Matchings and Independent Sets:

Problems, Conjectures and Results

Shmuel Friedland
University of Illinois at Chicago

Workshop on nonnegative matrices,

Maynooth, July 10-12, 2006

1 Outline of the lecture

1. Upper bounds for matchings in bipartite graphs
2. Lower bounds for matchings in bipartite graphs
3. Exact matching bounds for $\mathbf{2}$-regular graphs
4. LMC for regular graphs
5. Conjectural upper bounds for $\mathbf{3}$-regular graphs
6. Asymptotic growth of matchings & AUMC, ALMC
7. Graphs illustrations for dimensions $\mathbf{2, 3}$
8. Lower asymptotic bounds
9. Independent sets

B. Positive hyperbolic polynomials

1. Properties
2. Examples
3. Proof of cases of ALMC

2 Matchings

$G = (V, E)$ undirected graph with vertices V, edges E

matching in G: $M \subseteq E$

no two edges in M share a common endpoint

$e = (u, v) \in M$ is dimer

v not covered by M is monomer

M called monomer-dimer cover of G

M is perfect matching $\iff$ no monomers

M is k-matching $\iff \#M = k$

$\phi_k(G)$ number of k-matchings in G, $\phi_0(G) := 1$

$\Phi_G(x) := \sum_k \phi_k(G) x^k$ matching generating polyn.

roots of $\Phi_G(x)$ nonpositive [17].

$\Phi_{G_1 \cup G_2}(x) = \Phi_{G_1}(x)\Phi_{G_2}(x)$

$\Gamma(d, n)$ set of d-regular bipartite graphs on $2n$ vertices

3 Formulas for k-matchings

$A = A(G) \in \{0-1\}^{n\times n}$-adjacency matrix of $G = (V, E)$, $\#V = n$.

$\phi_k(G) = \mathrm{haff}_{\,k}(\mathrm{A}) :=$
$2^{-\mathrm{k}} \sum_{1\le \mathrm{i}_1<\ldots<\mathrm{i}_{2\mathrm{k}}\le \mathrm{n}} \frac{\partial^{2\mathrm{k}}}{\partial \mathrm{x}_{\mathrm{i}_1}\ldots \mathrm{x}_{\mathrm{i}_{2\mathrm{k}}}} (\mathrm{x}^{\mathrm{T}}\mathrm{A}\mathrm{x})^{\mathrm{k}}, \mathrm{k} \le \frac{\mathrm{n}}{2}$

$\mathrm{x} = (x_1, \ldots, x_n)^T$

$G = (V, E)$ bipartite $V = V_1 \cup V_2, E \subset V_1 \times V_2$, $B = B(G) \in \{0-1\}^{m\times n}$, $\#V_1 = m, V_2 = n$.

$\phi_k(G) = \mathrm{perm}_{\,\mathrm{k}}\mathrm{B} :=$
$\sum_{\alpha\in \mathrm{Q}_{\mathrm{k},\mathrm{m}},\beta\in \mathrm{Q}_{\mathrm{k},\mathrm{n}}} \mathrm{perm}\ \mathrm{B}[\alpha, \beta]$.

for general G: $\phi_k(G) =$
$2^{-k} \sum_{\alpha\in Q_{k,n},\beta\in Q_{k,n}\alpha\cap\beta=\emptyset,} \mathrm{perm}\ \mathrm{A}[\alpha, \beta]$

$Q_{k,n} := \{\alpha = \{i_1, \ldots, i_k\} :$
$1 \le i_1 < i_2 < \ldots < i_k \le n\}$.

4 Upper bounds for matchings

$A = [a_{ij}]_{i,j=1}^{n} \in \{0-1\}^{n\times n}$ represents bipartite graph G on n vertices in each class, with degree $r_i = \sum_{j=1}^{n} a_{ij}, i = 1, \ldots, n$ in first class

$\mathbf{perm\ A} = \#$ perfect matchings.

Minc-Bregman inequality -73: $\mathbf{perm\ A} \le \prod_{\mathbf{i=1}}^{\mathbf{n}} (\mathbf{r_i}!)^{\frac{1}{\mathbf{r_i}}}$

Cor: If G is a bipartite graph on n vertices in each class, each vertex in the first class of degree at most d then $\phi_n(G) \le (d!)^{\frac{n}{d}}$ If $d|n$ equality holds for

$\frac{n}{d}K_{d,d}$- union of $\frac{n}{d}$ complete d-bipartite graphs

UMC: under the above conditions

$$\phi_k(G) \le \phi_k(\frac{n}{d}K_{d,d}), k = 1, \ldots \qquad (4.1)$$

SUMC: Let G graph on $2n$ vertices and degree of each vertex at most d. Then (4.1) holds

Open even for $k = n$.

Known for $d = 2$ Friedland-Krop-Markström

5 Lower bounds for matchings

$\Omega_n \subset \mathbb{R}_+^{n\times n}$ set of doubly stochastic matrices

van der Waerden conjecture: for $A \in \Omega_n$
$\mathbf{perm\ A \geq perm\ J_n = \frac{n!}{n^n} \sim \sqrt{2\pi n}e^{-n} \geq e^{-n}}$
($J_n = [\frac{1}{n}] \in \Omega_n$) Friedland-79, Falikman, Egorichev-81.

Cor: $\phi_n(G) \geq (\frac{d}{e})^n$ for any $G \in \Gamma(d,n)$.

Tverberg conjecture (Friedland-82):
$\mathbf{perm}_{\ \mathbf{k}}(\mathbf{A}) \geq \mathbf{perm}_{\ \mathbf{k}}(\mathbf{J_n}) = \binom{\mathbf{n}}{\mathbf{k}}^2 \frac{\mathbf{k!}}{\mathbf{n^k}}$

Cor: $\phi_k(G) \geq \binom{n}{k}^2 \frac{k!d^k}{n^k}$ for any $G \in \Gamma(d,n)$.

Reason: $\frac{1}{d}B(G) \in \Omega_n$.

Voorhoeve-79 ($d = 3$) Schrijver-98

$\phi_n(G) \geq (\frac{(d-1)^{d-1}}{d^{d-2}})^n$ for $G \in \Gamma(d,n)$

Gurvits: $A \in \Omega_n$, each column has at most d nonzero entries: $\mathbf{perm\ A} \geq \frac{d!}{d^d}\left(\frac{d}{d-1}\right)^{d(d-1)}\left(\frac{d-1}{d}\right)^{(d-1)n}$.

Cor: $\phi_n(G) \geq \frac{d!}{d^d}\left(\frac{d}{d-1}\right)^{d(d-1)}(\frac{(d-1)^{d-1}}{d^{d-2}})^n$
for $G \in \Gamma(d,n)$

LMC: $\phi_k(G) \geq \binom{n}{k}^2 (\frac{nd-k}{nd})^{nd-k}(\frac{kd}{n})^k$

6 Graphs with $d \leq 2$

G-the degree of each vertex ≤ 2 is union of cycles, paths and isolated vertices G bipartite if each cycle in G is even

C_k, P_k cycle and path of length k,

$\Phi_{C_k}(x) = \Phi_{P_k}(x) + x\Phi_{P_{k-2}}(x)$

Friedland-Krop-Markström for 2-regular $G, \#V = n$

$\Phi_{C_i}(x)\Phi_{C_j}(x) - \Phi_{C_{i+j}}(x) = (-1)^i x^i \Phi_{C_{j-i}}(x)$

$\Phi_{C_i}(x)\Phi_{C_j}(x) \succ \Phi_{C_{i+j}}(x)$ if i even ($i \leq j$),

$\Phi_{C_i}(x)\Phi_{C_j}(x) \prec \Phi_{C_{i+j}}(x)$ if i odd $(i \leq j)$

$\Phi_G(x) \preceq \Phi_{C_4}(x)^{\frac{n}{4}} \texttt{ if } 4|n$

$\Phi_G(x) \preceq \Phi_{C_4}(x)^{\frac{n-5}{4}}\Phi_{C_5}(x) \texttt{ if } 4|n-1$

$\Phi_G(x) \preceq \Phi_{C_4}(x)^{\frac{n-6}{4}}\Phi_{C_6}(x) \texttt{ if } 4|n-2$

$\Phi_G(x) \preceq \Phi_{C_4}(x)^{\frac{n-7}{4}}\Phi_{C_7}(x) \texttt{ if } 4|n-3$

$\Phi_G(x) \succeq \Phi_{C_3}(x)^{\frac{n}{3}} \texttt{ if } 3|n$

$\Phi_G(x) \succeq \Phi_{C_3}(x)^{\frac{n-4}{3}}\Phi_{C_4}(x) \texttt{ if } 3|n-1$

$\Phi_G(x) \succeq \Phi_{C_3}(x)^{\frac{n-5}{3}}\Phi_{C_5}(x) \texttt{ if } 3|n-2$

$\Phi_G(x) \succeq \Phi_{C_{2n}}$ if $G \in \Gamma(2, n)$

7 LMC for regular graphs

K_m complete graph on m vertices

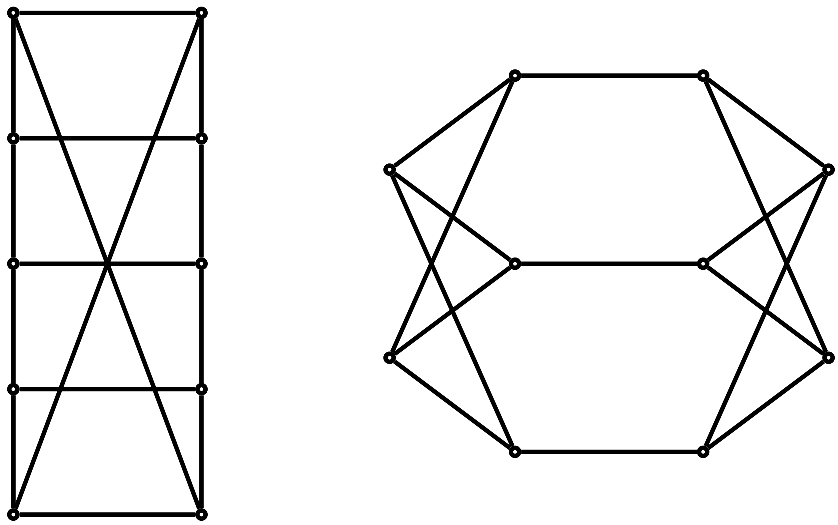
lK_m union of l copies of K_m

PROBLEM: Is $\Phi_G(x) \succeq \Phi_{lK_{d+1}}(x) = \Phi_{K_{d+1}}(x)^l$

for any d-regular graph on $l(d+1)$ vertices?

True for $d = 2$ (FKM)

8 UMC for $G \in \Gamma(3, n)$



M_{10} $\qquad$ G_1

$$\Phi_{M_{10}} = 1 + 15x + 75x^2 + 145x^3 + 95x^4 + 13x^5$$
$$\Phi_{G_1} = 1 + 15x + 75x^2 + 145x^3 + 96x^4 + 12x^5$$

Q_3 graph of 3-dimensional cube

Upper Matching Conjecture for $G \in \Gamma(3, n)$

- For $n \equiv 0 \pmod 3$, $\Phi_G \preceq \Phi_{\frac{n}{3}K_{3,3}}$, equality holding only if $G = \frac{n}{3}K_{3,3}$.
- For $n \equiv 1 \pmod 3$, $\Phi_G \preceq \Phi_{\frac{n-4}{3}K_{3,3} \cup Q_3}$, equality holding only if $G = \frac{n-4}{3}K_{3,3} \cup Q_3$.
- For $n \equiv 2 \pmod 3$, $\phi_G(k) \leqslant \max\left(\phi_{\frac{n-5}{3}K_{3,3} \cup M_{10}}(k), \phi_{\frac{n-5}{3}K_{3,3} \cup G_1}(k)\right)$, for $k = 1, \ldots, n$.

9 Asymptotic growth of matchings

$G_n = (V_n, E_n) \in \Gamma(d,n)$, $n = 1, 2, \ldots$
sequence of d-regular bipartite graphs with $\# V_n \to \infty$.
Let $k_n \in [0, \frac{\# V_n}{2}]$, $n = 1, 2, \ldots$ sequence of integers with $lim_{n\to\infty} \frac{2k_n}{\# V_n} = p \in (0, 1]$. upper and lower ($p$)-asymptotic growth:
$hu_d(p) : \limsup_{n\to\infty} \frac{\log \phi_{G_n}(k_n)}{\# V_n}$,
$hl_d(p) : \liminf_{n\to\infty} \frac{\log \phi_{G_n}(k_n)}{\# V_n}$ For
$G_n := C_{2m_{1,n}} \times \ldots \times C_{2m_{d,n}}$ and
$\lim_{n\to\infty} m_{i,n} = \infty$ for $i = 1, \ldots, d$,
$hu_d(p) = hl_d(p) = h_d(p)$ is p-dimer density
Hammersley-66

UMC, LMC yield AUMC, ALMC:
$hu_d(p) \le h_{K(d)}(p), \quad gh_d(p) \le hl_d(p)$
$gh_d(p) := \frac{1}{2}\big(p \log d - p \log p -$
$2(1-p)\log(1-p) + (d-p)\log\left(1 - \frac{p}{d}\right)\big)$
$P_{K(d)}(t) = \frac{\log \sum_{k=0}^{d} \binom{d}{k}^2 k!\, e^{2kt}}{2d}, t \in \mathbb{R}$.
$p(t) = P'_{K(d)}(t),\ t(p) = (P'_{K(d)})^{-1}(p)$,
$h_{K(d)}(p) = P_{K(d)}(t(p)) - t(p)p$ -Legendre trns
$K(d)$ countable union of $K_{d,d}$

10 Graph estimates for $h_2(p)$

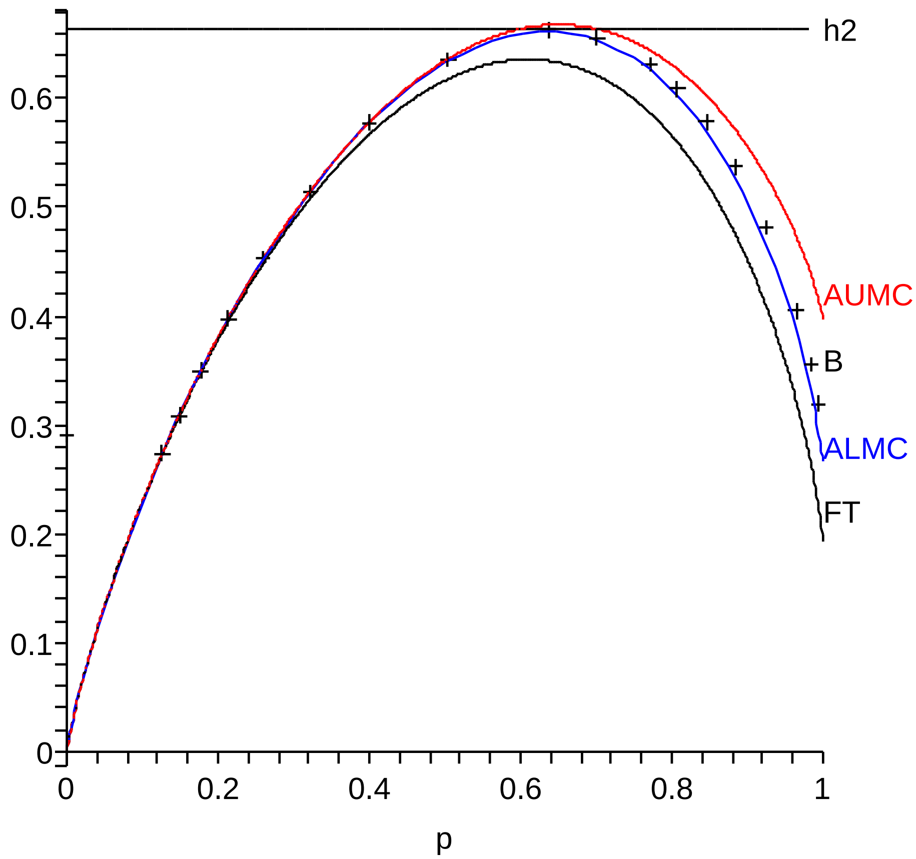


Figure 1: Monomer-dimer tiling of the $\mathbf{2}$-dimensional grid: entropy as a function of dimer density. FT is the Friedland-Tverberg lower bound, h2 is the true monomer-dimer entropy. B are Baxter's computed values. ALMC is the Asymptotic Lower Matching Conjecture. AUMC is the entropy of a countable union of $K_{4,4}$, conjectured to be an upper bound by the Asymptotic Upper Matching Conjecture.

11 Graph estimates for $h_3(p)$

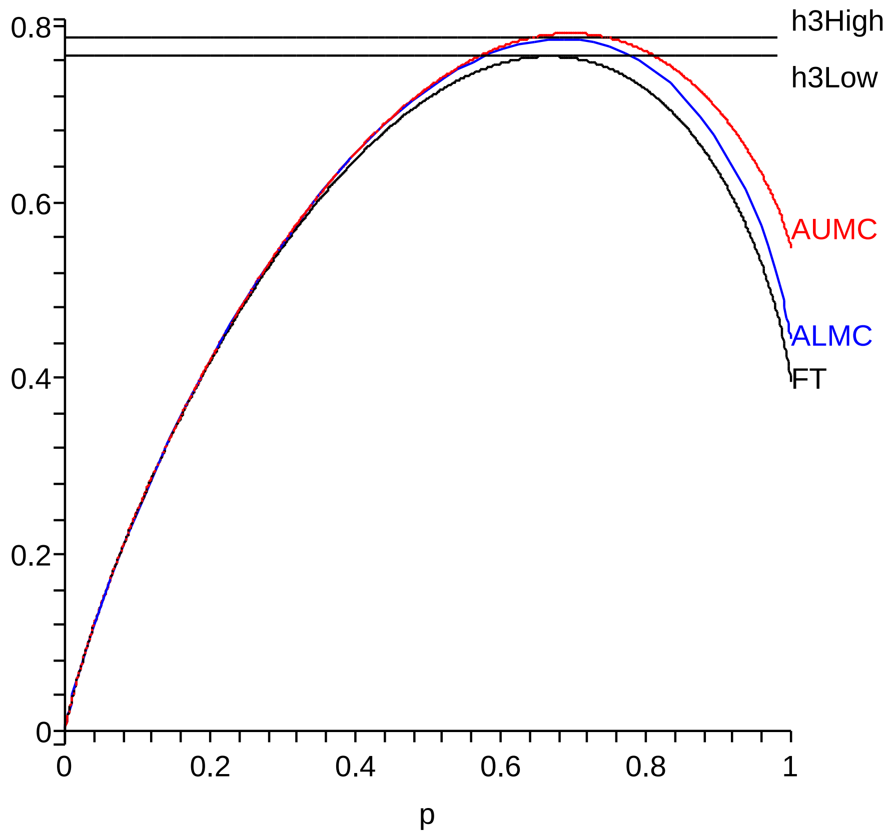


Figure 2: Monomer-dimer tiling of the $\mathbf{3}$-dimensional grid: entropy as a function of dimer density. FT is the Friedland-Tverberg lower bound, h3Low and h3High are the known bounds for the monomer-dimer entropy. ALMC is the Asymptotic Lower Matching Conjecture. AUMC is the entropy of a countable union of $K_{6,6}$, conjectured to be an upper bound by the Asymptotic Upper Matching Conjecture

12 The sharpness of the ALMC

Standard probabilistic model on $\Gamma(d,n)$:

$\sigma \in S_{nd}$ permutation on nd elements.

$e_1, ..., e_{nd}$-nd edges from $\{1, ..., n\}$ to $\{1, ..., n\}$. e_i connects vertex $\lceil \frac{i}{d} \rceil$ to $\lceil \frac{\sigma(i)}{d} \rceil$ for $i = 1, ..., nd$. The probability of G is $\frac{1}{(nd)!}$.

$\nu(d,n)$ the induced probability measure on $\Gamma(d,n)$

$\nu(d,n)$ invariant under the action of S_n on V_1 and V_2

$E_{\nu(d,n)}(\phi_k(G)) = \frac{\binom{n}{k}^2 d^{2k} k!(dn-k)!}{(dn)!}$

$\lim_{m\to\infty} \frac{\log E_{\nu(d,n_m)}(\phi_{k_m}(G))}{2n_m} = gh_d(p)$,

where $\lim_{m\to\infty} \frac{k_m}{n_m} = p \in [0,1]$

Same results holds for another probability on $\Gamma(d,n)$.

Identify G with $A(G) \in \mathbb{Z}_+^{n\times n}$.

$\pi : \Gamma(1,n)^d \longrightarrow \Gamma(d,n)$,

$(P_1, \ldots, P_d) \mapsto P_1 + \ldots + P_d$

$\mu(d,n)$ is the push forward of uniform probability $\eta_{d,n}$ on $\Gamma(1,n)^d$: $\mu(d,n)(G) = \eta_{d,n}(\pi^{-1}(G))$ FKM

13 Lower asymptotic bounds

Using results on positive hyperbolic polynomials Friedland-Gurvits showed:

Thm: $r \geq 3, s \geq 1, B_n \in \Omega_n, n = 1, 2, \ldots$ each column of B_n has at most r-nonzero entries.
$k_n \in [0, n] \cap \mathbb{N}, n = 1, 2, \ldots, lim_{n\to\infty} \frac{k_n}{n} = p \in (0, 1]$ then

$$\liminf_{n\to\infty} \frac{\log \operatorname{perm}_{k_n} B_n}{2n} \geqslant \frac{1}{2}(-p\log p - 2(1-p)\log(1-p)) + \frac{1}{2}(r+s-1)\log(1-\frac{1}{r+s}) - \frac{1}{2}(s-1+p)\log(1-\frac{1-p}{s})$$

Cor: d-ALMC holds for $p_s = \frac{d}{d+s}, s = 0, 1, \ldots,$

Con: under Thm assumptions

$$\liminf_{n\to\infty} \frac{\log \operatorname{perm}_{k_n} B_n}{2n} \geqslant gh_r(p) - \frac{p}{2}\log r$$

For $p_s = \frac{r}{r+s}, s = 0, 1, \ldots,$ conjecture holds

There is a finite version of above Thm

Above Thm gives a better lower bound for h_3

14 Independent sets

For $G = (V, E)$, $I \subset V$ independent if I anticlique

$\iota_k(G)$-number of independent sets of cardinality k

$\mathcal{I}_G(x) := \sum_{k \in \mathbb{Z}_+} \iota_k(G) x^k$

$\mathcal{I}_{G_1 \cup G_2}(x) = \mathcal{I}_{G_1}(x) \mathcal{I}_{G_2}(x)$

Con: $\mathcal{I}_G(x) \preceq \mathcal{I}_{mK_{d,d}}(x)$ for $G \in \Gamma(d, md)$ (Any d-regular G on $2md$ vertices?)

THM [15] $\mathcal{I}_G(x) \le \mathcal{I}_{mK_{d,d}}(x)$ for $G \in \Gamma(d, md), x \ge 1$

Line Graph $G' := (V', E')$ of $G = (V, E)$ given $V' = E$, $(e_1, e_2) \in E'$ iff e_1, e_2 have common vertex $\mathcal{I}_{G'} = \Phi_G(x)$

$v \in V$ induces clique of order $\deg v$ in G' Hence $\deg v \ge 3, v \in V$ then G' has triangle, not bipartite. If $\deg v \le 2, \forall v \in G$ then G' same type $G' \sim G$ if G 2-regular

Con. holds for 2-regular graphs

FKM imply lower bounds for $\iota_G(k)$ 2-regular

Lower bounds for $\iota_G(k), G \in \Gamma(d, n)$?

15 LISC for regular graphs

PROBLEM: Is

$$\mathcal{I}_G(x) \succeq \mathcal{I}_{lK_{d+1}}(x) = (1 + (d+1)x)^l$$

for any d-regular graph on $l(d+1)$ vertices?

True for $d = 2$ (FKM)

16 Positive Hyperbolic Polynomials

1. polynomial
   $p = p(\mathbf{x}) = p(x_1, \ldots, x_n) : \mathbb{R}^n \longrightarrow \mathbb{R}$ is
   positive hyperbolic (php) if:
   - p homogeneous polynomial of degree $m \geq 0$.
   - $p(\mathbf{x}) > 0$ for all $\mathbf{x} > \mathbf{0}$.
   - $\phi(t) := p(\mathbf{x} + t\mathbf{u}), t \in \mathbb{R}$
     has m-real t-roots for all $\mathbf{u} > \mathbf{0}$ and $\mathbf{x} \in \mathbb{R}^n$.
2. For
   $p : \mathbb{R}^n \longrightarrow \mathbb{R}, \mathbf{0} \neq \mathbf{u} = (u_1, \ldots, u_n)^\top \in \mathbb{R}^n$
   let $p_{\mathbf{u}} = p_{\mathbf{u}}(\mathbf{x}) := \sum_{i=1}^n u_i \frac{\partial p}{\partial x_i}(\mathbf{x})$.
3. $\deg_i p$ the degree of $p(\mathbf{x})$ with respect to variable x_i
4. $\mathbf{e}_i := (\delta_{i1}, \ldots, \delta_{in})^\top \in \mathbb{R}^n, \ i = 1, \ldots, n$
5. $\mathbf{1} := (1, \ldots, 1)^\top \in \mathbb{R}^n$

17 Properties positive hyperbolic pol.

$p : \mathbb{R}^n \longrightarrow \mathbb{R}$ is php of degree m:

1. For $\mathbf{u} \gneqq 0, \mathbf{x}$ fixed $\phi(t) = p(\mathbf{x} + t\mathbf{u})$. If $p(\mathbf{u}) > 0$ then $\phi(t)$ has m real t roots

   and $p_{\mathbf{u}}(\mathbf{x})$ is php of degree $m - 1$.

   $\mathbf{y} \geq \mathbf{x} \geq 0 \Rightarrow p(\mathbf{y}) \geq p(\mathbf{x}) \geq 0$.

2. $\mathbf{u} \gneqq 0, \mathbf{x} \gneqq 0$ and $p(\mathbf{u}) = 0$.

   either $\phi(t) > 0 \forall t \geq 0$

   or $p(\mathbf{x}) = 0$ and $\phi(t) \equiv 0$.

   If $p(\mathbf{x}) > 0$ and $\phi(t) \neq Const$ then all roots of $\phi(t)$ real and negative.

   if $p_{\mathbf{u}} \neq 0$ then $p_{\mathbf{u}}$ is a php of degree $m - 1$.

3. If $q((x_1, \ldots, x_{n-1})) :=$ $p((x_1, x_2, \ldots, x_{n-1}, 0)) \neq 0$

   then q is php of degree m in $\mathbb{R}^{n-1}$.

   In particular, $r((x_1, \ldots, x_{n-1})) :=$ $\frac{\partial p}{\partial x_n}((x_1, \ldots, x_{n-1}, 0))$ is either 0 or php in $n - 1$ variables of degree $m - 1$.

4. The coefficient of each monomial in php is nonnegative.

18 Examples of php

1. $A = (a_{ij})_{i=j=1}^{m,n} \in \mathbb{R}_+^{m\times n}$ ($A \geq 0$) and each row of A is nonzero

   $p_{k,A}(x) := \sum_{1\leq i_1<\ldots<i_k\leq m} \prod_{j=1}^{k} (A\mathrm{x})_{i_j}$

   is php of degree $k \in [1, m]$:

   $p_{k,A} = \frac{\partial^{m-k}}{\partial y_1 \ldots \partial y_{m-k}} \prod_{j=1}^{m} ((A\mathrm{x})_j + \sum_{i=1}^{m-k} y_i)$

2. $A_1, \ldots, A_n \in \mathbb{C}^{m\times m}$ hermitian, nonnegative definite and $A_1 + \ldots + A_m$ is positive definite

   $A(\mathrm{x}) := \sum_{i=1}^{n} x_i A_i$

   - $p(\mathrm{x}) = \det\ \mathrm{A}(\mathrm{x})$ is php
   - $p_k(\mathrm{x})$-the sum of all $k \times k$ principle minors of $A(\mathrm{x})$ is php:

     $p_k(\mathrm{x}) = \frac{1}{k!}\frac{\partial^k}{\partial t^k} \det\ (\mathrm{A}(\mathrm{x}) + \mathrm{tI_m})(\mathrm{x}, 0)$

19 Capacity

Capacity for php $p(\mathrm{x}) : \mathbb{R}^n \longrightarrow \mathbb{R}$:

$\mathrm{Cap}\ p := \inf_{\mathrm{x}>0, x_1 \dots x_n = 1} p(\mathrm{x}) = \inf_{\mathrm{x}>0} \frac{p(\mathrm{x})}{(x_1 \dots x_n)^{\frac{m}{n}}} \geq 0$

1. $\mathrm{Cap}\ (p) = 0$ for $p = x_1^{m_1} \dots x_n^{m_n}$
   $0 \lneq (m_1, \dots, m_n) \neq k\mathbf{1}$
2. $A = [a_{ij}] \in \mathbb{R}_+^{m \times n}$ doubly stochastic:

   each row has sum 1 each column has sum $\frac{m}{n}$.

   (a) $p_{m,A} = \prod_{i=1}^m (\sum_{j=1}^n a_{ij} x_j) \geq \prod_{i=1}^m \prod_{j=1}^n x_i^{a_{ij}} = (x_1 \dots x_n)^{\frac{m}{n}} \Rightarrow \mathrm{Cap}\ (p_{m,A}) \geq 1$

   $p_{m,A}(\mathbf{1}) = 1 \Rightarrow \mathrm{Cap}\ (p_{m,A}) = 1$

   (b) $\binom{m}{k}^{-1} p_{k,A} \geq p_{m,A}^{\frac{m}{n}} \geq (x_1 \dots x_n)^{\frac{k}{n}} \Rightarrow \mathrm{Cap}\ (p_{k,A}) = \binom{m}{k}$
3. $A \in \mathbb{R}_+^{n \times n}$ irreducible $A := D_1 B D_2$, B doubly stochastic matrix and D_1, D_2 diagonal pos. def.

   Sinkhorn.

   $\mathrm{Cap}\ p_{k,A} = (\det \mathrm{D_1 D_2})^{\frac{k}{n}}$

20 Mixed discriminants

$(A_1, \dots, A_n) \in \mathbf{H}^n_{m,+}$ doubly stochastic tuple:

$\mathrm{tr}\, A_i = \frac{m}{n}, i = 1, \dots, n$, and $\sum_{i=1}^n A_i = I_m$.

if $B_i = \mathrm{diag}(b_{1i}, \dots, b_{mi}), i = 1, \dots, n$, then $B = (b_{ji})_{j,i=1}^{m,n} \in \mathbb{R}_+^{m \times n}$ is d.s.

For d.s. tuple $(A_1, \dots, A_n)$

$\mathrm{Cap}\ (p_k) \geq \binom{m}{k}$

If each $A_i > 0$ equality holds iff $A_i = \frac{1}{n} I_m, i = 1, \dots, n$.

Prf:

$U^*(x)A(\mathrm{x})U(x) = \mathrm{diag}(\lambda_1(\mathrm{x}), \dots, \lambda_n(\mathrm{x}))$,

$\lambda_1(\mathrm{x}), \dots, \lambda_n(\mathrm{x}) > 0$.

$C_i = U^* A_i U = (c_{jk,i})_{j,k=1}^m$ for $i = 1, \dots, n$.

Then $(C_1, \dots, C_n)$ d.s. tuple. $(c_{jj,1}, \dots, c_{jj,n})$ prob. vec. $\lambda_j(\mathrm{x}) = \sum_{i=1} x_i c_{jj,i}$. arithmetic-geometric in. $\lambda_j(\mathrm{x}) \geq \prod_{i=1}^n x_i^{c_{jj,i}}$ for $j = 1, \dots, m$

$\det\ \mathrm{A}(\mathrm{x}) \geq \prod_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{x}_{\mathrm{i}}^{\mathrm{tr}\ \mathrm{A}_{\mathrm{i}}} = (\mathrm{x}_1 \dots \mathrm{x}_{\mathrm{n}})^{\frac{\mathrm{m}}{\mathrm{n}}}$ Hence $\mathrm{Cap}\ (p) \geq 1$.

21 The main inequality

Gurvits: Let $(u_1, \ldots, u_n), (v_1, \ldots, v_n) > 0$,
$f(t) := \prod_{i=1}^{k}(u_i t + v_i)$, $K(f) := \inf_{t>0} \frac{f(t)}{t}$.

Then $f'(0) = K$ for $k = 1$ and
$f'(0) \geq (\frac{k-1}{k})^{k-1} K$ for $k \geq 2$.

For $k \geq 2$ equality holds iff $\frac{v_1}{u_1} = \ldots = \frac{v_k}{u_k}$.

F-G: $p : R^n \to \mathbb{R}$ phd $\deg p = m \in [1, n], \deg_i p \leq r_i \in [1, m], i = 1, \ldots, n$.

Rearrange $r_1, \ldots, r_n$ to $1 \leq r_1^* \leq r_2^* \leq \ldots \leq r_n^*$.
$k \in [1, n]$ is the smallest integer $r_k^* > m - k$.

$$\sum_{1 \leq i_1 < \ldots < i_m \leq n} \frac{\partial^m}{\partial x_{i_1} \ldots \partial x_{i_m}} p(0, \ldots, 0) \geq \frac{n^{n-m}}{(n-m)!} \frac{(n-k+1)!}{(n-k+1)^{n-k+1}} \times \quad (FG)$$
$$\prod_{j=1}^{k-1} \left(\frac{r_j^* + n - m - 1}{r_j^* + n - m}\right)^{r_j^* + n - m - 1} \mathrm{Cap}\ p$$

Gurvits: $A \in \mathbb{R}_+^{n \times n}$ d.s. each column contains at most $r \in [1, n]$ nonzero entries:

$$\mathbf{perm\ A} \geq \frac{\mathbf{r!}}{\mathbf{r^r}}\left(\frac{\mathbf{r-1}}{\mathbf{r}}\right)^{\mathbf{(r-1)(n-r)}} = \frac{\mathbf{r!}}{\mathbf{r^r}}\left(\frac{\mathbf{r}}{\mathbf{r-1}}\right)^{\mathbf{r(r-1)}}\left(\frac{\mathbf{r-1}}{\mathbf{r}}\right)^{\mathbf{(r-1)n}}$$

Improvement of Schrijver for perfect matchings in $\Gamma(r, n)$

22 Lower bounds for sparse matrices

(FG) yields the Tverberg conjecture with $p_{k,A}(\mathbf{x})$

(FG) does not yield ALMC

Reason: (FG) proven using $p(\mathbf{x})\big(\frac{x_1+\ldots+x_n}{n}\big)^{n-m}$

FG: $p : R^n \to \mathbb{R}$ php $\deg p = m \in [1, n), \deg_i p \le r_i \in [1, m], i = 1, \ldots, n$.

Rearrange $r_1, \ldots, r_n$ to $1 \le r_1^* \le r_2^* \le \ldots \le r_n^*$.

for $s \in \mathbb{N}$ let $k \in [1, n]$ first integer $r_k^* + s > n - k$:

$$\sum_{1\le i_1<\ldots<i_m\le n} \frac{\partial^m}{\partial x_{i_1}\ldots\partial x_{i_m}} p(0, \ldots, 0) \ge \frac{(sn)!}{s^{n-m}(n-m)!((s-1)n+m)!} \frac{(n-k+1)!}{(n-k+1)^{n-k+1}} \prod_{j=1}^{k-1} \big(\frac{r_j^*+s-1}{r_j^*+s}\big)^{r_j^*+s-1} \mathrm{Cap}\ p$$

Prf: apply (FG) to $pp_{n-m,\frac{1}{s}A}$ for $A(G), G \in \Gamma(s, n)$ and average on $\Gamma(s, n)$

Cor: $\deg_i p \le r \in [1, m], i = 1, \ldots, n, s \in \mathbb{N}, k = n - r - s + 1 \ge 1$

$$\sum_{1\le i_1<\ldots<i_m\le n} \frac{\partial^m}{\partial x_{i_1}\ldots\partial x_{i_m}} p(0, \ldots, 0) \ge \frac{(sn)!}{s^{n-m}(n-m)!((s-1)n+m)!} \frac{(r+s)!}{(r+s)^{r+s}} \times \big(\frac{r+s-1}{r+s}\big)^{(r+s-1)(n-r-s)} \mathrm{Cap}\ p$$

23 Matching in general graphs

$G = (V, E)$ non bipartite graph $B = B(G)$

number of m-matchings: $\mathrm{haff}\ _{\mathrm{m}}\mathrm{B} = 2^{-\mathrm{m}}$ $\sum_{\alpha,\beta\subset\{1,\ldots,2n\},\#\alpha=\#\beta=m,\alpha\cap\beta=\emptyset} \mathrm{perm}\ \mathrm{B}[\alpha,\beta] =$ $2^{-m}\sum_{1\le i_1<\ldots<i_{2m}\le 2n} \frac{\partial^{2m}}{\partial x_{i_1}\ldots\partial x_{i_{2m}}}(\mathrm{x}^T B\mathrm{x})^m$

for $0 \neq B \in S_n(\mathbb{R}_+)$ $\mathrm{x}^T B\mathrm{x}$ php iff $\lambda_2(B) \le 0$

Thm: $B \in S_{2n}(\mathbb{R}_+)$ irreducible, $\lambda_2(B) \le 0$. Let $K := \mathrm{Cap}\ (\mathrm{x}^T B\mathrm{x})$. Then for $m \in [1, n]$ $\mathrm{haff}\ _{\mathrm{m}}\mathrm{B} \ge \binom{2\mathrm{n}}{2\mathrm{m}}\frac{\mathrm{K}^{\mathrm{m}}(2\mathrm{m})!}{2^{\mathrm{m}}(2\mathrm{n})!}$.

Lem: $0 \neq B \in S_n(\mathbb{R}_+)$ irreducible. $0 < K := \min \mathrm{x}^T B\mathrm{x}$, subject $\mathrm{x} = (x_1, \ldots, x_n)^T > 0, x_1 \ldots x_n = 1$, achieved at unique d $:= (d_1, \ldots, d_n) > 0, d_1 \ldots d_n = 1, n > 2$: $D := \mathrm{diag}(d_1, \ldots, d_n)$. Then $\frac{n}{K}DBD$ d.s.

Thm $G = (V, E)$ connected, $\mathrm{x}^T A(G)\mathrm{x}$ php iff G is complete k-partite

References

[1] R.J. Baxter, Dimers on a rectangular lattice, *J. Math. Phys.* 9 (1968), 650–654.

[2] L.M. Bregman, Some properties of nonnegative matrices and their permanents, *Soviet Math. Dokl.* 14 (1973), 945-949.

[3] G.P. Egorichev, Proof of the van der Waerden conjecture for permanents, *Siberian Math. J.* 22 (1981), 854–859.

[4] D.I. Falikman, Proof of the van der Waerden conjecture regarding the permanent of doubly stochastic matrix, *Math. Notes Acad. Sci. USSR* 29 (1981), 475–479.

[5] S. Friedland, A lower bound for the permanent of doubly stochastic matrices, *Ann. of Math.* 110 (1979), 167-176.

[6] S. Friedland, A proof of a generalized van der Waerden conjecture on permanents, *Lin. Multilin. Algebra* 11 (1982), 107–120.

[7] S. Friedland, E. Krop, P.H. Lundow and K. Markström, Computational Validations of the Asymptotic Matching

Conjectures, preprint 2005.

[8] S. Friedland, E. Krop and K. Markström, The limit value of the averages of monomer-dimer partitions of r-regular bipartite graphs, in preparation.

[9] S. Friedland and U.N. Peled, Theory of Computation of Multidimensional Entropy with an Application to the Monomer-Dimer Problem, *Advances of Applied Math.* 34(2005), 486-522.

[10] S. Friedland and U.N. Peled, The pressure associated with multidimensional SOFT, *in preparation*.

[11] L. Garding, An inequality for hyperbolic polynomials, *J. Mathematics and Mechanics*, 8 (1959), 957-965.

[12] L. Gurvits, Hyperbolic polynomials approach to van der Waerden/Schrijver-Valiant like conjectures: Sharper bounds, simpler proofs and algorithmic applications, arXIv:math.CO/0510452, 21 October, 2005

[13] J.M. Hammersley, Existence theorems and Monte Carlo methods for the monomer-dimer problem, in *Reseach papers in statistics: Festschrift for J. Neyman*, edited by F.N. David, Wiley, London, 1966, 125–146.

[14] O.J. Heilmann and E.H. Lieb, Theory of monomer-dimer systems., *Comm. Math. Phys.* 25 (1972), 190–232.

[15] J. Kahn, Entropy, independent sets and antichains, a new approach to dedekind's problem, *Procedings Amer. Math. Soc.* 130 (2001), 371-378.

[16] A.G. Khovanskii, Analogue of the Alexander-Fenchel inequalities for hyperbolic forms, Soviet Math. Dokl. 29 (1984), 710-713.

[17] L. Lovász and M.D. Plummer, *Matching Theory*, North-Holland Mathematical Studies, vol. 121, North-Holland, Amsterdam, 1986.

[18] R. Sinkhorn, A relationship between arbitrary positive matrices and doubly stochastic matrices, *Ann. Math. Statist.* 35 (1964), 876-879.

[19] B.L. van der Waerden, Aufgabe 45, *Jber Deutsch. Math.-Vrein.* 35 (1926), 117.

[20] A. Schrijver, Counting 1-factors in regular bipartite graphs, *J. Comb. Theory* B 72 (1998), 122–135.

[21] H. Tverberg, On the permanent of bistochastic matrix, *Math. Scand.* 12 (1963), 25-35.

[22] M. Voorhoeve, A lower bound for the permanent of certain $\mathbf{(0,1)}$ matrices, *Indag. Math.* 41 (1979), 83-86.

[23] I.M. Wanless, A lower bound on the maximum permanent in $\boldsymbol{\Lambda_n^k}$, *Linear Algebra Appl.* 373 (2003), 153–167.