0, k^m] \cap \mathbb{Z}$ let $x(i,m) \in [0,1]$ be the unique solution of $F^{\circ m}(x(i,m)) = i$. Clearly, if $i = i'k$ then $x(i', m-1) = x(i,m)$. Moreover

$$0 = x(0,m) < x(1,m) < ... < x(k^m, m) = 1.$$

We claim that the set $T := \cup_{m=1}^{\infty} \cup_{i=0}^{k^m} \{x(i,m)\}$ is dense in I. This is equivalent to the statement that for any $0 \le x < y \le 1$ there exists $x(i,m)$ such that $x < x(i,m) < y$. Assume to the contrary that there exist $0 \le x < y \le 1$ such that for any $m \ge 1$ and $i \in [0, k^m] \cap \mathbb{Z}$ the condition $x(i,m) \notin (x,y)$ holds. Hence

$0 < F^{\circ m}(y) - F^{\circ m}(x) < 1,\ m = 1, ...$ Choose x', y' such that $x < x' < y' < y$. As $F^{\circ m}$ is expansive

$$F^{\circ m}(y') - F^{\circ m}(x') =$$
$$F^{\circ m}(y) - F^{\circ m}(x) - (F^{\circ m}(y) - F^{\circ m}(y')) - (F^{\circ m}(x') - F^{\circ m}(x)) <$$
$$1 - \epsilon, \quad \epsilon = (y - y' + x' - x) > 0.$$

Since F is expansive it follows that

$$0 < F^{\circ m}(y') - F^{\circ m}(x') < F^{\circ (m+1)}(y') - F^{\circ (m+1)}(x') < 1 - \epsilon, \quad m = 0, 1, ...$$

Hence

$$\lim_{m \to \infty} F^{\circ m}(y') - F^{\circ m}(x') = a, \quad 0 < a \leq 1 - \epsilon.$$

Let

$$p_m := \lfloor F^{\circ m}(x') \rfloor,\ u_m := F^{\circ m}(x') - p_m \in [0, 1),\ v_m := F^{\circ m}(y') - p_m, \quad m = 0, 1, ...$$

Choose a subsequence $u_{m_j},\ j = 1, ...$ which converges to $u \in I$. Then $v_{m_j},\ j = 1, ...$ converges to $u + a$. Observe that

$$F(v) - F(u) =$$
$$\lim_{j \to \infty} F(v_{m_j}) - F(u_{m_j}) = \lim_{j \to \infty} F(F^{\circ m_j}(y') - p_{m_j}) - F(F^{\circ m_j}(x') - p_{m_j}) =$$
$$\lim_{j \to \infty} F(F^{\circ m_j}(y')) - p_{m_j} k - (F(F^{\circ m_j}(x')) - p_{m_j} k) =$$
$$\lim_{j \to \infty} F^{\circ (m_j+1)}(y') - F^{\circ (m_j+1)}(x') = a = v - u.$$

This contradicts the expansiveness of F. Define H on the following dense countable set $S := \cup_{m=1}^{\infty} \cup_{i=0}^{k^m} \{\frac{i}{k^m}\}$:

$$H(\frac{i}{k^m}) = x(i, m), \quad i = 0, ..., k^m,\ m = 1, ... \tag{2.3}$$

Note that if $i = i'k$ then $H(\frac{i}{k^m}) = H(\frac{i'}{k^{m-1}}) = x(i', m-1) = x(i, m)$. So H is well defined on S. Furthermore H is an increasing function on S. As S and T are dense in I H has a unique continuous extension to I. Clearly the function H is increasing on I with $H(0) = 0,\ H(1) = 1$. Extend H to $\mathbb{R}$ by (2.1). For $i \in [0, k^m] \cap \mathbb{Z}$ such that $i = j + i_j k^{m-1}$ with $j \in [0, m^{k-1}] \cap \mathbb{Z},\ i_j \in [0, k] \cap \mathbb{Z}$ we have

$$H(G_k(\frac{i}{k^m})) = H(\frac{i}{k^{m-1}}) = H(\frac{j}{k^{m-1}} + i_j) = H(\frac{j}{k^{m-1}}) + i_j = x(j, m-1) + i_j.$$

Observe next that $F(H(\frac{i}{k^m})) = F(x(i,m))$. We claim that $F(x(i,m)) = x(j, m-1) + i_j$. Indeed

$$F^{\circ(m-1)}(x(j,m-1)+i_j) = F^{\circ(m-1)}(x(j,m-1)) + i_j k^{m-1} =$$
$$j + i_j k^{m-1} = i = F^{\circ(m-1)}(F(x(i,m)).$$

Hence (2.2) holds on S. Since S is dense in I (2.2) holds on I. Use the "periodic" properties of F, G_k, H to deduce (2.2) on $\mathbb{R}$.

It is left to show that H is unique. Recall that H is the identity map on $\mathbb{Z}$. Assume that (2.2) holds. Then $H \circ G_k^{\circ m} = F^{\circ m} \circ H$. Clearly

$$H(G_k^{\circ m}(\frac{i}{k^m})) = H(i) = i = F^{\circ m}(x(i,m)) = F^{\circ m}(H(\frac{i}{k^m})), \quad i \in [0, k^m] \cap \mathbb{Z}.$$

Hence $H(\frac{i}{k^m}) = x(i,m)$. $\square$

Theorem 2.2 *Let F be a continuous increasing function on $\mathbb{R}$ satisfying (2.1) for an integer $k \geq 2$. Let f be the inverse function of F. Then the orientation preserving k-covering map $\tilde{F} : S^1 \to S^1$ preserves the Lebesgue measure λ if and only if there exists k nonnegative measurable functions $p_1, ..., p_k$ such that*

$$\begin{aligned} &0 < \int_a^b p_i d\lambda \quad \text{for all } 0 \leq a < b \leq 1, \quad i = 1, ...k, \\ &\sum_{i=1}^k p_i(x) = 1, \quad \text{a.e. in } I, \\ &f(x+i-1) = \int_0^x p_i d\lambda + \sum_{j=0}^{i-1} \int_0^1 p_j d\lambda, \quad p_0(x) = 0, \ x \in I, \ i = 1, ..., k. \end{aligned} \tag{2.4}$$

In particular, $\tilde{F}$ is λ-preserving and is invertible with respect to λ if and only if there exists a k-λ-dense partition $\mathcal{A} = \{A_1, ..., A_k\}$ of I such that $p_i = \chi_{A_i}$ a.e. for $i = 1, ..., k$. In this case $\phi_{\mathcal{A}}$ is the λ inverse of $\tilde{F}$.

Proof. Clearly, for $0 \leq x < y \leq 1$

$$\tilde{F}^{-1}(x,y) = \cup_{i=1}^k (f(x+i-1), f(y+i-1)). \tag{2.5}$$

Then $\tilde{F}$ is λ-preserving if and only if $\lambda(\tilde{F}^{-1}(x,y)) = y - x$. Hence for each $i \in < k >$ $0 < f(y+i-1) - f(x+i-1) < y - x$. Therefore $0 \leq \frac{df(x+i-1)}{dx} = p_i \leq 1$ for some measurable function on I for $i = 1, ..., k$. In particular the last equality of

(2.4) holds. Since $f(x)$ is increasing in the interval $[0,k]$ we deduce the first equality of (2.4). The second equality of (2.4) is equivalent to the assumption that $\mathbb{F}$ is λ-preserving.

Vice versa, suppose that we are given k nonnegative measurable function $p_1,...,p_k$ which satisfy the first two conditions of (2.4). Define $f:[0,k]\to\mathbb{R}$ by the last condition of (2.4). Then f is an increasing function which maps $[0,k]$ on I. Let $F: I\to[0,k]$ be the inverse of of f. Then $\tilde{F}$ is an orientation preserving k-covering of S^1 which preserves λ. Note that for any set $B\subset I$, which is a finite union of intervals, the last equality of (2.4) and (2.5) yield

$$\lambda(f(B+i-1))=\int_B p_i d\lambda,\ i=1,...,k,\quad \lambda(B)=\lambda(\tilde{F}^{-1}(B))=\sum_{i=1}^k \lambda(f(B+i-1)). \tag{2.6}$$

Hence the above equalities hold for any measurable set $B\subset I$. Suppose furthermore that $p_i(x)=\chi_{A_i}$ a.e. for some measurable set $A_i\subset I$ for $i=1,...,k$. The first two conditions of (2.4) are equivalent to the assumption that $\mathcal{A}=\{A_1,...,A_k\}$ can be chosen to be k-λ-dense partition. (2.6) yields

$$\tilde{F}^{-1}(B)=\int_B \chi_{A_i} d\lambda,\quad \text{for any measurable set } B\subset A_i,\ i\in<k>. \tag{2.7}$$

Hence $\tilde{F}$ has the λ inverse $\phi_{\mathcal{A}}$ given by

$$\phi_{\mathcal{A}}(x)=f(x+i-1)\quad \text{for } x\in A_i,\quad i\in<k>. \tag{2.8}$$

Assume finally that $\tilde{F}$ preserves λ and $\tilde{F}$ has λ inverse ψ. In particular (2.4) holds. As $\tilde{F}^{-1}(x)=\cup_{i=1}^k f(x+i-1)$, the existence of ψ implies the partition of I to k measurable pairwise distinct sets $A_1,...,A_k$, such that for $\psi(x)=f(x+i-1)$ $x\in A_i$. Let B be a measurable subset of A_i. Since $\tilde{F}$ preserves λ the first equality of (2.6) implies

$$\lambda(B)=\lambda(\tilde{F}^{-1}(B))=\lambda(\psi(B))=\lambda(f(B+i-1))=\int_B p_i d\lambda\le\int_B d\lambda=\lambda(B).$$

Hence $p_i|B=1$ a.e.. The second condition of (2.4) yields $p_i=\chi_{A_i}$ a.e. for $i=1,...,k$. The first condition of (2.4) implies that $\mathcal{A}=\{A_1,...,A_k\}$ is k-λ-dense partition of I. $\square$

Theorem 2.2 was inspired by Parry's paper [3].

Theorem 2.3 *Let $\mathcal{A}=\{A_1,...,A_k\}$ be k-λ-dense partition with $k\ge 2$. Let $f_{\mathcal{A}}$ be given by* (1.4) *and $F_{\mathcal{A}}$ be the inverse of $f_{\mathcal{A}}$. Then $F_{\mathcal{A}}$ is expansive, $\tilde{F}_{\mathcal{A}}$ is an* *orientation preserving k covering of* S^1 *which preserves* λ*. The generalized interval exchange* $\phi_{\mathcal{A}}$ *given by* (1.2) *is the* λ *inverse of* $\tilde{F}_{\mathcal{A}}$*. Furthermore*

$$h_\lambda(\tilde{F}_{\mathcal{A}}) = h_\lambda(\phi_{\mathcal{A}}) = 0. \tag{2.9}$$

Proof. Assume that $x, y \in [j-1, j],\ j \in \mathbb{Z}$ and $x < y$. Let $j \equiv i \bmod k$ for some $i \in < k >$. Since $\mathcal{A}$ is λ-dense

$$y - x = \int_x^y d\lambda = \sum_{p=1}^{k} \int_x^y \chi_p d\lambda > \int_x^y \chi_i d\lambda = f(y) - f(x).$$

Hence $F(v) - F(u) > v - u$ for any $v > u$. The proof of Theorem 2.2 and the definitions of $f_{\mathcal{A}}$ and $\phi_{\mathcal{A}}$ yield that $\tilde{F}_{\mathcal{A}}$ is λ preserving and $\phi_{\mathcal{A}}$ is the λ inverse of $\tilde{F}_{\mathcal{A}}$. As $F_{\mathcal{A}}$ is expansive by Theorem 2.1 $F_{\mathcal{A}}$ is conjugate to G_k. In particular $\tilde{F}_{\mathcal{A}}$ is conjugate to $\tilde{G}_k$. λ is conjugate to nonatomic probability measure ω, whose support is $\overline{I}$ and which is $\tilde{G}_k$-invariant. As $\tilde{G}_k$ has the standard Markov partition $M_i = [\frac{i-1}{k}, \frac{i}{k}),\ i = 1, ..., k$, we deduce that $\tilde{F}_{\mathcal{A}}$ is equivalent to complete $\mathbb{Z}_+$ shift on k symbols. Let $\mathcal{M} = \{H(M_1), ..., H(M_k)\})$ the Markov partition for $\tilde{F}_{\mathcal{A}}$. Then $\mathcal{F} = \vee_{i=0}^{\infty} \tilde{F}^{-i}\mathcal{M}$ is the σ-subalgebra generated by the cylinders, which is equivalent to the Borel algebra for any nonatomic probability measure ν. Since $\tilde{F}$ is λ invertible it follows that $h_\lambda(\tilde{F}) = 0$ (cf.[6, Cor. 4.18.1]). □

In the next section we show that for any k-partition $\mathcal{A}$ of I $h_\lambda(\phi_{\mathcal{A}}) = 0$.

Problem 2.4 *Let* $\mathcal{A} = \{A_1, ..., A_k\}$ *be k-partition of* I*. When* $\phi_{\mathcal{A}}$ *is ergodic?*

Corollary 2.5 *Let* $\mathcal{A} = \{A_1, ..., A_p\}, \mathcal{B} = \{B_1, ..., B_q\}$ *be two* p, q*-*λ*-dense partitions of* I *with* $p, q \geq 2$*. Then*

$$h_\lambda(\phi_{\mathcal{A}} \circ \phi_{\mathcal{B}}) = h_\lambda(\tilde{F}_{\mathcal{B}} \circ \tilde{F}_{\mathcal{A}}) = 0. \tag{2.10}$$

Proof. $F := F_{\mathcal{B}} \circ F_{\mathcal{A}}$ is a continuous increasing expansive function on $\mathbb{R}$ satisfying (2.1) for $k = pq$. Furthermore $\tilde{F}$ preserves λ. Theorem 2.2 implies that $F = F_{\mathcal{C}}$ for some k-λ-dense partition of I. Hence (2.10) holds. □

Problem 2.6 *Let* $\mathcal{A} = \{A_1, ..., A_p\}, \mathcal{B} = \{B_1, ..., B_q\}$ *be two* p *and* q*-*λ *dense partitions of* I *with* $p, q \geq 2$*. Estimate from above*

$$h_\lambda(\phi_{\mathcal{A}}^{-1} \circ \phi_{\mathcal{B}}) = h_\lambda(\phi_{\mathcal{B}}^{-1} \circ \phi_{\mathcal{A}}). \tag{2.11}$$

Theorem 2.7 *Let* $F : \mathbb{R} \to \mathbb{R}$ *be a measurable function satisfying* (2.1) *a.e. for some* $k \in \mathbb{Z}$*. Assume that*

$$F \circ G_m = G_m \circ F, \quad |m| \in [2, \infty) \cap \mathbb{Z}. \tag{2.12}$$

Then $F = G_k = kx$ *a.e..*

Proof. Let $E(x) = F(x) - kx$. Then $E(x+1) = E(x)$ a.e. in $\mathbb{R}$. Let j be a positive integer. Since F and G_k commute with G_m it follows that $E \circ G_{m^j} = G_{m^j} \circ E$. Hence

$$m^j E(x) = E(m^j x) = E(m^j x + 1) = E(m^j(x + \frac{1}{m^j})) = m^j E(x + \frac{1}{m^j}) \Rightarrow$$
$$E(x + \frac{1}{m^j}) = E(x).$$

Since j is an arbitrary positive integer it follows that E is constant a.e.. The condition $E(mx) = mE(x)$ yields that $E = 0$ a.e.. □

The above theorem is related to a theorem (unpublished) of Jean-Paul Thouvenot:

Theorem 2.8 *Let $p, q \in \mathbb{Z} \backslash \{-1, 0, 1\}$ and assume that p and q are multiplicatively independent, i.e. p and q are not integer powers of some integer r. Let $T : S^1 \to S^1$ be measurable λ-preserving. Assume that T commutes with $\tilde{G}_p$ and $\tilde{G}_q$. Then $T = \tilde{G}_k$ for some $k \in \mathbb{Z}^*$.*

Proof of Theorem 1.1. Suppose first that there exist 2 and 3-λ-dense partitions $\mathcal{A}$ and $\mathcal{B}$ of I such that (1.7) holds. Theorem 2.3 yields that $F_{\mathcal{A}}$ is expansive. Theorem 2.1 yields that $H^{-1} \circ F_{\mathcal{A}} \circ H = G_2$. Let $F := H^{-1} \circ F_{\mathcal{B}} \circ H$. Then F is a continuous function on $\mathbb{R}$ satisfying (2.1) with $k = 3$ which commutes with G_2. Theorem 2.7 yields that $F = G_3$. As $\tilde{F}_{\mathcal{A}}, \tilde{F}_{\mathcal{B}}$ preserve the Lebesgue measure λ it follows that $\tilde{G}_2, \tilde{G}_3$ preserve the probability measure $\omega = (H^{-1})^* \lambda$, which is nonnatomic and whose support is $\overline{I}$. As $\tilde{F}_{\mathcal{A}}, \tilde{F}_{\mathcal{B}}$ are λ-invertible (Theorem 2.3), $\tilde{G}_2, \tilde{G}_3$ are ω-invertible. Hence $\omega \neq \lambda$, which contradicts the $2-3$ conjecture.

Assume now that $2-3$ conjecture is false. Then there exists a nonatomic probability measure ω which is $\tilde{G}_2, \tilde{G}_3$ invariant. According to [2] the support of ω is $\overline{I}$. Rudolph's theorem [4] claims that $h_\omega(\tilde{G}_2) = h_\omega(\tilde{G}_3) = 0$. Hence $\tilde{G}_2, \tilde{G}_3$ are ω-invertible (cf.[6, Cor. 4.14.3]). Let

$$H(x) = \int_0^x d\omega, \quad x \in I.$$

Then $H(x)$ is strictly increasing function on I with $H(0) = 0$, $H(1) = 1$. Extend H to $\mathbb{R}$ using (2.1) with $k = 1$. Let $F_k = H \circ G_k \circ H^{-1}$, $k = 2, 3$. Then $F_2 \circ F_3 = F_3 \circ F_2$. Furthermore $\tilde{F}_2, \tilde{F}_3$ preserve λ and are λ invertible. Theorem 2.2 implies that $F_2 = F_{\mathcal{A}}$ and $F_3 = F_{\mathcal{B}}$ for some 2 and 3-λ-dense partitions $\mathcal{A}$ and $\mathcal{B}$ of I. □

3 $h_\lambda(\phi_{\mathcal{A}}) = 0$

Let $F : \mathbb{R} \to \mathbb{R}$ is be a nondecreasing function, which may be discontinuous. Then F has a countable number of point of discontinuities. We will assume the normalization that F is right continuous. Assume now that F is an increasing function on $\mathbb{R}$ which is not bounded from below and above. Then there exists a unique continuous nondecreasing function $f : \mathbb{R} \to \mathbb{R}$, which unbounded from below and above, such that $f \circ F = \mathrm{Id}$. We call f the inverse of F. Vice versa, if $f : \mathbb{R} \to \mathbb{R}$ is a continuous nondecreasing function, which is not bounded from below and above, then there exists a unique increasing function $F : \mathbb{R} \to \mathbb{R}$ such that $f \circ F = \mathrm{Id}$. We call F the inverse of f.

Let $k \in \mathbb{N}$ and assume that F is an increasing function on $\mathbb{R}$ which is continuous at the integer points $\mathbb{Z}$ and satisfies (2.1). Then we can define a measurable map $\tilde{F} : S^1 \to S^1$. We call $\tilde{F}$ an almost k-covering map.

Theorem 3.1 *Let F be an increasing function on $\mathbb{R}$ continuous on $\mathbb{Z}$ and satisfying (2.1) for an integer $k \geq 2$. Let f be the inverse function of F. Then almost k-covering map $\tilde{F} : S^1 \to S^1$ preserves the Lebesgue measure λ if and only if there exists k nonnegative measurable functions $p_1, ..., p_k$ such that*

$$\sum_{i=1}^{k} p_i(x) = 1, \quad \text{a.e. in } I,$$

$$f(x + i - 1) = \int_0^x p_i d\lambda + \sum_{j=0}^{i-1} \int_0^1 p_j d\lambda, \quad p_0(x) = 0,\ x \in I,\ i = 1, ..., k.$$

In particular, $\tilde{F}$ is λ-preserving and is invertible with respect to λ if and only if there exists a k-partition $\mathcal{A} = \{A_1, ..., A_k\}$ of I such that $p_i = \chi_{A_i}$ a.e. for $i = 1, ..., k$. In this case $\phi_{\mathcal{A}}$ is the λ inverse of $\tilde{F}$.

The proof of this theorem follows from simple modifications of the proof of Theorem 2.2 and is left to the reader.

Let $U, V \in \Sigma$. In what follows we use the notation:

$$U \sim V \iff \lambda(U \Delta V) = 0, \quad U \not\sim V \iff \lambda(U \Delta V) > 0.$$

Let $J \subset \mathbb{R}$ be an interval of positive Lebesgue measure (open, closed or half open). Let $\mathcal{A} = \{A_1, ..., A_k\}$ and $\mathcal{B} = \{B_1, ..., B_m\}$ be two partitions of J. Recall that $\mathcal{A}$ and $\mathcal{B}$ are equivalent if there exist permutationa $\mu :< k > \to < k >$, $\nu :< m > \to < m >$ and positive integer p such that

$$A_{\mu(i)} \sim B_{\nu(i)},\ i = 1, ..., p,\ A_{\mu(i)} \sim B_{\nu(j)}) \sim \emptyset \text{ for } i > p \text{ and } j > p.$$

Theorem 3.2 *Let $k \geq 1$ and assume that $\mathcal{A} = \{A_1, ..., A_k\}$ is a partition of $I = [0,1)$. Let $f_{\mathcal{A}}$ be the continuous nondecreasing function defined by (1.3-1.4). Let $F_{\mathcal{A}} : \mathbb{R} \to \mathbb{R}$ be the increasing function which is the inverse of $f_{\mathcal{A}}$. Let $\tilde{F}_{\mathcal{A}}$ be almost k-covering of S^1 preserving λ and whose λ inverse is $\phi_{\mathcal{A}}$. Let $0 = \beta_0 \leq \beta_1 \leq ... \leq \beta_k = 1$ be defined in (1.1). Let $\mathcal{B} = \{[\beta_0, \beta_1), [\beta_1, \beta_2), ..., [\beta_{k-1}, \beta_k)\}$ be a partition of S^1 to k intervals. Then the partition $\mathcal{B}_n := \mathcal{B} \vee \phi_{\tilde{\mathcal{A}}}\mathcal{B} \vee ... \vee \phi_{\tilde{\mathcal{A}}}^n \mathcal{B}$ is equivalent to a partition of $[0,1)$ to intervals $\mathcal{C}_n := \{J_{n,1}, ..., J_{n,\ell(n)}\}$ with the following properties:*
(a) $\ell(0) = k,\ J_{0,j} = [\beta_{j-1}, \beta_j),\ j = 1, ..., k.$
(b) *$\mathcal{C}_n$ is obtained from $\mathcal{C}_{n-1}$ by subdividing each interval $J_{n-1,j}$ to a finite number of subintervals for each $n \in \mathbb{N}$.*
Then one of the following conditions holds:
(c) *The partitions $\mathcal{C}_n,\ n = 0, 1, ...,$ separate points on S^1.*
(d) *The partitions $\mathcal{C}_n,\ n = 0, 1, ...,$ do not separate points on S^1. Then there exists a nonempty countable $\mathcal{J}$ with the following properties. For each $j \in \mathcal{J}$ there exist $m_j \in \mathbb{N}$ pairwise disjoint open intervals $I_{j,1}, ..., I_{j,m_j} \subset S^1$ of equal length such that $\phi_{\mathcal{A}}$ acts on $\{I_{j,1}, ..., I_{j,m_j}\}$ as an orientation preserving cyclic interval exchange up to a set of zero measure:*

$$\begin{aligned} &\phi_{\mathcal{A}}(I_{j,p}) \subset \overline{I}_{j,p+1}, \\ &I_{j,p+1} \sim \phi_{\mathcal{A}}(I_{i,p}),\ p = 1, ..., m_j,\ (I_{j,m_j+1} = I_{i,1}), \text{ for any } j \in \mathcal{J}, \\ &I_{j,p} \cap I_{j',p'} = \emptyset \text{ for any } j \neq j' \text{and } p \in <m_j>,\ p' \in <m_{j'}> . \end{aligned} \tag{3.1}$$

Let $X = \cup_{j \in \mathcal{J}} \cup_{p=1}^{m_j} \overline{I}_{j,q}$. Then the restriction of the partitions $\mathcal{C}_n,\ n = 0, 1, ...$ to $S \backslash X$ separate the points in $S \backslash X$.

Hence in both of the cases the measure entropy $h_\lambda(\phi_{\mathcal{A}})$ equals to zero.

Proof. For $k = 1$ $\tilde{F}_{\mathcal{A}} = \mathrm{Id}$ and the theorem is trivial. Without a loss of generality we may assume that $k \geq 2$ and $\lambda(A_i) > 0$ for $i = 1, ..., k$.

Let $J \subset \mathbb{R}$ be an interval. From the definition of $f_{\mathcal{A}}$ it follows that $f_{\mathcal{A}}(J)$ is an interval. Let $J \subset [0,1)$. Define $I_i = f_{\mathcal{A}}(J + i - 1) \cap [\beta_{j-1}, \beta_j)$ for $i = 1, ..., k$. Then $I_1, ..., I_k$ are pairwise distinct intervals, which may be empty or consisting of one point. From the definition of $\phi_{\mathcal{A}}$ it follows that $\phi_{\mathcal{A}}(J) \sim \cup_{i=1}^k I_i$. Hence $\mathcal{B}_n$ is equivalent to a partition $\mathcal{C}_n$ of $[0,1)$ to disjoint intervals. Furthermore $\mathcal{C}_n$ is the refinement of $\mathcal{C}_{n-1}$. Hence (a) and (b) hold.

Assume first that the partitions $\mathcal{C}_n,\ n = 0, 1, ...,$ separate points. Hence $\vee_{n=0}^\infty \mathcal{C}_n$ is equivalent to the Borel σ-algebra on S^1 up to sets of zero measure. Therefore $\vee_{n=0}^\infty \phi_{\mathcal{A}}^n \mathcal{B}$ is equivalent to the Borel σ-algebra on S^1 up to sets of zero measure. As $\tilde{F}_{\mathcal{A}}^{-1} = \phi_{\mathcal{A}}$ we deduce that $h_\lambda(\tilde{F}_{\mathcal{A}}) = 0$, e.g. [6, Cor.4.18.1], which implies that $h_\lambda(\phi_{\mathcal{A}}) = 0$.

Assume now that $\mathcal{C}_n,\ 0,1,...$, do not separate points. That is there is at least one nested set of intervals $J_{1,q_1} \supset J_{2,q_2} \supset ...$ such that $\cap_{i=1}^{\infty} \overline{J}_{i,q_i} = K = \overline{K}_o,\ K_o = (a,b),\ 0 \le a < b \le 1$. Note that for each $i \ge 2$ there exists $J^1_{i-1,q^1_{i-1}}$ such that $J_{i,q_i} \backslash \phi_{\mathcal{A}}(J^1_{i-1,q^1_{i-1}}) \sim \emptyset$. Then $J^1_{1,q^1_1} \supset J^1_{2,q^1_2} \supset ...$ is nested set of intervals such that $\cap_{i=1}^{\infty} \overline{J}^1_{i,q^1_i} = K^1$ is a closed interval in S. Clearly $\lambda(K \backslash \phi_{\mathcal{A}}(K^1)) = 0$. Hence $\lambda(K^1) \ge \lambda(K)$, i.e. $K^1 = \overline{K}^1_o,\ K^1_o = (a_1, b_1),\ 0 \le a_1 < b_1 \le 1,\ b_1 - a_1 \ge b - a$. Since K and K^1 are intersection of nested sequences of the intervals in the partitions $\mathcal{C}_n,\ n = 1,...$, it follows that either $K_o = K^1_o$ or $K_o \cap K^1_o = \emptyset$. Repeating this argument we obtain for each integer $p \ge 2$ a sequence of nested intervals $J^p_{1,q^p_1} \supset J^p_{2,q^2_2} \supset ...$ such that $\cap_{i=1}^{\infty} \overline{J}^p_{i,q^p_i} = K^p$ is a closed interval in S. Furthermore $\lambda(K^{p-1} \backslash \phi_{\mathcal{A}}(K^p)) = 0$. Hence $K^p = \overline{K}^p_o,\ K^p_o = (a_p, b_p),\ 0 \le a_p < b_p \le 1,\ b_p - a_p \ge b_{p-1} - a_{p-1}$ for $p = 2, 3, ...,$. Let $K = K^0$. Then for any $0 \le r < p$ either $K^r_o = K^p_0$ or $K^r_o \cap K^p_o = \emptyset$. Consider the sequence of open intervals $K^0_o, K^1_o, K^1_o, ...$ in $(0,1)$, whose length is a nondecresing sequence. Then it is impossible that all these open intervals are pairwise disjoint. So assume that $K^r_o \cap K^p_o \ne \emptyset$ for some $0 \le r < p$. Hence $K^r_o = K^p_o$. If $K^{r+1}_o = K^r_o$ we choose $p = r + 1$. Otherwise we can assume without loss of generality that $K^j_o \ne K^r_o$ for $j = p-1, ..., r+1$. Clearly $\lambda(K^j) = \lambda(K^r),\ j = p-1, ..., r+1$. Therefore up a zero measure $\phi_{\mathcal{A}}$ acts the orientation preserving interval exchange $K^r_o = K^p_o \to K^{p-1}_o \to ... \to K^r_o$ of $p - r$ distinct open intervals in $(0,1)$. Obviuosly K^0_o appears among this $p - r$ intervals.

Clearly all maximal open intervals K_o whose points are not separated by $\mathcal{C}_n,\ n = 0, ...$, is a countable set of pairwise disjoint intervals of $(0,1)$. If we group each K_o with the other $p - r - 1$ intervals as above, we obtain a countable set $\mathcal{J}$ of such groups as described in the theorem. Let $X = \cup_{j \in \mathcal{J}} \cup_{p=1}^{m_j} \overline{I}_{j,q}$. Then $\phi_{\mathcal{A}}(X) = X$ (up to zero measure sets). Clearly $h_\lambda(\phi_{\mathcal{A}}|_X) = 0$. Then $Y = S^1 \backslash X$ is $\phi_{\mathcal{A}}$ invariant set (up to a set of zero measure). $\mathcal{C}_n \cap Y,\ n = 0, ...$, separates the points on Y. The arguments in the beginning of the proof of the theorem yield that $h_\lambda(\phi_{\mathcal{A}}|_Y) = 0$. Hence $h_\lambda(\phi_{\mathcal{A}}) = 0$.

4 The condition $f_{\mathcal{A}} \circ f_{\mathcal{B}} = f_{\mathcal{A}} \circ f_{\mathcal{B}}$

Lemma 4.1 *Let $\mathcal{A}$, $\mathcal{B}$ be 2 and 3 partitions of $I = [0,1)$ respectively. Then*

$$f_{\mathcal{A}} \circ f_{\mathcal{B}} = f_{\mathcal{C}}, \quad f_{\mathcal{B}} \circ f_{\mathcal{A}} = f_{\mathcal{D}} \tag{4.1}$$

for some 6-partitions $\mathcal{C}$, $\mathcal{D}$ of I. Suppose furthermore that $\mathcal{A}$ and $\mathcal{B}$ are λ-dense partitions. Then $\mathcal{C}$ and $\mathcal{D}$ are λ-dense partitions.

Proof. Clearly

$$f'_{\mathcal{A}} = \chi_A, \quad f'_{\mathcal{B}} = \chi_B,$$
$$(f_{\mathcal{A}} \circ f_{\mathcal{B}})' = \chi_{f_{\mathcal{B}}^{-1}(A)}\chi_B, \quad (f_{\mathcal{B}} \circ f_{\mathcal{A}})' = \chi_{f_{\mathcal{A}}^{-1}(B)}\chi_A,$$
$$f_{\mathcal{A}} \circ f_{\mathcal{B}}(x+6) = f_{\mathcal{A}} \circ f_{\mathcal{B}}(x) + 1, \quad f_{\mathcal{B}} \circ f_{\mathcal{A}}(x+6) = f_{\mathcal{B}} \circ f_{\mathcal{A}}(x) + 1.$$

Let

$$\begin{aligned} B_{i,j} &:= \{x \in B_i : \quad f_{\mathcal{B}}(i-1+x) \in A_j\}, \text{ for } i=1,2,3, \ j=1,2, \\ A_{j,i} &:= \{x \in A_j : \quad f_{\mathcal{A}}(j-1+x) \in B_i\}, \text{ for } i=1,2,3, \ j=1,2, \end{aligned} \tag{4.2}$$

We claim that

$$\mathcal{C} := \{B_{1,1}, B_{2,1}, B_{3,1}, B_{1,2}, B_{2,2}, B_{3,2}\}, \quad \mathcal{D} := \{A_{1,1}, A_{2,1}, A_{1,2}, A_{2,2}, A_{1,3}, A_{2,3}\} \tag{4.3}$$

are 6-partitions of I and (4.1) holds. Since $\mathcal{B}$ is a partition of I $B_{i,j} \cap B_{p,q} = \emptyset$ for $i \neq p$. As $\mathcal{A}$ is a partition of I $B_{i,j} \cap B_{i,p} = \emptyset$ for $j \neq p$. As $f_{\mathcal{B}}([0,3]) = [0,1]$ and $f_{\mathcal{B}}(B \cap [0,3])$ has measure 1 it follows that $\mathcal{C}$ is a 6-partition of I. Similar arguments show that $\mathcal{D}$ is a 6 partition of I. Let $C, D \subset \mathbb{R}$ be the induced sets by $\mathcal{C}$, $\mathcal{D}$ respectively. The definition of $\mathcal{C}$ and a straightforward calculation shows that $(f_{\mathcal{A}} \circ f_{\mathcal{B}})' = \chi_C$. As $f_{\mathcal{A}} \circ f_{\mathcal{B}}(0) = 0$ we deduce the first equality of (4.1). The second equality of (4.1) follows similarly.

Suppose now that $\mathcal{A}$ and $\mathcal{B}$ are λ-dense partitions. Then $f_{\mathcal{A}}$ and $f_{\mathcal{B}}$ are increasing. Hence $f_{\mathcal{A}} \circ f_{\mathcal{B}}$ and $f_{\mathcal{B}} \circ f_{\mathcal{A}}$ are also increasing. The equalites (4.1) yield that $\mathcal{C}$ and $\mathcal{D}$ are λ-dense partitions. □

For a set $A \subset \mathbb{R}$ we denote

$$A(s,t) := A \cap [s,t], \quad s \le t,$$
$$A(t) := A \cap [0,t], \quad 0 \le t.$$

Let $\mathcal{A} = \{A_1, ..., A_k\}$ and $\mathcal{A}' = \{A'_1, ..., A'_k\}$ be two k-partitions of $[0,1)$. We say that $\mathcal{A}$ and $\mathcal{A}'$ are strongly equivalent, and denote it by $\mathcal{A} \sim \mathcal{B}$ if $A_i \sim A'_i$ for $i = 1, ..., k$.

Lemma 4.2 *Let $\mathcal{A}$ and $\mathcal{B}$ be* 2 *and* 3 *partitions of* $[0,1]$ *respectively. Let $A, B \in \Sigma$ be defined by $\mathcal{A}, \mathcal{B}$ using (1.3) respectively. Then the following are equivalent*
(a) (1.8) *holds.*
(b) *The partitions $\mathcal{C}$ and $\mathcal{D}$ given in* (4.3) *are both strongly equivalent to the partition*

$$\mathcal{A} \cdot \mathcal{B} := \{A_1 \cap B_1, A_2 \cap B_2, A_1 \cap B_3, A_2 \cap B_1, A_1 \cap B_2, A_2 \cap B_3\}. \tag{4.4}$$

(c)

$$A(f_{\mathcal{B}}(s), f_{\mathcal{B}}(t)) \sim f_{\mathcal{B}}(A(s,t) \cap B(s,t)), \quad \text{for all } s \le t,$$
$$B(f_{\mathcal{A}}(s), f_{\mathcal{A}}(t)) \sim f_{\mathcal{A}}(A(s,t) \cap B(s,t)), \quad \text{for all } s \le t. \tag{4.5}$$

Proof. Assume (a). Then (4.1) implies that $\mathcal{C} \sim \mathcal{D}$. Furthermore $C \sim D \subset A \cap B$. A straightforward argument yields that $A \cap B$ is induced by a partition $\mathcal{A} \cdot \mathcal{B}$. As $1 = \lambda(C(6)) = \lambda(A \cap B \cap [0,6]$ we deduce that $C \sim A \cap B$ and $\mathcal{C} \sim \mathcal{D} \sim \mathcal{A} \cdot \mathcal{B}$.

Assume (b). Then (4.1) implies (a).

Assume (a) and (b). Use the definition of $\mathcal{C}$ and the condition $C \sim A \cap B$ and to deduce the first condition in (4.5) with $s = 0$ and $t \ge 0$. Hence the first condition of (4.5) holds for any $0 \le s \le t$. Use the the condition (1.5) for $f_{\mathcal{B}}$ with $k = 3$ to deduce the condition of (4.5) for any $s \le t$. The second condition in (4.5) is derived similarly.

Assume (c). Recall that $f_{\mathcal{B}}$ maps any measurable set $E \subset B$ to a set E' of the same measure. Furthermore the complement of B (B^c) is mapped to a set of zero measure. Hence

$$f_{\mathcal{B}}(B(t)) \sim f_{\mathcal{B}}([0,t]) = [0, f_{\mathcal{B}}(t)] \Rightarrow \lambda(A(t) \cap B(t)) = \lambda(f_{\mathcal{B}}(A(t) \cap B(t)).$$

Similar conditions hold for $f_{\mathcal{A}}([0,t])$. Assume first that (4.5) holds for $s = 0$ and any $t \ge 0$. Then

$$f_{\mathcal{A}}(f_{\mathcal{B}}(t)) = \lambda(A(f_{\mathcal{B}}(t)) = \lambda(f_{\mathcal{B}}(A(t) \cap B(t)) = \lambda(A(t) \cap B(t)) =$$
$$\lambda(f_{\mathcal{A}}(A(t) \cap B(t)) = \lambda(B(f_{\mathcal{A}}(t)) = f_{\mathcal{B}}(f_{\mathcal{A}}(t)).$$

Hence (1.8) holds for any $t \ge 0$. Since the two functions appearing in (1.8) satisfy (1.5) we deduce (1.8) for all $t \in \mathbb{R}$. □

It is straightforward to show that the condition (1.8) yields the condition (1.9). In the next section we show that the condition (1.9) is sometimes weaker than (1.8). Recall that a (k-)partition $\mathcal{A} = \{A_1, ..., A_k\}$ of I is called a regular (k-)partition if $\lambda(A_i) > 0$ for $i = 1, ..., k$. The following Proposition is straightforward.

Proposition 4.3 *Let*

$$\mathcal{A} = \{[0,t), [t,1)\}, \ \mathcal{B} = \{[0,t), \emptyset, [t,1)\} \quad \text{for } t \in [0,1]. \tag{4.6}$$

Then (1.8) *holds. Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be* 2 *and* 3*-partitions of* I *which are not strongly equivalent to the corresponding two partitions given in* (4.6)*. Assume that* (1.8) *holds. Then* $\mathcal{A}$ *and* $\mathcal{B}$ *are regular partitions of* I*.*

5 Interval exchanges

In this section we consider only partitions of the interval $I = [0, 1)$ induced by the partition of I to n intervals of equal length $\frac{1}{n}$. Let $\mathcal{J} := \{J_1, ..., J_n\}$ be a partition of I to $n \geq 2$ half closed intervals of length $\frac{1}{n}$ arranged in an increasing order. Let $2 \leq k \leq n$ and let $\Omega_1, ..., \Omega_k$ be a partition of $< n >$ to k disjoint (possibly empty) sets. Set

$$A_j = \cup_{l \in \Omega_j} J_l, \quad j = 1, ..., k.$$

Then $\mathcal{A} = \{A_1, ..., A_k\}$ is called a k-n-partition of I. $\mathcal{A}$ is a regular k-n-partition of I if and only if each Ω_j is a nonempty set. Then $\phi_{\mathcal{A}}$ is an interval exchange. $\phi_{\mathcal{A}}$ induces the following permutation $\sigma :< n >\to< n >$:

$$\phi_{\mathcal{A}}(J_i) = J_{\sigma(i)}, \quad i = 1, ..., n.$$

σ maps the nonempty set Ω_j to the set $[\gamma_{j-1} + 1, \gamma_{j-1} + |\Omega_j|] \cap \mathbb{Z}$ monotonically for $j = 1, ..., k$. Here

$$\gamma_0 = 0, \quad \gamma_j = \sum_{l=1}^{j} |\Omega_l|, \quad j = 1, ..., k.$$

Any k-n-interval partition $\mathcal{A}$ induces a unique regular m-n-interval partition $\mathcal{A}'$ with $1 \leq m \leq n$, by discarding the empty sets. Clearly, $\phi_{\mathcal{A}} = \phi_{\mathcal{A}'}$, that is $\mathcal{A}$ and $\mathcal{A}'$ induce the same interval exchange on I. Equivalently, $\mathcal{A}$ and $\mathcal{A}'$ induce the same permutation $\sigma :< n >\to< n >$. Any permutation σ on $< n >$ we identify with the ordered set of the elements of $< n >$:

$$\{i_1, i_2, ..., i_n\} = \{\sigma^{-1}(1), \sigma^{-1}(2), ..., \sigma^{-1}(n)\}. \tag{5.1}$$

It is easy to show that σ given in the above form is iduced by a unique minimal regular m-n-interval partition, where m is exactly the number of $j \leq n-1$ for which $i_j > i_{j+1}$.

Lemma 5.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be 2-n-interval and 3-n-interval regular partitions of I respectively. Assume that the condition* (1.8) *holds. Suppose furthermore that the induced permutations σ, η fixes either* 1 *or n. Then there exist* 2-$(n-1)$*-interval and* 3-$(n-1)$*-interval partitions $\mathcal{A}'$ and $\mathcal{B}'$ satisfying the condition* (1.8).

Proof. Since $\mathcal{B}$ is a regular 3-n partition of I we obtain that $n \geq 3$. Let

$\Gamma_1 := \{1 \leq i_1 < i_2 < ... < i_p\}$,
$\Gamma_2 = \{1 \leq i_{p+1} < i_{p+2} < ... < i_n\}$,

$$
\begin{aligned}
&1 \le p < n,\ \Gamma_1 \cup \Gamma_2 =< n >,\\
&\Delta_1 := \{1 \le j_1 < j_2 < ... < j_q\},\\
&\Delta_2 = \{1 \le j_{q+1} < j_{q+2} < ... < j_{q'}\},\\
&\Delta_3 = \{1 \le j_{q'+1} < j_{q'+2} < ... < j_n\},\\
&1 \le q < q' < n,\ \Delta_1 \cup \Delta_2 \cup \Delta_3 =< n >,\\
&A_i = \cup_{m\in\Gamma_i}[\frac{m-1}{n}, \frac{m}{n}),\ i = 1,2, \quad B_j = \cup_{m\in\Delta_j}[\frac{m-1}{n}, \frac{m}{n}),\ j = 1,2,3.
\end{aligned} \tag{5.2}
$$

Assume first that σ, η fix 1. Then $i_1 = j_1 = 1$. Let

$$
\begin{aligned}
\Gamma_1' &= \{i_2 - 1, ..., i_p - 1\},\\
\Gamma_2' &= \{i_{p+1} - 1, ..., i_n - 1\},\\
\Delta_1' &= \{j_2 - 1, ..., j_q - 1\},\\
\Delta_2' &= \{j_{q+1} - 1, j_{q+2} - 1, ..., j_{q'} - 1\},\\
\Delta_3' &= \{j_{q'+1} - 1, j_{q'+2} - 1, ..., j_n - 1\}.
\end{aligned}
$$

Let $\mathcal{A}', \mathcal{B}'$ be induced by $\{\Gamma_1', \Gamma_2'\}, \{\Delta_1', \Delta_2', \Delta_3'\}$ respectively. A straightforward argument using Lemma 4.2 shows that

$$
f_{\mathcal{A}} \circ f_{\mathcal{B}} = f_{\mathcal{B}} \circ f_{\mathcal{A}} \Rightarrow f_{\mathcal{A}'} \circ f_{\mathcal{B}'} = f_{\mathcal{B}'} \circ f_{\mathcal{A}'}. \tag{5.3}
$$

(Another way to deduce the above implication is to collaps each interval $[m, m+\frac{1}{n}) \subset \mathbb{R}$, $m \in \mathbb{Z}$ to a point to obtain R. Then (1.8) holds also on R, which is equivalent to $f_{\mathcal{A}'} \circ f_{\mathcal{B}'} = f_{\mathcal{B}'} \circ f_{\mathcal{A}'}$.)

Assume now that σ, η fix n. Then $i_{p'} = j_{q''} = n$. Let $\Gamma_2' = \Gamma_2\backslash\{n\}, \Delta_3' = \Delta_3\backslash\{n\}$. Let $\mathcal{A}', \mathcal{B}'$ be induced by $\{\Gamma_1, \Gamma_2'\}, \{\Delta_1, \Delta_2, \Delta_3'\}$ respectively. Then (5.3) holds. □

Lemma 5.2 *Let $\mathcal{A}$ and $\mathcal{B}$ be regular 2-n and 3-n-partitions induced by the regular 2-n and 3-n-partitions of $< n >$ given in* (5.2). *Let the partition $\mathcal{C} = \mathcal{A} \cdot \mathcal{B}$, given by* (4.4), *be induced by*

$$
\begin{aligned}
\Omega_1 &= \Gamma_1 \cap \Delta_1 = \{k_1, ...k_{r_{11}}\}, \quad r_{11} \ge 0,\\
\Omega_2 &= \Gamma_2 \cap \Delta_2 = \{k_{r_{11}+1}, ..., k_{r_{22}}\}, \quad r_{22} \ge r_{11},\\
\Omega_3 &= \Gamma_1 \cap \Delta_3 = \{k_{r_{22}+1}, ..., k_{r_{13}}\}, \quad r_{13} \ge r_{22},\\
\Omega_4 &= \Gamma_2 \cap \Delta_1 = \{k_{r_{13}+1}, ..., k_{r_{21}}\}, \quad r_{21} \ge r_{13},\\
\Omega_5 &= \Gamma_1 \cap \Delta_2 = \{k_{r_{21}+1}, ..., k_{r_{12}}\}, \quad r_{12} \ge r_{21},\\
\Omega_6 &= \Gamma_2 \cap \Delta_3 = \{k_{r_{12}+1}, ..., k_{r_{23}}\}, \quad n = r_{23} \ge r_{12}.
\end{aligned} \tag{5.4}
$$

Assume that (1.8) *holds. Then*

$$q = r_{22} \le p = r_{13} \le q' = r_{21}. \tag{5.5}$$

$$k_u = i_{j_u} = j_{i_u}, \quad u = 1, ..., n. \tag{5.6}$$

$$\begin{aligned} &j_{r_{11}} \le p < j_{r_{11}+1} \le j_q, \\ &j_{q+1} \le j_p \le p < j_{p+1} \le j_{q'}, \\ &j_{q'+1} \le j_{r_{12}} \le p < j_{r_{12}+1}, \end{aligned} \tag{5.7}$$

$$\begin{aligned} &i_{r_{11}} \le q < i_{r_{11}+1} \le i_q \le q' < i_{q+1} \le i_p, \\ &i_{p+1} \le i_{q'} \le q < i_{q'+1} \le i_{r_{12}} \le q' < i_{r_{12}+1}. \end{aligned} \tag{5.8}$$

If one the below equalities hold

$$0 = r_{11},\ r_{11} = q,\ q = p,\ p = q',\ q' = r_{12},\ r_{12} = n, \tag{5.9}$$

then the above corresponding inequalities are vacuous.

Proof. Lemma 4.2 yields

$$\begin{aligned} &f_{\mathcal{A}}(A(2) \cap B(2)) = B(1) = B_1 \Rightarrow \frac{r_{22}}{n} = \lambda(f_{\mathcal{A}}(A(2) \cap B(2))) = \lambda(B_1) = \frac{q}{n}, \\ &f_{\mathcal{A}}(A(4) \cap B(4)) = B(2) = B_1 \cup (1 + B_2) \Rightarrow \\ &\frac{r_{21}}{n} = \lambda(f_{\mathcal{A}}(A(2) \cap B(2))) = \lambda(B_1) + \lambda(B_2) = \frac{q'}{n}, \\ &f_{\mathcal{B}}(A(3) \cap B(3)) = A(1) = A_1 \Rightarrow \frac{r_{13}}{n} = \lambda(f_{\mathcal{B}}(A(3) \cap B(3))) = \lambda(A_1) = \frac{p}{n}. \end{aligned}$$

Hence (5.5) holds.

Let σ, η be the permutations of $< n >$ induced by $\{\Gamma_1, \Gamma_2\}, \{\Delta_1, \Delta_2, \Delta_3\}$ respectively. Consider $k_u, \in \Gamma_i \cap \Delta_j$ for some $i \in< 2 >, j \in< 3 >$. Then $k_u = i_l$ for $l \in< p >$ if $i = 1$ and $l > p$ if $i = 2$. k_u corresponds to the interval $[t_u - \frac{1}{n}, t_u] \in A(2m_{ij}) \cap B(2m_{ij})$ for the smallest integer $m_{ij} \in< 3 >$. Then $f_{\mathcal{A}}(A(t_u) \cap B(t_u)) = B(f_{\mathcal{A}}(t_u))$ is of total length $\frac{u}{n}$. So the interval $[t_u - \frac{1}{n}, t_u)$ is mapped on the interval $[m_{ij} - 1 + \frac{j_u - 1}{n}, m_{ij} - 1 + \frac{j_u}{n}) \in m_{ij} - 1 + B_{m_{ij}}$. Hence $\sigma(i_l) = l = j_u$. This proves the first equality in (5.6). Observe next that $k_u = j_v$. Use the identity $f_{\mathcal{B}}(A(t_u) \cap B(t_u)) = A(f_{\mathcal{B}}(t_u))$ to deduce the the equality $v = i_u$.

If $r_{11} > 0$ then $k_{r_{11}} \in \Omega_1 \subset \Gamma_1$. As $k_{r_{11}} = i_{j_{r_{11}}}$ it follows that $j_{r_{11}} \leq p$. If $q = r_{22} > r_{11}$ then $k_{r_{11}+1} \in \Omega_2 \subset \Gamma_2$. As $k_{r_{11}+1} = i_{j_{r_{11}+1}}$ it follows that $j_{r_{11}+1} > p$. If $q = r_{22} < r_{13} = p$ then $\Omega_3 \neq \emptyset$. Then $k_{q+1}, k_p \in \Gamma_1$. As $k_{q+1}) = i_{q+1}, k_p = i_{j_p}$ it follows that $j_{q+1} \leq j_p \leq p$. If $p = r_{13} < r_{21} = q'$ then $\Omega_4 \neq \emptyset$. Then $k_{p+1} \in \Gamma_2$. As $k_{p+1} = i_{j_{p+1}}$ it follows that $j_{p+1} > p$. If $q' = r_{21} < r_{12}$ then $\Omega_5 \neq \emptyset$. Then $k_{q'+1}, k_{r_{12}} \in \Gamma_1$. As $k_{q'+1} = i_{q'+1}, k_{r_{12}} = i_{j_{r_{12}}}$ it follows that $j_{q'+1} \leq j_{r_{12}} \leq p$. If $r_{12} < r_{23}$ then $\Omega_6 \neq \emptyset$. Then $k_{r_{12}+1} \in \Gamma_2$. As $k_{r_{12}+1} = i_{j_{r_{12}+1}}$ it follows that $j_{r_{12}+1} > p$. These arguments prove (5.7).

Recalling that Ω_i is also a subset of the corresponding Δ_j and combining the above arguments with the equality $k_u = j_{i_u}$ we deduce (5.8). $\square$

Corollary 5.3 *Let the assumptions of Lemma 5.2 hold. Then*

$$\begin{aligned} &q + q' = 2p,\ r_{11} = q - q' + p,\ r_{12} = q' - q + p, \\ &1 \leq q < q' < n,\ q \leq p \leq q',\ 2q \geq p,\ 3p - 2q \leq n. \end{aligned} \tag{5.10}$$

Corollary 5.4 *Let $\mathcal{A}$ and $\mathcal{B}$ be* 2-n *and* 3-n-*partitions which are not of the form* (4.6)*. Assume that $n \leq 3$. Then* (1.8) *does not hold.*

Let $n = 3$ and assume that σ is the cyclic permutation on $< 3 >$. Let $\eta = \sigma^2$. A straightforward calculation shows that for $\mathcal{A} = \{A_1, A_2\}$ and $\mathcal{B} = \{B_1, B_2, B_3\}$:

$$A_1 = \{J_2, J_3\},\ A_2 = \{J_1\}, \quad B_1 = J_3,\ B_2 = J_1,\ B_3 = J_2,$$

$\phi_{\mathcal{A}}$ and $\phi_{\mathcal{B}}$ are inducing the permutations σ and η of $< 3 >$ respectively. Hence (1.9) holds. In view of Corollary 5.4 (1.8) does not hold.

Lemma 5.5 *The following regular* 2-4 *and* 3-4-*interval partitions*

$$\mathcal{A} = \{\{J_2, J_4\}, \{J_1, J_3\}\}, \quad \mathcal{B} = \{\{J_3\}, \{J_1, J_4\}, \{J_2\}\} \tag{5.11}$$

are the unique regular 2-4 *and* 3-4-*interval partitions for which* (1.8) *holds. The induced permutations σ, η are cyclic permutation with $\eta = \sigma^{-1}$.*

The proof of the lemma is left to the reader. Combine Lemma 5.1 with Lemma 5.5 to obtain:

Corollary 5.6 *Let p, n be nonnegative integers such that $0 \le p \le n-4$. Then the following regular 2-n and 3-n-partitions satisfy* (1.8)*:*

$$\mathcal{A} :=$$
$$\{\{[0, \frac{p}{n}), [\frac{p+1}{n}, \frac{p+2}{n}), [\frac{p+3}{n}, \frac{p+4}{n})\}, \{[\frac{p}{n}, \frac{p+1}{n}), [\frac{p+2}{n}, \frac{p+3}{n}), [\frac{p+4}{n}, 1)\}\},$$
$$\mathcal{B} :=$$
$$\{\{[0, \frac{p}{n}), [\frac{p+2}{n}, \frac{p+3}{n})\}, \{[\frac{p}{n}, \frac{p+1}{n})\}, \{[\frac{p+3}{n}, \frac{p+4}{n})\}, \{[\frac{p+1}{n}, \frac{p+2}{n}), [\frac{p+4}{n}, 1)\}\}.$$

The corresponding permutations σ, η satisfy $\eta = \sigma^{-1}$.

For $n = 2m$ with $m \ge 2$ and $3 \nmid 2m+1$, there exist regular $2-n$ and $3-n$ partitions of I, induced by the commuting maps G_2, G_3, for which (1.8) holds.

Lemma 5.7 *Let $m \ge 2$ be an integer and assume that $2m+1$ is not divisible by 3. Let $\sigma_1, \eta_1 :< 2m > \to < 2m >$ are given by the maps $x \to 2x, x \to 3x$ modulo $2m+1$ restricted to $< 2m >$. Then σ_1 and η_1 commute. Let $\mathcal{A}_{2m}, \mathcal{B}_{2m}$ be the regular $2-2m, 3-2m$ partitions induced by*

$$\Gamma_1 := \{\sigma_1(1), \sigma_1(2), \ldots, \sigma_1(m)\},\ \Gamma_2 := \{\sigma_1(m+1), \sigma_1(m+2), \ldots, \sigma(2m)\},$$
$$\Delta_1 = \{\eta_1(1), \ldots, \eta_1(\lfloor \frac{2m+1}{3} \rfloor)\},\ \Delta_2 = \{\eta_1(\lfloor \frac{2m+1}{3} \rfloor + 1), \ldots, \}, \eta_1(\lfloor \frac{4m+2}{3} \rfloor)\},$$
$$\Delta_3 = \{\eta_1(\lfloor \frac{4m+2}{3} \rfloor + 1), \ldots, \eta_1(2m)\}.$$

Then $\phi_{\mathcal{A}_{2m}} \circ \phi_{\mathcal{B}_{2m}} = \phi_{\mathcal{B}_{2m}} \circ \phi_{\mathcal{A}_{2m}}$.

The proof is left to the reader. Note that

$$\lim_{m\to\infty} \phi_{\mathcal{A}_{2m}}(x) = \frac{x}{2} = G_2^{-1}(x), \quad \lim_{m\to\infty} \phi_{\mathcal{B}_{2m}}(x) = \frac{x}{3} = G_3^{-1}(x).$$

Thus Lemma 5.7 does not give in the limit a contradiction to the 2-3 conjecture.

We do not know for which $m \ge 3$ the converse to Lemma 5.7 holds. That is, assume that $m \ge 3$, $3 \nmid 2m+1$ and $\mathcal{A} = \{A_1, A_2\}, \mathcal{B} = \{B_1, B_2, B_3\}$ are regular $2-2m, 3-2m$ partitions. Suppose furthermore that $J_{2m} \in A_1, J_1 \in A_2$ and (1.8) holds. Are $\mathcal{A}, \mathcal{B}$ equal to $\mathcal{A}_{2m}, \mathcal{B}_{2m}$ respectively?

Another way to find a counterexample to the 2-3 conjecture is to study the ergodic measures invariant under $\tilde{G}_2, \tilde{G}_3$, which are supported on a finite number of points. It is straightforward to show that such measure is equi-distributed on an orbit of the action of the permutations σ_1, η_1 given in Lemma 5.7. It seems that this approach is not straightforward related to the problem of the converse to Lemma 5.7 we discussed above.

References

[1] P. Arnoux, D.S. Ornstein and B. Weiss, Cutting and stacking, interval exchanges and geometric models, *Israel J. Math.* 50 (1985), 160-168.

[2] H. Furstenberg, Disjointness in ergodic theory, minimal sets, and a problem in diophantine approximation, *Math. Sys. Theory* 1 (1967), 1-49.

[3] W. Parry, In general a degree two map is an automorphism, *Contemporary Math.* 135 (1992), 219-224.

[4] D. Rudolph, $\times 2$ and $\times 3$ invariant measures and entropy, *Ergodic Theory & Dynamical Systems* 10 (1991), 395-406.

[5] Jean-Paul Thouvenot, *private communication.*

[6] P. Walters, *An Introduction to Ergodic Theory*, Springer-Verlag, 1982.