Matchings and Independent Sets: Problems, Conjectures and Results

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1 Outline of the lecture

1. Upper bounds for matchings in bipartite graphs
2. Lower bounds for matchings in bipartite graphs
3. Exact matching bounds for 2-regular graphs
4. LMC for regular graphs
5. Conjectural upper bounds for 3-regular graphs
6. Asymptotic growth of matchings & AUMC, ALMC
7. Graphs illustrations for dimensions 2, 3
8. Lower asymptotic bounds
9. Independent sets

B. Positive hyperbolic polynomials

1. Properties
2. Examples
3. Proof of cases of ALMC
2 Matchings

$G = (V, E)$ undirected graph with vertices $V$, edges $E$

matching in $G$: $M \subseteq E$

no two edges in $M$ share a common endpoint

e $= (u, v) \in M$ is dimer

$v$ not covered by $M$ is monomer

$M$ called monomer-dimer cover of $G$

$M$ is perfect matching $\iff$ no monomers

$M$ is $k$-matching $\iff \#M = k$

$\phi_k(G)$ number of $k$-matchings in $G$, $\phi_0(G) := 1$

$\Phi_G(x) := \sum_k \phi_k(G)x^k$ matching generating polyn.

roots of $\Phi_G(x)$ nonpositive [17].

$\Phi_{G_1 \cup G_2}(x) = \Phi_{G_1}(x)\Phi_{G_2}(x)$

$\Gamma(d, n)$ set of $d$-regular bipartite graphs on $2n$ vertices
3 Formulas for $k$-matchings

$$A = A(G) \in \{0 - 1\}^{n \times n}$$-adjacency matrix of $G = (V, E), \#V = n.$

$$\phi_k(G) = \text{haff } k(A) := 2^{-k} \sum_{1 \leq i_1 < \ldots < i_{2k} \leq n} \frac{\partial^{2k}}{\partial x_{i_1} \ldots x_{i_{2k}}} (x^T A x)^k, k \leq \frac{n}{2}$$

$$x = (x_1, \ldots, x_n)^T$$

$G = (V, E)$ bipartite $V = V_1 \cup V_2, E \subset V_1 \times V_2,$ $B = B(G) \in \{0 - 1\}^{m \times n}, \#V_1 = m, V_2 = n.$

$$\phi_k(G) = \text{perm } kB := \sum_{\alpha \in Q_{k,m}, \beta \in Q_{k,n}} \text{perm } B[\alpha, \beta].$$

for general $G$: $\phi_k(G) =$

$$2^{-k} \sum_{\alpha \in Q_{k,n}, \beta \in Q_{k,n}} \alpha \cap \beta = \emptyset, \text{perm } A[\alpha, \beta]$$

$$Q_{k,n} := \{ \alpha = \{i_1, \ldots, i_k\} : 1 \leq i_1 < i_2 < \ldots < i_k \leq n \}.$$
4 Upper bounds for matchings

$A = [a_{ij}]_{i,j=1}^{n} \in \{0 - 1\}^{n \times n}$ represents bipartite graph $G$ on $n$ vertices in each class, with degree $r_i = \sum_{j=1}^{n} a_{ij}, i = 1, \ldots, n$ in first class

$\text{perm } A = \# \text{ perfect matchings.}$

Minc-Bregman inequality -73: $\text{perm } A \leq \prod_{i=1}^{n} (r_i!)^\frac{1}{r_i}$

Cor: If $G$ is a bipartite graph on $n$ vertices in each class, each vertex in the first class of degree at most $d$ then $\phi_n(G) \leq (d!)^\frac{n}{d}$ If $d | n$ equality holds for $\frac{n}{d} K_{d,d}$ - union of $\frac{n}{d}$ complete $d$-bipartite graphs

UMC: under the above conditions

$\phi_k(G) \leq \phi_k(\frac{n}{d} K_{d,d}), k = 1, \ldots \quad (4.1)$

SUMC: Let $G$ graph on $2n$ vertices and degree of each vertex at most $d$. Then (4.1) holds

Open even for $k = n$.

Known for $d = 2$ Friedland-Krop-Markström
5 Lower bounds for matchings

\( \Omega_n \subset \mathbb{R}^{n \times n}_{+} \) set of doubly stochastic matrices

dan der Waerden conjecture: for \( A \in \Omega_n \)
\[
\text{perm } A \geq \text{perm } J_n = \frac{n!}{n^n} \sim \sqrt{2\pi n} e^{-n} \geq e^{-n}
\]
(\( J_n = \left[ \frac{1}{n} \right] \in \Omega_n \)) Friedland-79, Falikman, Egorichev-81.

Cor: \( \phi_n(G) \geq \left( \frac{d}{e} \right)^n \) for any \( G \in \Gamma(d, n) \).

Tverberg conjecture (Friedland-82):
\[
\text{perm}_k(A) \geq \text{perm}_k(J_n) = \binom{n}{k}^2 \frac{k!}{n^k}
\]
Cor: \( \phi_k(G) \geq \left( \frac{n}{k} \right)^2 \frac{k!d^k}{n^k} \) for any \( G \in \Gamma(d, n) \).

Reason: \( \frac{1}{d} B(G) \in \Omega_n \).

Voorhoeve-79 (\( d = 3 \)) Schrijver-98
\[
\phi_n(G) \geq \left( \frac{(d-1)d^{-1}}{dd-2} \right)^n \text{ for } G \in \Gamma(d, n)
\]

Gurvits: \( A \in \Omega_n \), each column has at most \( d \) nonzero entries: \( \text{perm } A \geq \frac{d!}{d^d} \left( \frac{d}{d-1} \right)^{d(d-1)} \left( \frac{d-1}{d} \right)^{(d-1)n} \).

Cor: \( \phi_n(G) \geq \frac{d!}{d^d} \left( \frac{d}{d-1} \right)^{d(d-1)} \left( \frac{d-1}{d} \right)^{(d-1)n} \)
for \( G \in \Gamma(d, n) \)

LMC: \( \phi_k(G) \geq \left( \frac{n}{k} \right)^2 \left( \frac{nd-k}{nd} \right) nd-k \left( \frac{kd}{n} \right)^k \)
6 Graphs with \( d \leq 2 \)

\( G \)-the degree of each vertex \( \leq 2 \) is union of cycles, paths and isolated vertices \( G \) bipartite if each cycle in \( G \) is even

\( C_k, P_k \) cycle and path of length \( k \),

\[
\Phi_{C_k}(x) = \Phi_{P_k}(x) + x\Phi_{P_{k-2}}(x)
\]

Friedland-Krop-Markström for \( 2 \)-regular \( G \), \( \#V = n \)

\[
\Phi_{C_i}(x)\Phi_{C_j}(x) - \Phi_{C_{i+j}}(x) = (-1)^i x^i \Phi_{C_{j-i}}(x)
\]

\[
\Phi_{C_i}(x)\Phi_{C_j}(x) \geq \Phi_{C_{i+j}}(x) \text{ if } i \text{ even } (i \leq j),
\]

\[
\Phi_{C_i}(x)\Phi_{C_j}(x) < \Phi_{C_{i+j}}(x) \text{ if } i \text{ odd } (i \leq j)
\]

\[
\Phi_G(x) \leq \Phi_{C_4}(x) \frac{n}{4} \text{ if } 4 \mid n
\]

\[
\Phi_G(x) \leq \Phi_{C_4}(x) \frac{n-5}{4} \Phi_{C_5}(x) \text{ if } 4 \mid n - 1
\]

\[
\Phi_G(x) \leq \Phi_{C_4}(x) \frac{n-6}{4} \Phi_{C_6}(x) \text{ if } 4 \mid n - 2
\]

\[
\Phi_G(x) \leq \Phi_{C_4}(x) \frac{n-7}{4} \Phi_{C_7}(x) \text{ if } 4 \mid n - 3
\]

\[
\Phi_G(x) \geq \Phi_{C_3}(x) \frac{n}{3} \text{ if } 3 \mid n
\]

\[
\Phi_G(x) \geq \Phi_{C_3}(x) \frac{n-4}{3} \Phi_{C_4}(x) \text{ if } 3 \mid n - 1
\]

\[
\Phi_G(x) \geq \Phi_{C_3}(x) \frac{n-5}{3} \Phi_{C_5}(x) \text{ if } 3 \mid n - 2
\]

\[
\Phi_G(x) \geq \Phi_{C_{2n}} \text{ if } G \in \Gamma(2, n)
\]
7 LMC for regular graphs

$K_m$ complete graph on $m$ vertices

$lK_m$ union of $l$ copies of $K_m$

PROBLEM: Is $\Phi_G(x) \geq \Phi_{lK_{d+1}}(x) = \Phi_{K_{d+1}}(x)^l$

for any $d$-regular graph on $l(d + 1)$ vertices?

True for $d = 2$ (FKM)
8 UMC for $G \in \Gamma(3, n)$

$\Phi_{M_{10}} = 1 + 15x + 75x^2 + 145x^3 + 95x^4 + 13x^5$

$\Phi_{G_1} = 1 + 15x + 75x^2 + 145x^3 + 96x^4 + 12x^5$
$Q_3$ graph of 3-dimensional cube

Upper Matching Conjecture for $G \in \Gamma(3, n)$

- For $n \equiv 0 \pmod{3}$, $\Phi_G \preceq \Phi_{n/3}K_{3,3}$, equality holding only if $G = \frac{n}{3}K_{3,3}$.
- For $n \equiv 1 \pmod{3}$, $\Phi_G \preceq \Phi_{n-4/3}K_{3,3} \cup Q_3$, equality holding only if $G = \frac{n-4}{3}K_{3,3} \cup Q_3$.
- For $n \equiv 2 \pmod{3}$, $\phi_G(k) \leq \max \left( \phi_{n-5/3}K_{3,3} \cup M_{10}(k), \phi_{n-5/3}K_{3,3} \cup G_1(k) \right)$, for $k = 1, \ldots, n$. 
9 Asymptotic growth of matchings

\( G_n = (V_n, E_n) \in \Gamma(d, n), n = 1, 2, \ldots \)

sequence of \( d \)-regular bipartite graphs with \( \#V_n \to \infty \).

Let \( k_n \in [0, \frac{\#V_n}{2}] \), \( n = 1, 2, \ldots \) sequence of integers
with \( \lim_{n \to \infty} \frac{2k_n}{\#V_n} = p \in (0, 1] \). upper and lower
\((p)\)-asymptotic growth:

\[
hu_d(p) : \limsup_{n \to \infty} \frac{\log \phi_{G_n}(k_n)}{\#V_n},
\]

\[
hl_d(p) : \liminf_{n \to \infty} \frac{\log \phi_{G_n}(k_n)}{\#V_n}.
\]

For \( G_n := C_{2m_1,n} \times \ldots \times C_{2m_d,n} \) and

\[
\lim_{n \to \infty} m_i,n = \infty \text{ for } i = 1, \ldots, d,
\]

\( hu_d(p) = hl_d(p) = h_d(p) \) is \( p \)-dimer density

Hammersley-66

UMC, LMC yield AUMC, ALMC:

\[
hu_d(p) \leq h_{K(d)}(p), \quad gh_d(p) \leq hl_d(p)
\]

\[
gh_d(p) := \frac{1}{2} \left( p \log d - p \log p - 2(1 - p) \log(1 - p) + (d - p) \log \left( \frac{1 - p}{d} \right) \right)
\]

\[
P_{K(d)}(t) = \frac{\log \sum_{k=0}^{d} \binom{d}{k} k! e^{2kt}}{2d}, \quad t \in \mathbb{R}.
\]

\[
p(t) = P'_{K(d)}(t), \quad t(p) = (P'_{K(d)})^{-1}(p),
\]

\[
h_{K(d)}(p) = P_{K(d)}(t(p)) - t(p)p
\]

-Legendre trns

\( K(d) \) countable union of \( K_{d,d} \)
10  Graph estimates for $h_2(p)$

![Graph showing Monomer-dimer tiling of the 2-dimensional grid: entropy as a function of dimer density. FT is the Friedland-Tverberg lower bound, $h_2$ is the true monomer-dimer entropy. B are Baxter’s computed values. ALMC is the Asymptotic Lower Matching Conjecture. AUMC is the entropy of a countable union of $K_{4,4}$, conjectured to be an upper bound by the Asymptotic Upper Matching Conjecture.](image)

Figure 1: Monomer-dimer tiling of the 2-dimensional grid: entropy as a function of dimer density. FT is the Friedland-Tverberg lower bound, $h_2$ is the true monomer-dimer entropy. B are Baxter’s computed values. ALMC is the Asymptotic Lower Matching Conjecture. AUMC is the entropy of a countable union of $K_{4,4}$, conjectured to be an upper bound by the Asymptotic Upper Matching Conjecture.
Figure 2: Monomer-dimer tiling of the 3-dimensional grid: entropy as a function of dimer density. FT is the Friedland-Tverberg lower bound, h3Low and h3High are the known bounds for the monomer-dimer entropy. ALMC is the Asymptotic Lower Matching Conjecture. AUMC is the entropy of a countable union of $K_{6,6}$, conjectured to be an upper bound by the Asymptotic Upper Matching Conjecture.
The sharpness of the ALMC

Standard probabilistic model on $\Gamma(d, n)$:

$\sigma \in S_{nd}$ permutation on $nd$ elements.

e_1, \ldots, e_{nd-nd}$ edges from $\{1, \ldots, n\}$ to $\{1, \ldots, n\}$.

e_i connects vertex $\left\lceil \frac{id}{d} \right\rceil$ to $\left\lceil \frac{\sigma(i)}{d} \right\rceil$ for $i = 1, \ldots, nd$. The probability of $G$ is $\frac{1}{(nd)!}$.

$\nu(d, n)$ the induced probability measure on $\Gamma(d, n)$

$\nu(d, n)$ invariant under the action of $S_n$ on $V_1$ and $V_2$

$E_{\nu(d,n)}(\phi_k(G)) = \frac{C^{2k}}{(dn-k)!} \frac{C^{2k}}{(dn)!}$

$\lim_{m \to \infty} \frac{\log E_{\nu(d,n,m)}(\phi_{km}(G))}{2n_m} = gh_d(p)$,

where $\lim_{m \to \infty} \frac{k_m}{n_m} = p \in [0, 1]$

Same results holds for another probability on $\Gamma(d, n)$.

Identify $G$ with $A(G) \in \mathbb{Z}^{n \times n}$.

$\pi : \Gamma(1, n)^d \to \Gamma(d, n)$,

$(P_1, \ldots, P_d) \mapsto P_1 + \ldots + P_d$

$\mu(d, n)$ is the push forward of uniform probability $\eta_{d,n}$ on $\Gamma(1, n)^d$: $\mu(d, n)(G) = \eta_{d,n}(\pi^{-1}(G))$ FKM
Lower asymptotic bounds

Using results on positive hyperbolic polynomials Friedland-Gurvits showed:

**Thm:** $r \geq 3, s \geq 1, B_n \in \Omega_n, n = 1, 2, \ldots$ each column of $B_n$ has at most $r$-nonzero entries.

$k_n \in [0, n] \cap \mathbb{N}, n = 1, 2, \ldots, \lim_{n \to \infty} \frac{k_n}{n} = p \in (0, 1]$ then

$$
\liminf_{n \to \infty} \frac{\log \text{perm}_{2n} k_n B_n}{2n} \geq \\
\frac{1}{2} (-p \log p - 2(1 - p) \log(1 - p)) + \\
\frac{1}{2} (r + s - 1) \log(1 - \frac{1}{r+s}) - \\
\frac{1}{2} (s - 1 + p) \log(1 - \frac{1-p}{s})
$$

**Cor:** $d$-ALMC holds for $p_s = \frac{d}{d+s}, s = 0, 1, \ldots$,

**Con:** under Thm assumptions

$$
\liminf_{n \to \infty} \frac{\log \text{perm}_{2n} k_n B_n}{2n} \geq gh_r(p) - \frac{p}{2} \log r
$$

For $p_s = \frac{r}{r+s}, s = 0, 1, \ldots$, conjecture holds

There is a finite version of above Thm

Above Thm gives a better lower bound for $h_3$
Independent sets

For \( G = (V, E) \), \( I \subset V \) independent if \( I \) anticlique

\( \nu_k(G) \)-number of independent sets of cardinality \( k \)

\( I_G(x) := \sum_{k \in \mathbb{Z}_+} \nu_k(G) x^k \)

\( I_{G_1 \cup G_2}(x) = I_{G_1}(x) I_{G_2}(x) \)

Con: \( I_G(x) \leq I_{mK_{d,d}}(x) \) for \( G \in \Gamma(d, md) \) (Any \( d \)-regular \( G \) on \( 2md \) vertices?)

THM [15] \( I_G(x) \leq I_{mK_{d,d}}(x) \) for \( G \in \Gamma(d, md), x \geq 1 \)

Line Graph \( G' := (V', E') \) of \( G = (V, E) \) given

\( V' = E, (e_1, e_2) \in E' \) iff \( e_1, e_2 \) have common vertex

\( I_{G'}(x) = \Phi_G(x) \)

\( v \in V \) induces clique of order \( \deg v \) in \( G' \) Hence

\( \deg v \geq 3, v \in V \) then \( G' \) has triangle, not bipartite. If

\( \deg v \leq 2, \forall v \in G \) then \( G' \) same type \( G' \sim G \) if \( G \)

2-regular

Con. holds for 2-regular graphs

FKM imply lower bounds for \( \nu_G(k) \) 2-regular

Lower bounds for \( \nu_G(k), G \in \Gamma(d, n) \)?
15 LISC for regular graphs

PROBLEM: Is
\[ \mathcal{I}_G(x) \geq \mathcal{I}_{lK_{d+1}}(x) = (1 + (d + 1)x)^l \]
for any \( d \)-regular graph on \( l(d + 1) \) vertices?

True for \( d = 2 \) (FKM)
16 Positive Hyperbolic Polynomials

1. polynomial

\[ p = p(x) = p(x_1, \ldots, x_n) : \mathbb{R}^n \rightarrow \mathbb{R} \] is positive hyperbolic (php) if:

- \( p \) homogeneous polynomial of degree \( m \geq 0 \).
- \( p(x) > 0 \) for all \( x > 0 \).
- \( \phi(t) := p(x + tu), t \in \mathbb{R} \)
  has \( m \)-real \( t \)-roots for all \( u > 0 \) and \( x \in \mathbb{R}^n \).

2. For

\[ p : \mathbb{R}^n \rightarrow \mathbb{R}, \quad 0 \neq u = (u_1, \ldots, u_n)^\top \in \mathbb{R}^n \]
let \( p_u = p_u(x) := \sum_{i=1}^n u_i \frac{\partial p}{\partial x_i}(x) \).

3. \( \deg_i p \) the degree of \( p(x) \) with respect to variable \( x_i \)

4. \( e_i := (\delta_{i1}, \ldots, \delta_{in})^\top \in \mathbb{R}^n, \quad i = 1, \ldots, n \)

5. \( 1 := (1, \ldots, 1)^\top \in \mathbb{R}^n \)
17 Properties positive hyperbolic pol.

\( p : \mathbb{R}^n \to \mathbb{R} \) is php of degree \( m \):

1. For \( u \geq 0, x \) fixed \( \phi(t) = p(x + tu) \). If \( p(u) > 0 \) then \( \phi(t) \) has \( m \) real \( t \) roots and \( p_u(x) \) is php of degree \( m - 1 \).
   \[ y \geq x \geq 0 \Rightarrow p(y) \geq p(x) \geq 0. \]

2. \( u \geq 0, x \geq 0 \) and \( p(u) = 0 \).
   
   either \( \phi(t) > 0 \forall t \geq 0 \)
   
   or \( p(x) = 0 \) and \( \phi(t) \equiv 0 \).
   
   If \( p(x) > 0 \) and \( \phi(t) \neq Const \) then all roots of \( \phi(t) \) real and negative.
   
   if \( p_u \neq 0 \) then \( p_u \) is a php of degree \( m - 1 \).

3. If \( q((x_1, \ldots, x_{n-1})) := p((x_1, x_2, \ldots, x_{n-1}, 0)) \neq 0 \)
   
   then \( q \) is php of degree \( m \) in \( \mathbb{R}^{n-1} \).
   
   In particular, \( r((x_1, \ldots, x_{n-1})) := \frac{\partial p}{\partial x_n}((x_1, \ldots, x_{n-1}, 0)) \) is either 0 or php in \( n - 1 \) variables of degree \( m - 1 \).

4. The coefficient of each monomial in php is nonnegative.
18  **Examples of php**

1. \( A = (a_{ij})_{i=j=1}^{m,n} \in \mathbb{R}_{+}^{m \times n} \) (\( A \geq 0 \)) and each row of \( A \) is nonzero

\[
p_{k,A}(x) := \sum_{1 \leq i_1 < \ldots < i_k \leq m} \prod_{j=1}^{k} (Ax)_{i_j}
\]

is php of degree \( k \in [1, m] \):

\[
p_{k,A} = \frac{\partial^{m-k}}{\partial y_1 \ldots \partial y_{m-k}} \prod_{j=1}^{m} ((Ax)_j + \sum_{i=1}^{m-k} y_i)
\]

2. \( A_1, \ldots, A_n \in \mathbb{C}^{m \times m} \) hermitian, nonnegative definite and \( A_1 + \ldots + A_m \) is positive definite

\( A(x) := \sum_{i=1}^{n} x_i A_i \)

- \( p(x) = \det A(x) \) is php
- \( p_k(x) \)-the sum of all \( k \times k \) principle minors of \( A(x) \) is php:

\[
p_k(x) = \frac{1}{k!} \frac{\partial^k}{\partial t^k} \det (A(x) + tI_m)(x, 0)
\]
19    Capacity

Capacity for php $p(x) : \mathbb{R}^n \rightarrow \mathbb{R}$:

$\text{Cap } p := \inf_{x > 0, x_1 \ldots x_n = 1} p(x) = \inf_{x > 0} \frac{p(x)}{(x_1 \ldots x_n)^{\frac{m}{n}}} \geq 0$

1. $\text{Cap } (p) = 0$ for $p = x_1^{m_1} \ldots x_n^{m_n}$
   $0 \leq (m_1, \ldots, m_n) \neq k1$

2. $A = [a_{ij}] \in \mathbb{R}_{+}^{m \times n}$ doubly stochastic:
   each row has sum 1 each column has sum $\frac{m}{n}$.

   (a) $p_{m,A} = \prod_{i=1}^{m} (\sum_{j=1}^{n} a_{ij} x_j) \geq \prod_{i=1}^{m} \prod_{j=1}^{n} x_i^{\frac{a_{ij}}{n}} = (x_1 \ldots x_n)^{\frac{m}{n}} \Rightarrow$
   $\text{Cap } (p_{m,A}) \geq 1$
   $p_{m,A}(1) = 1 \Rightarrow \text{Cap } (p_{m,A}) = 1$

   (b) $(\frac{m}{k})^{-1} p_{k,A} \geq p_{m,A} \geq (x_1 \ldots x_n)^{\frac{k}{n}} \Rightarrow$
   $\text{Cap } (p_{k,A}) = (\frac{m}{k})$

3. $A \in \mathbb{R}_{+}^{n \times n}$ irreducible $A := D_1 B D_2$, $B$ doubly stochastic matrix and $D_1, D_2$ diagonal pos. def.
   Sinkhorn.

   $\text{Cap } p_{k,A} = (\det D_1 D_2)^{\frac{k}{n}}$
Mixed discriminants

\((A_1, \ldots, A_n) \in H_{m,+}^n\) doubly stochastic tuple:

\[ \text{tr } A_i = \frac{m}{n}, i = 1, \ldots, n, \text{ and } \sum_{i=1}^{n} A_i = I_m. \]

if \(B_i = \text{diag}(b_{1i}, \ldots, b_{mi}), i = 1, \ldots, n,\) then

\[ B = (b_{ji})_{j,i=1}^{m,n} \in \mathbb{R}_{+}^{m \times n} \text{ is d.s.} \]

For d.s. tuple \((A_1, \ldots, A_n)\)

\[ \text{Cap } (p_k) \geq \binom{m}{k} \]

If each \(A_i > 0\) equality holds iff

\[ A_i = \frac{1}{n} I_m, i = 1, \ldots, n. \]

Prf:

\[ U^*(x) A(x) U(x) = \text{diag}(\lambda_1(x), \ldots, \lambda_n(x)), \]

\(\lambda_1(x), \ldots, \lambda_n(x) > 0.\)

\[ C_i = U^* A_i U = (c_{jk,i})_{j,k=1}^{m} \text{ for } i = 1, \ldots, n. \]

Then \((C_1, \ldots, C_n)\) d.s. tuple. \((c_{jj,1}, \ldots, c_{jj,n})\)

prob. vec. \(\lambda_j(x) = \sum_{i=1}^{n} x_i c_{jj,i}.\) arithmetic-geometric

in. \(\lambda_j(x) \geq \prod_{i=1}^{n} x_i^{c_{jj,i}} \text{ for } j = 1, \ldots, m\)

\[ \det A(x) \geq \prod_{i=1}^{n} x_i^{\text{tr} A_i} = (x_1 \ldots x_n)^{\frac{m}{n}} \]

Hence \(\text{Cap } (p) \geq 1.\)
The main inequality

Gurvits: Let \((u_1, \ldots, u_n), (v_1, \ldots, v_n) > 0\),
\[
f(t) := \prod_{i=1}^{k} (u_i t + v_i), \quad K(f) := \inf_{t>0} \frac{f(t)}{t}.
\]
Then \(f'(0) = K\) for \(k = 1\) and \(f'(0) \geq \left(\frac{k-1}{k}\right)^{k-1} K\) for \(k \geq 2\).
For \(k \geq 2\) equality holds iff \(\frac{v_1}{u_1} = \ldots = \frac{v_k}{u_k}\).

F-G: \(p : \mathbb{R}^n \to \mathbb{R}\) phd \(\deg p = m \in [1, n]\), \(\deg_i p \leq r_i \in [1, m], i = 1, \ldots, n\).
Rearrange \(r_1, \ldots, r_n\) to \(1 \leq r_1^* \leq r_2^* \leq \ldots \leq r_n^*\).
\(k \in [1, n]\) is the smallest integer \(r_k^* > m - k\).
\[
\sum_{1 \leq i_1 < \ldots < i_m \leq n} \frac{\partial^m}{\partial x_{i_1} \ldots \partial x_{i_m}} p(0, \ldots, 0) \geq
\]
\[
\frac{n^{n-m}}{(n-m)!} \left(\frac{n-k+1}{(n-k+1)^{n-k+1}}\right) \times (FG)
\]
\[
\prod_{j=1}^{k-1} \left(\frac{r_j^* + n - m - 1}{r_j^* + n - m}\right)^{r_j^* + n - m - 1} \text{Cap } p
\]

Gurvits: \(A \in \mathbb{R}^{n \times n}_+\) d.s. each column contains at most \(r \in [1, n]\) nonzero entries:
\[
\text{perm } A \geq \frac{r!}{r^r} \left(\frac{r-1}{r}\right)^{(r-1)(n-r)} = \\
\frac{r!}{r^r} \left(\frac{r}{r-1}\right)^{r(r-1)} \left(\frac{r-1}{r}\right)^{(r-1)n}
\]
Improvement of Schrijver for perfect matchings in \(\Gamma(r, n)\)
Lower bounds for sparse matrices

(FG) yields the Tverberg conjecture with $p_{k,A}(x)$

(FG) does not yield ALMC

Reason: (FG) proven using $p(x)\left(\frac{x_1+\ldots+x_n}{n}\right)^{n-m}$

FG: $p: R^n \rightarrow \mathbb{R}$, php $\deg p = m \in [1, n)$, $\deg_i p \leq r_i \in [1, m]$, $i = 1, \ldots, n$.

Rearrange $r_1, \ldots, r_n$ to $1 \leq r_1^* \leq r_2^* \leq \ldots \leq r_n^*$.

for $s \in \mathbb{N}$ let $k \in [1, n]$ first integer $r_k^* + s > n - k$:

$$\sum_{1 \leq i_1 < \ldots < i_m \leq n} \frac{\partial^m}{\partial x_{i_1} \ldots \partial x_{i_m}} p(0, \ldots, 0) \geq$$

$$\frac{(sn)!}{s^{n-m}(n-m)!( (s-1)n+m)!} \frac{(n-k+1)!}{(n-k+1)^{n-k+1}} \prod_{j=1}^{k-1} \left(\frac{r_j^* + s - 1}{r_j^* + s}\right)^{r_j^* + s - 1} \text{Cap } p$$

Prf: apply (FG) to $pp_{n-m, \frac{1}{s}A}$ for $A(G), G \in \Gamma(s, n)$ and average on $\Gamma(s, n)$

Cor: $\deg_i p \leq r \in [1, m]$, $i = 1, \ldots, n$, $s \in \mathbb{N}$, $k = n - r - s + 1 \geq 1$

$$\sum_{1 \leq i_1 < \ldots < i_m \leq n} \frac{\partial^m}{\partial x_{i_1} \ldots \partial x_{i_m}} p(0, \ldots, 0) \geq$$

$$\frac{(sn)!}{s^{n-m}(n-m)!( (s-1)n+m)!} \frac{(r+s)!}{(r+s)^{r+s}} \times \frac{(r+s)^{r+s}}{(r+s-1)(n-r-s)} \text{Cap } p$$
Matching in general graphs

\[ G = (V, E) \text{ non bipartite graph } B = B(G) \]

number of \( m \)-matchings: 
\[ \text{haff } mB = 2^{-m} \sum_{\alpha, \beta \subseteq \{1, \ldots, 2n\}, \# \alpha = \# \beta = m, \alpha \cap \beta = \emptyset} \text{perm } B[\alpha, \beta] = 2^{-m} \sum_{1 \leq i_1 < \ldots < i_{2m} \leq 2n} \frac{\partial^{2m}}{\partial x_{i_1} \ldots \partial x_{i_{2m}}} (x^T B x)^m \]

for \( 0 \neq B \in S_n(\mathbb{R}^+) \) \( x^T B x \) php iff \( \lambda_2(B) \leq 0 \)

Thm: \( B \in S_{2n}(\mathbb{R}^+) \) irreducible, \( \lambda_2(B) \leq 0 \). Let 
\( K := \text{Cap } (x^T B x) \). Then for \( m \in [1, n] \)
\[ \text{haff } mB \geq \left( \frac{2n}{2m} \right) \frac{K^m(2m)!}{2^m(2n)!}. \]

Lem: \( 0 \neq B \in S_n(\mathbb{R}^+) \) irreducible. 
\[ 0 < K := \min x^T B x, \text{ subject } \]
\[ x = (x_1, \ldots, x_n)^T > 0, x_1 \ldots x_n = 1, \text{ achieved at unique } \]
\[ d := (d_1, \ldots, d_n) > 0, d_1 \ldots d_n = 1, n > 2: \]
\[ D := \text{diag}(d_1, \ldots, d_n). \text{ Then } \frac{n}{K} DBD \text{ d.s.} \]

Thm \( G = (V, E) \) connected, \( x^T A(G)x \) php iff \( G \) is complete \( k \)-partite
References


