# Matchings
# and positive hyperbolic polynomials

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# 1 Outline of the lecture

A. Matchings in general and bipartite graphs

1. Haffnians and permanents
2. Upper bounds for matchings in bipartite graphs
3. Lower bounds for matchings in bipartite graphs
4. Exact matching bounds for $\mathbf{2}$-regular graphs
5. Conjectural upper bounds for $\mathbf{3}$-regular graphs
6. Asymptotic growth of matchings & AUMC, ALMC
7. Graphs illustrations for dimensions $\mathbf{2, 3}$

B. Positive hyperbolic polynomials

1. Properties
2. Examples
3. Lower bounds
4. Proof of cases of ALMC

## 2 Matchings

$G = (V, E)$ undirected graph with vertices $V$, edges $E$.

matching in $G$: $M \subseteq E$

no two edges in $M$ share a common endpoint.

$e = (u, v) \in M$ is dimer

$v$ not covered by $M$ is monomer.

$M$ called monomer-dimer cover of $G$.

$M$ is perfect matching $\iff$ no monomers.

$M$ is $k$-matching $\iff$ $\#M = k$.

$\phi_G(k)$ number of $k$-matchings in $G$, $\phi_G(0) := 1$

$\Phi_G(x) := \sum_k \phi_G(k) x^k$ matching generating polyn.

roots of $\Phi_G(x)$ nonpositive [16].

$\Phi_{G_1 \cup G_2}(x) = \Phi_{G_1}(x) \Phi_{G_2}(x)$

$\Gamma(d, n)$ set of $d$-regular bipartite graphs on $2n$ vertices

## 3 Formulas for $k$-matchings

$A = A(G) \in \{0-1\}^{n\times n}$-adjacency matrix of $G = (V, E)$, $\#V = n$.

$\phi_k(G) = \mathrm{haff}_k(\mathrm{A}) :=$
$2^{-\mathrm{k}} \sum_{1\le \mathrm{i}_1 < \ldots < \mathrm{i}_{2\mathrm{k}} \le \mathrm{n}} \frac{\partial^{2\mathrm{k}}}{\partial \mathrm{x}_{\mathrm{i}_1} \ldots \mathrm{x}_{\mathrm{i}_{2\mathrm{k}}}} (\mathrm{x}^{\mathrm{T}}\mathrm{A}\mathrm{x})^{\mathrm{k}}, \mathrm{k} \le \frac{\mathrm{n}}{2}$

$\mathrm{x} = (x_1, \ldots, x_n)^T$

$G = (V, E)$ bipartite $V = V_1 \cup V_2, E \subset V_1 \times V_2$, $B = B(G) \in \{0-1\}^{m\times n}$, $\#V_1 = m, V_2 = n$.

$\phi_k(G) = \mathrm{perm}_{\mathrm{k}}(\mathrm{B}) :=$
$\sum_{\alpha \in \mathrm{Q}_{\mathrm{k},\mathrm{m}}, \beta \in \mathrm{Q}_{\mathrm{k},\mathrm{n}}} \mathrm{perm}\ \mathrm{B}[\alpha, \beta].$

for general $G$: $\phi_k(G) =$
$2^{-k} \sum_{\alpha \in Q_{k,n}, \beta \in Q_{k,n} \alpha\cap\beta=\emptyset,} \mathrm{perm}\ \mathrm{A}[\alpha, \beta]$

$Q_{k,n} := \{\alpha = \{i_1, \ldots, i_k\} :$

$1 \le i_1 < i_2 < \ldots < i_k \le n\}$.

# 4 Upper bounds for matchings

$A = [a_{ij}]_{i,j=1}^{n} \in \{0-1\}^{n\times n}$ represents bipartite graph $G$ on $n$ vertices in each class, with degree $r_i = \sum_{j=1}^{n} a_{ij}, i = 1,\ldots,n$ in first class

$\mathbf{perm\ A} = \#$ perfect matchings.

Minc-Bregman inequality -73: $\mathbf{perm\ A} \le \prod_{i=1}^{n} (r_i!)^{\frac{1}{r_i}}$

Cor: If $G$ is a bipartite graph on $n$ vertices in each class, each vertex in the first class of degree at most $d$ then $\phi_n(G) \le (d!)^{\frac{n}{d}}$ If $d|n$ equality holds for

$\frac{n}{d}K_{d,d}$- union of $\frac{n}{d}$ complete $d$-bipartite graphs

UMC: under the above conditions

$$\phi_k(G) \le \phi_k(\frac{n}{d}K_{d,d}), k = 1,\ldots \tag{4.1}$$

SUMC: Let $G$ graph on $2n$ vertices and degree of each vertex at most $d$. Then (4.1) holds

Open even for $k = n$.

Known for $d = 2$ Friedland-Krop-Markström

# 5 Lower bounds for matchings

$\Omega_n \subset \mathbb{R}_+^{n\times n}$ set of doubly stochastic matrices

van der Waerden conjecture: for $A \in \Omega_n$

$\mathrm{perm}\ \mathbf{A} \geq \mathrm{perm}\ \mathbf{J_n} = \frac{\mathbf{n!}}{\mathbf{n^n}} \sim \sqrt{2\pi \mathbf{n}}\mathbf{e}^{-\mathbf{n}} \geq \mathbf{e}^{-\mathbf{n}}$

($J_n = [\frac{1}{n}] \in \Omega_n$) Friedland-79, Falikman, Egorichev-81.

Cor: $\phi_n(G) \geq (\frac{d}{e})^n$ for any $G \in \Gamma(d, n)$.

Tverberg conjecture (Friedland-82):

$\mathrm{perm}\ _{\mathbf{k}}(\mathbf{A}) \geq \mathrm{perm}\ _{\mathbf{k}}(\mathbf{J_n}) = \binom{\mathbf{n}}{\mathbf{k}}^2 \frac{\mathbf{k!}}{\mathbf{n^k}}$

Cor: $\phi_k(G) \geq \binom{n}{k}^2 \frac{k! d^k}{n^k}$ for any $G \in \Gamma(d, n)$.

Reason: $\frac{1}{d} B(G) \in \Omega_n$.

Voorhoeve-79 ($d = 3$) Schrijver-98

$\phi_n(G) \geq (\frac{(d-1)^{d-1}}{d^{d-2}})^n$ for $G \in \Gamma(d, n)$

Gurvits: $A \in \Omega_n$, each column has at most $d$ nonzero entries: $\mathrm{perm}\ \mathbf{A} \geq \frac{\mathbf{d!}}{\mathbf{d^d}} \left(\frac{\mathbf{d}}{\mathbf{d-1}}\right)^{\mathbf{d(d-1)}} \left(\frac{\mathbf{d-1}}{\mathbf{d}}\right)^{\mathbf{(d-1)n}}$.

Cor: $\phi_n(G) \geq \frac{d!}{d^d} \left(\frac{d}{d-1}\right)^{d(d-1)} (\frac{(d-1)^{d-1}}{d^{d-2}})^n$

for $G \in \Gamma(d, n)$

LMC: $\phi_k(G) \geq \binom{n}{k}^2 (\frac{nd-k}{nd})^{nd-k} (\frac{kd}{n})^k$

# 6 Graphs with $d \leq 2$

$G$-the degree of each vertex $\leq 2$ is union of cycles, paths and isolated vertices $G$ bipartite if each cycle in $G$ is even

$C_k, P_k$ cycle and path of length k,

$\Phi_{C_k}(x) = \Phi_{P_k}(x) + x\Phi_{P_{k-2}}(x)$

Friedland-Krop-Markström for $2$-regular $G, \#V = n$

$\Phi_{C_i}(x)\Phi_{C_j}(x) - \Phi_{C_{i+j}}(x) = (-1)^i x^i \Phi_{C_{j-i}}(x)$

$\Phi_{C_i}(x)\Phi_{C_j}(x) \succ \Phi_{C_{i+j}}(x)$ if $i$ even ($i \leq j$),

$\Phi_{C_i}(x)\Phi_{C_j}(x) \prec \Phi_{C_{i+j}}(x)$ if $i$ odd $(i \leq j)$

$\Phi_G(x) \preceq \Phi_{C_4}(x)^{\frac{n}{4}}$ if $4|n$

$\Phi_G(x) \preceq \Phi_{C_4}(x)^{\frac{n-5}{4}}\Phi_{C_5}(x)$ if $4|n-1$

$\Phi_G(x) \preceq \Phi_{C_4}(x)^{\frac{n-6}{4}}\Phi_{C_6}(x)$ if $4|n-2$

$\Phi_G(x) \preceq \Phi_{C_4}(x)^{\frac{n-7}{4}}\Phi_{C_7}(x)$ if $4|n-3$

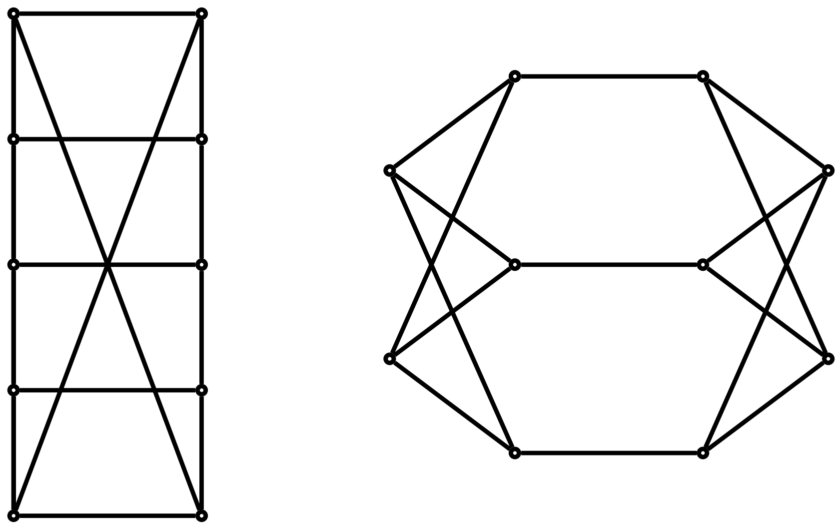
$\Phi_G(x) \succeq \Phi_{C_3}(x)^{\frac{n}{3}}$ if $3|n$

$\Phi_G(x) \succeq \Phi_{C_3}(x)^{\frac{n-4}{3}}\Phi_{C_4}(x)$ if $3|n-1$

$\Phi_G(x) \succeq \Phi_{C_3}(x)^{\frac{n-5}{3}}\Phi_{C_5}(x)$ if $3|n-2$

$\Phi_G(x) \succeq \Phi_{C_{2n}}$ if $G \in \Gamma(2, n)$

## 7 UMC for $G \in \Gamma(3,n)$



$M_{10}$ $G_1$

$$\Phi_{M_{10}} = 1 + 15x + 75x^2 + 145x^3 + 95x^4 + 13x^5$$
$$\Phi_{G_1} = 1 + 15x + 75x^2 + 145x^3 + 96x^4 + 12x^5$$

$Q_3$ graph of $3$-dimensional cube

Upper Matching Conjecture for $G \in \Gamma(3, n)$

- For $n \equiv 0 \pmod 3$, $\Phi_G \preceq \Phi_{\frac{n}{3}K_{3,3}}$, equality holding only if $G = \frac{n}{3}K_{3,3}$.
- For $n \equiv 1 \pmod 3$, $\Phi_G \preceq \Phi_{\frac{n-4}{3}K_{3,3}\cup Q_3}$, equality holding only if $G = \frac{n-4}{3}K_{3,3} \cup Q_3$.
- For $n \equiv 2 \pmod 3$, $\phi_G(k) \leqslant \max\left(\phi_{\frac{n-5}{3}K_{3,3}\cup M_{10}}(k), \phi_{\frac{n-5}{3}K_{3,3}\cup G_1}(k)\right)$, for $k = 1, \dots, n$.

# 8 Asymptotic growth of matchings

$G_n = (V_n, E_n) \in \Gamma(d, n)$, $n = 1, 2, \ldots$
sequence of $d$-regular bipartite graphs with $\#V_n \to \infty$.
Let $k_n \in [0, \frac{\#V_n}{2}]$, $n = 1, 2, \ldots$ sequence of integers with $lim_{n\to\infty} \frac{2k_n}{\#V_n} = p \in (0, 1]$. upper and lower ($p$)-asymptotic growth:
$hu_d(p) : \limsup_{n\to\infty} \frac{\log \phi_{G_n}(k_n)}{\#V_n}$,
$hl_d(p) : \liminf_{n\to\infty} \frac{\log \phi_{G_n}(k_n)}{\#V_n}$ For
$G_n := C_{2m_{1,n}} \times \ldots \times C_{2m_{d,n}}$ and
$\lim_{n\to\infty} m_{i,n} = \infty$ for $i = 1, \ldots, d$,
$hu_d(p) = hl_d(p) = h_d(p)$ is $p$-dimer density
Hammersley-66

UMC, LMC yield AUMC, ALMC:

$$hu_d(p) \le h_{K(d)}(p), \quad gh_d(p) \le hl_d(p)$$

$$gh_d(p) := \frac{1}{2}\big(p \log d - p \log p - 2(1-p)\log(1-p) + (d-p)\log\big(1 - \frac{p}{d}\big)\big)$$

$$P_{K(d)}(t) = \frac{\log \sum_{k=0}^{d} \binom{d}{k}^2 k!\, e^{2kt}}{2d}, t \in \mathbb{R}.$$

$$h_{K(d)}(p(t)) = P_{K(d)}(t) - tp(t)$$

$K(d)$ countable union of $K_{d,d}$

# 9 Graph estimates for $h_2(p)$

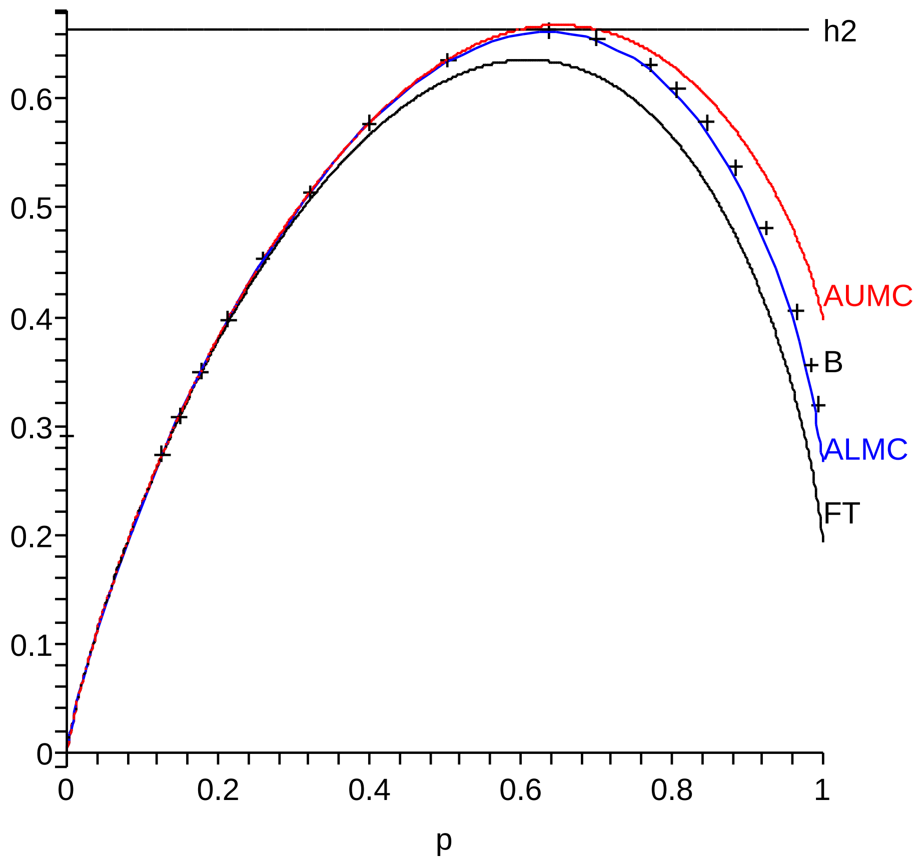



Figure 1: Monomer-dimer tiling of the $\mathbf{2}$-dimensional grid: entropy as a function of dimer density. FT is the Friedland-Tverberg lower bound, h2 is the true monomer-dimer entropy. B are Baxter's computed values. ALMC is the Asymptotic Lower Matching Conjecture. AUMC is the entropy of a countable union of $K_{4,4}$, conjectured to be an upper bound by the Asymptotic Upper Matching Conjecture.

# 10 Graph estimates for $h_3(p)$

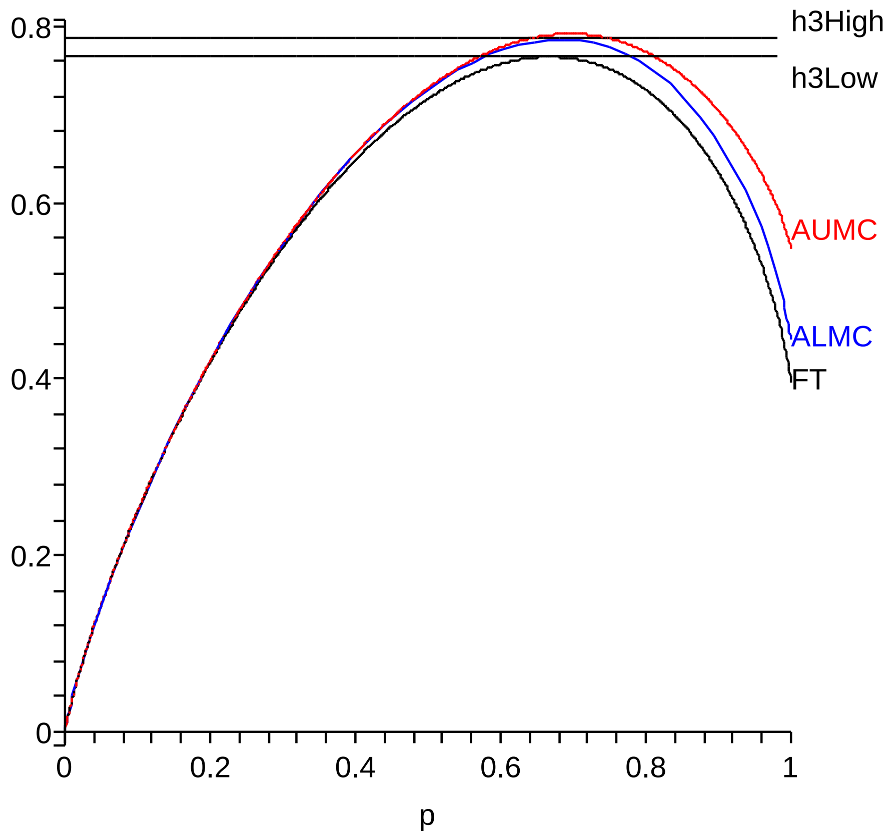



Figure 2: Monomer-dimer tiling of the $3$-dimensional grid: entropy as a function of dimer density. FT is the Friedland-Tverberg lower bound, h3Low and h3High are the known bounds for the monomer-dimer entropy. ALMC is the Asymptotic Lower Matching Conjecture. AUMC is the entropy of a countable union of $K_{6,6}$, conjectured to be an upper bound by the Asymptotic Upper Matching Conjecture.

## 11 The sharpness of the ALMC

Standard probabilistic model on $\Gamma(d,n)$:

$\sigma \in S_{nd}$ permutation on $nd$ elements.

$e_1,...,e_{nd}$-$nd$ edges from $\{1,...,n\}$ to $\{1,...,n\}$. $e_i$ connects vertex $\lceil \frac{i}{d} \rceil$ to $\lceil \frac{\sigma(i)}{d} \rceil$ for $i=1,...,nd$. The probability of $G$ is $\frac{1}{(nd)!}$.

$\nu(d,n)$ the induced probability measure on $\Gamma(d,n)$

$\nu(d,n)$ invariant under the action of $S_n$ on $V_1$ and $V_2$

$E_{\nu(d,n)}(\phi_k(G)) = \frac{\binom{n}{k}^2 d^{2k} k!(dn-k)!}{(dn)!}$

$\lim_{m\to\infty} \frac{\log E_{\nu(d,n_m)}(\phi_{k_m}(G))}{2n_m} = gh_d(p)$,

where $\lim_{m\to\infty} \frac{k_m}{n_m} = p \in [0,1]$

Same results holds for another probability on $\Gamma(d,n)$.

Identify $G$ with $A(G) \in \mathbb{Z}_+^{n\times n}$.

$\pi : \Gamma(1,n)^d \to \Gamma(d,n)$,

$(P_1,\ldots,P_d) \mapsto P_1 + \ldots + P_d$

$\mu(d,n)$ is the push forward of uniform probability $\eta_{d,n}$ on $\Gamma(1,n)^d$: $\mu(d,n)(G) = \eta_{d,n}(\pi^{-1}(G))$ FKM

## 12 Positive Hyperbolic Polynomials

1. polynomial

   $p = p(\mathbf{x}) = p(x_1, \ldots, x_n) : \mathbb{R}^n \longrightarrow \mathbb{R}$ is

   positive hyperbolic (php) if:

   - $p$ homogeneous polynomial of degree $m \geq 0$.
   - $p(\mathbf{x}) > 0$ for all $\mathbf{x} > \mathbf{0}$.
   - $\phi(t) := p(\mathbf{x} + t\mathbf{u}), t \in \mathbb{R}$

     has $m$-real $t$-roots for all $\mathbf{u} > \mathbf{0}$ and $\mathbf{x} \in \mathbb{R}^n$.

2. For

   $p : \mathbb{R}^n \longrightarrow \mathbb{R}, \mathbf{0} \neq \mathbf{u} = (u_1, \ldots, u_n)^\top \in \mathbb{R}^n$

   let $p_{\mathbf{u}} = p_{\mathbf{u}}(\mathbf{x}) := \sum_{i=1}^{n} u_i \frac{\partial p}{\partial x_i}(\mathbf{x})$.

3. $\deg_i p$ the degree of $p(\mathbf{x})$ with respect to variable $x_i$

4. $\mathbf{e}_i := (\delta_{i1}, \ldots, \delta_{in})^\top \in \mathbb{R}^n, \; i = 1, \ldots, n$

5. $\mathbf{1} := (1, \ldots, 1)^\top \in \mathbb{R}^n$

## 13 Properties positive hyperbolic pol.

$p : \mathbb{R}^n \longrightarrow \mathbb{R}$ is php of degree $m$:

1. For $\mathbf{u} \gneqq \mathbf{0}, \mathbf{x}$ fixed $\phi(t) = p(\mathbf{x} + t\mathbf{u})$. If $p(\mathbf{u}) > 0$ then $\phi(t)$ has $m$ real $t$ roots

   and $p_{\mathbf{u}}(\mathbf{x})$ is php of degree $m - 1$.
   $\mathbf{y} \geq \mathbf{x} \geq \mathbf{0} \Rightarrow p(\mathbf{y}) \geq p(\mathbf{x}) \geq 0$.

2. $\mathbf{u} \gneqq \mathbf{0}, \mathbf{x} \gneqq \mathbf{0}$ and $p(\mathbf{u}) = 0$.

   either $\phi(t) > 0 \forall t \geq 0$

   or $p(\mathbf{x}) = 0$ and $\phi(t) \equiv 0$.

   If $p(\mathbf{x}) > 0$ and $\phi(t) \neq Const$ then all roots of $\phi(t)$ real and negative.

   if $p_{\mathbf{u}} \neq 0$ then $p_{\mathbf{u}}$ is a php of degree $m - 1$.

3. If $q((x_1, \ldots, x_{n-1})) :=$
   $p((x_1, x_2, \ldots, x_{n-1}, 0)) \neq 0$

   then $q$ is php of degree $m$ in $\mathbb{R}^{n-1}$.

   In particular, $r((x_1, \ldots, x_{n-1})) :=$
   $\frac{\partial p}{\partial x_n}((x_1, \ldots, x_{n-1}, 0))$ is either $0$ or php in $n - 1$ variables of degree $m - 1$.

4. The coefficient of each monomial in php is nonnegative.

## 14 Examples of php

1. $A = (a_{ij})_{i=j=1}^{m,n} \in \mathbb{R}_+^{m\times n}$ ($A \geq 0$) and each row of $A$ is nonzero

   $p_{k,A}(x) := \sum_{1\leq i_1<\ldots<i_k\leq m} \prod_{j=1}^{k} (A\mathrm{x})_{i_j}$

   is php of degree $k \in [1, m]$:

   $p_{k,A} =$
   $\frac{\partial^{m-k}}{\partial y_1 \ldots \partial y_{m-k}} \prod_{j=1}^{m} ((A\mathrm{x})_j + \sum_{i=1}^{m-k} y_i)$

2. $A_1, \ldots, A_n \in \mathbb{C}^{m\times m}$ hermitian, nonnegative definite and $A_1 + \ldots + A_m$ is positive definite

   $A(\mathrm{x}) := \sum_{i=1}^{n} x_i A_i$

   - $p(\mathrm{x}) = \det \mathrm{A}(\mathrm{x})$ is php
   - $p_k(\mathrm{x})$-the sum of all $k \times k$ principle minors of $A(\mathrm{x})$ is php:

     $p_k(\mathrm{x}) = \frac{1}{k!}\frac{\partial^k}{\partial t^k} \det (\mathrm{A}(\mathrm{x}) + \mathrm{t}\mathrm{I}_\mathrm{m})(\mathrm{x}, 0)$

## 15 Capacity

Capacity for php $p(\mathbf{x}) : \mathbb{R}^n \longrightarrow \mathbb{R}$:

$\text{Cap } p := \inf_{\mathbf{x}>0, x_1 \ldots x_n = 1} p(\mathbf{x}) = \inf_{\mathbf{x}>0} \frac{p(\mathbf{x})}{(x_1 \ldots x_n)^{\frac{m}{n}}} \geq 0$

1. $\text{Cap } (p) = 0$ for $p = x_1^{m_1} \ldots x_n^{m_n}$
   $0 \lneqq (m_1, \ldots, m_n) \neq k1$

2. $A = [a_{ij}] \in \mathbb{R}_+^{m \times n}$ doubly stochastic:

   each row has sum $1$ each column has sum $\frac{m}{n}$.

   (a) $p_{m,A} = \prod_{i=1}^m (\sum_{j=1}^n a_{ij} x_j) \geq \prod_{i=1}^m \prod_{j=1}^n x_i^{a_{ij}} = (x_1 \ldots x_n)^{\frac{m}{n}} \Rightarrow \text{Cap } (p_{m,A}) \geq 1$
   $p_{m,A}(1) = 1 \Rightarrow \text{Cap } (p_{m,A}) = 1$

   (b) $\binom{m}{k}^{-1} p_{k,A} \geq p_{m,A}^{\frac{m}{n}} \geq (x_1 \ldots x_n)^{\frac{k}{n}} \Rightarrow \text{Cap } (p_{k,A}) = \binom{m}{k}$

3. $A \in \mathbb{R}_+^{n \times n}$ irreducible $A := D_1 B D_2$, $B$ doubly stochastic matrix and $D_1, D_2$ diagonal pos. def.

   Sinkhorn.

   $\text{Cap } p_{k,A} = (\det \mathrm{D_1 D_2})^{\frac{k}{n}}$

## 16 Mixed discriminants

$(A_1,\ldots,A_n)\in \mathbf{H}^n_{m,+}$ doubly stochastic tuple:

$\mathrm{tr}\, A_i=\frac{m}{n}, i=1,\ldots,n,$ and $\sum_{i=1}^n A_i=I_m$.

if $B_i=\mathrm{diag}(b_{1i},\ldots,b_{mi}), i=1,\ldots,n$, then
$B=(b_{ji})_{j,i=1}^{m,n}\in\mathbb{R}_+^{m\times n}$ is d.s.

For d.s. tuple $(A_1,\ldots,A_n)$

$\mathrm{Cap}\ (p_k)\ge\binom{m}{k}$

If each $A_i>0$ equality holds iff
$A_i=\frac{1}{n}I_m, i=1,\ldots,n$.

Prf:
$U^*(x)A(\mathrm{x})U(x)=\mathrm{diag}(\lambda_1(\mathrm{x}),\ldots,\lambda_n(\mathrm{x}))$,
$\lambda_1(\mathrm{x}),\ldots,\lambda_n(\mathrm{x})>0$.
$C_i=U^*A_iU=(c_{jk,i})_{j,k=1}^m$ for $i=1,\ldots,n$.
Then $(C_1,\ldots,C_n)$ d.s. tuple. $(c_{jj,1},\ldots,c_{jj,n})$ prob. vec. $\lambda_j(\mathrm{x})=\sum_{i=1}x_ic_{jj,i}$. arithmetic-geometric in. $\lambda_j(\mathrm{x})\ge\prod_{i=1}^n x_i^{c_{jj,i}}$ for $j=1,\ldots,m$
$\det\ \mathrm{A}(\mathrm{x})\ge\prod_{\mathrm{i}=1}^{\mathrm{n}}\mathrm{x}_\mathrm{i}^{\mathrm{tr}\ \mathrm{A_i}}=(\mathrm{x}_1\ldots\mathrm{x}_\mathrm{n})^{\frac{\mathrm{m}}{\mathrm{n}}}$ Hence
$\mathrm{Cap}\ (p)\ge1$.

## 17 The main inequality

Gurvits: Let $(u_1,\dots,u_n),(v_1,\dots,v_n)>0$, $f(t):=\prod_{i=1}^{k}(u_it+v_i), K(f):=\inf_{t>0}\frac{f(t)}{t}$.

Then $f'(0)=K$ for $k=1$ and $f'(0)\ge(\frac{k-1}{k})^{k-1}K$ for $k\ge 2$.

For $k\ge 2$ equality holds iff $\frac{v_1}{u_1}=\ldots=\frac{v_k}{u_k}$.

F-G: $p:R^n\to\mathbb{R}$ phd $\deg p=m\in[1,n], \deg_i p\le r_i\in[1,m], i=1,\dots,n$.

Rearrange $r_1,\dots,r_n$ to $1\le r_1^*\le r_2^*\le\ldots\le r_n^*$. $k\in[1,n]$ is the smallest integer $r_k^*>m-k$.
$\sum_{1\le i_1<\ldots<i_m\le n}\frac{\partial^m}{\partial x_{i_1}\ldots\partial x_{i_m}}p(0,\dots,0)\ge \frac{n^{n-m}}{(n-m)!}\frac{(n-k+1)!}{(n-k+1)^{n-k+1}}\times \quad (FG)$
$\prod_{j=1}^{k-1}(\frac{r_j^*+n-m-1}{r_j^*+n-m})^{r_j^*+n-m-1}\,\mathrm{Cap}\;p$

Gurvits: $A\in\mathbb{R}_+^{n\times n}$ d.s. each column contains at most $r\in[1,n]$ nonzero entries:

$\mathbf{perm\ A}\ge\frac{\mathbf{r!}}{\mathbf{r^r}}\left(\frac{\mathbf{r-1}}{\mathbf{r}}\right)^{\mathbf{(r-1)(n-r)}}= \frac{\mathbf{r!}}{\mathbf{r^r}}\left(\frac{\mathbf{r}}{\mathbf{r-1}}\right)^{\mathbf{r(r-1)}}\left(\frac{\mathbf{r-1}}{\mathbf{r}}\right)^{\mathbf{(r-1)n}}$

Improvement of Schrijver for perfect matchings in $\Gamma(r,n)$

## 18 Lower bounds for sparse matrices

(FG) yields the Tverberg conjecture with $p_{k,A}(\mathbf{x})$

(FG) does not yield ALMC

Reason: (FG) proven using $p(\mathbf{x})\left(\frac{x_1+\ldots+x_n}{n}\right)^{n-m}$

FG: $p: R^n \to \mathbb{R}$ php $\deg p = m \in [1,n), \deg_i p \le r_i \in [1,m], i = 1,\ldots,n$.

Rearrange $r_1,\ldots,r_n$ to $1 \le r_1^* \le r_2^* \le \ldots \le r_n^*$.

for $s \in \mathbb{N}$ let $k \in [1,n]$ first integer $r_k^* + s > n - k$:

$$\sum_{1\le i_1<\ldots<i_m\le n} \frac{\partial^m}{\partial x_{i_1}\ldots\partial x_{i_m}} p(0,\ldots,0) \ge \frac{(sn)!}{s^{n-m}(n-m)!((s-1)n+m)!}\frac{(n-k+1)!}{(n-k+1)^{n-k+1}} \prod_{j=1}^{k-1}\left(\frac{r_j^*+s-1}{r_j^*+s}\right)^{r_j^*+s-1} \mathrm{Cap}\ p$$

Prf: apply (FG) to $pp_{n-m,\frac{1}{s}A}$ for $A(G), G \in \Gamma(s,n)$ and average on $\Gamma(s,n)$

Cor: $\deg_i p \le r \in [1,m], i = 1,\ldots,n, s \in \mathbb{N}, k = n - r - s + 1 \ge 1$

$$\sum_{1\le i_1<\ldots<i_m\le n} \frac{\partial^m}{\partial x_{i_1}\ldots\partial x_{i_m}} p(0,\ldots,0) \ge \frac{(sn)!}{s^{n-m}(n-m)!((s-1)n+m)!}\frac{(r+s)!}{(r+s)^{r+s}} \times \left(\frac{r+s-1}{r+s}\right)^{(r+s-1)(n-r-s)} \mathrm{Cap}\ p$$

# 19 Lower asymptotic bounds

Thm: $r \geq 3, s \geq 1, B_n \in \Omega_n, n = 1, 2, \ldots$ each column of $B_n$ has at most $r$-nonzero entries.
$k_n \in [0, n] \cap \mathbb{N}, n = 1, 2, \ldots, lim_{n\to\infty} \frac{k_n}{n} = p \in (0, 1]$ then

$$\liminf_{n\to\infty} \frac{\log \mathrm{perm}_{\ k_n} B_n}{2n} \geqslant \frac{1}{2}\left(-p \log p - 2(1-p)\log(1-p)\right) + \frac{1}{2}(r+s-1)\log(1-\frac{1}{r+s}) - \frac{1}{2}(s-1+p)\log(1-\frac{1-p}{s})$$

Cor: $d$-ALMC holds for $p_s = \frac{d}{d+s}, s = 0, 1, \ldots,$

Con: under Thm assumptions

$$\liminf_{n\to\infty} \frac{\log \mathrm{perm}_{\ k_n} B_n}{2n} \geqslant gh_r(p) - \frac{p}{2}\log r$$

For $p_s = \frac{r}{r+s}, s = 0, 1, \ldots,$ conjecture holds

## 20 Matching in general graphs

$G=(V,E)$ non bipartite graph $B=B(G)$

number of $m$-matchings: $\mathrm{haff}\ _{\mathrm{m}}\mathrm{B}=2^{-\mathrm{m}}$
$\sum_{\alpha,\beta\subset\{1,\ldots,2n\},\#\alpha=\#\beta=m,\alpha\cap\beta=\emptyset}\mathrm{perm}\ \mathrm{B}[\alpha,\beta]=$
$2^{-m}\sum_{1\le i_1<\ldots<i_{2m}\le 2n}\frac{\partial^{2m}}{\partial x_{i_1}\ldots\partial x_{i_{2m}}}(\mathrm{x}^T B\mathrm{x})^m$

for $0\ne B\in S_n(\mathbb{R}_+)$ $\mathrm{x}^T B\mathrm{x}$ php iff $\lambda_2(B)\le 0$

Thm: $B\in S_{2n}(\mathbb{R}_+)$ irreducible, $\lambda_2(B)\le 0$. Let
$K:=\mathrm{Cap}\ (\mathrm{x}^T B\mathrm{x})$. Then for $m\in[1,n]$
$\mathrm{haff}\ _{\mathrm{m}}\mathrm{B}\ge\binom{2\mathrm{n}}{2\mathrm{m}}\frac{\mathrm{K}^{\mathrm{m}}(2\mathrm{m})!}{2^{\mathrm{m}}(2\mathrm{n})!}$.

Lem: $0\ne B\in S_n(\mathbb{R}_+)$ irreducible.
$0<K:=\min\mathrm{x}^T B\mathrm{x}$, subject
$\mathrm{x}=(x_1,\ldots,x_n)^T>0, x_1\ldots x_n=1$, achieved
at unique
d $:=(d_1,\ldots,d_n)>0, d_1\ldots d_n=1, n>2$:
$D:=\mathrm{diag}(d_1,\ldots,d_n)$. Then $\frac{n}{K}DBD$ d.s.

Thm $G=(V,E)$ connected, $\mathrm{x}^T A(G)\mathrm{x}$ php iff $G$ is
complete $k$-partite

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