Counting matchings in graphs with applications to the monomer-dimer models

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- Summary and open problems
Figure: Matching on the two dimensional grid: Bipartite graph on 60 vertices, 101 edges, 24 dimers, 12 monomers
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- $M$ is $k$-matching $\iff$ $\#M = k$. 
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Bipartite graphs

Figure: An example of a bipartite graph

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Incidence matrix
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For $G = (\langle 2n \rangle, E)$ bipartite $G \in \mathcal{G}(r, 2n) \iff \frac{1}{r} B(G) \in \Omega_n \iff G$ is a disjoint (edge) union of $r$ perfect matchings
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$$r^k \min_{C \in \Omega_n} \text{perm}_k C \leq \phi(k, G) \text{ for any } G \in \mathcal{G}(r, 2n)$$
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van der Waerden permanent conjecture 1926:

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Tverberg permanent conjecture 1963:

\[ \min_{C \in \Omega_n} \text{perm}_k \ C = \text{perm}_k \frac{1}{n} J_n \left( = \binom{n}{k}^2 \frac{k!}{n^k} \right) \]

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There are new simple proofs using nonnegative hyperbolic polynomials e.g. Friedland-Gurvits 2008
Lower matching bounds for 0 – 1 matrices
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Voorhoeve-1979 ($d = 3$) Schrijver-1998

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Cor: \[ \phi(n, G) \geq \frac{r!}{r^r} \left( \frac{r}{r-1} \right)^{r(r-1)} \left( \frac{(r-1)^{r-1}}{r^{r-2}} \right)^n \]
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Gurvits 2006: $A \in \Omega_n$, each column has at most $r$ nonzero entries:

\[ \text{perm} \ A \geq \frac{r!}{r^r} \left( \frac{r}{r-1} \right)^{r(r-1)} \left( \frac{r - 1}{r} \right)^{(r-1)n}. \]

Cor: \[ \phi(n, G) \geq \frac{r!}{r^r} \left( \frac{r}{r-1} \right)^{r(r-1)} \left( \frac{(r - 1)^{r-1}}{rr^{r-2}} \right)^n \]

Con FKM 2006: \[ \phi(k, G) \geq \binom{n}{k}^2 \left( \frac{nr - k}{nr} \right)^{nr-k} \left( \frac{kr}{n} \right)^k, \quad G \in \mathcal{G}(r, 2n) \]
Lower matching bounds for 0–1 matrices

Voorhoeve-1979 ($d = 3$) Schrijver-1998

$$
\phi(n, G) \geq \left( \frac{(r - 1)^{r-1}}{r^{r-2}} \right)^n \quad \text{for} \quad G \in \mathcal{G}(r, 2n)
$$

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\text{perm } A \geq \frac{r!}{r^r} \left( \frac{r}{r - 1} \right)^{r(r-1)} \left( \frac{r - 1}{r} \right)^{(r-1)n}.
$$

Cor:

$$
\phi(n, G) \geq \frac{r!}{r^r} \left( \frac{r}{r - 1} \right)^{r(r-1)} \left( \frac{(r - 1)^{r-1}}{r^{r-2}} \right)^n
$$

Con FKM 2006:

$$
\phi(k, G) \geq \left( \binom{n}{k} \right)^2 \left( \frac{nr - k}{nr} \right)^{nr-k} \left( \frac{nr}{n} \right)^k, \quad G \in \mathcal{G}(r, 2n)
$$

F-G 2008 showed weaker inequalities
Upper matching bounds for 0–1 matrices
Assume $A \in \{0, 1\}^{n \times n}$. 
Upper matching bounds for 0 – 1 matrices

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- $r_i$ is the $i$-th row sum of $A$. 
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- $\phi(qr, G) \leq \phi(qr, qK_r, r)$ for any $G \in \mathcal{G}(r, 2qr)$
- **Con FKM 2006**: $\phi(k, G) \leq \phi(k, qK_r, r)$ for any $G \in \mathcal{G}(r, 2qr)$ and $k = 1, \ldots, qr$
- $c_4(G)$ - The number of 4-cycles in $G$
Upper matching bounds for 0 – 1 matrices

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- \( c_4(G) \) - The number of 4-cycles in \( G \)
- Thm: For any \( r \)-regular graph \( G = (V, E) \),

\[
c_4(G) \leq \frac{r \# V (r - 1)^2}{2} \frac{4}{4}
\]

Equality iff \( G = qK_{r,r} \)
Upper matching bounds for 0 – 1 matrices

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$$c_4(G) \leq \frac{r\#V}{2} \frac{(r - 1)^2}{4}$$

Equality iff $G = qK_{r,r}$

- Prf: Any edge in $e \in E$ can be in at most $(r - 1)^2$ different 4-cycles.
An example

Figure: Edge neighborhood of $V_2W_2$ of 4-regular graph on 8 vertices
$G = (V, E)$ Non-bipartite graph on $2n$ vertices

$$\phi(n, G) \leq \prod_{v \in V} \left( \frac{1}{2 \deg v} \right)^{\frac{1}{2 \deg v}} \left( (\deg v)! \right)^{\frac{1}{2 \deg v}}$$

If $\deg v > 0$, $\forall v \in V$ equality holds iff $G$ is a disjoint union of complete balanced bipartite graphs
Exact values for small matchings

For $G \in \mathcal{G}(r, 2n)$
Exact values for small matchings

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1. $\phi(1, G) = nr$
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2. $\phi(2, G) = \binom{nr}{2} - 2n\binom{r}{2} = \frac{nr(nr-(2r-1))}{2}$
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2. $\phi(2, G) = \frac{nr}{2} - 2n\binom{r}{2} = \frac{nr(nr - (2r - 1))}{2}$
3. $\phi(3, G) = \frac{nr}{3} - 2n\binom{r}{3} - nr(r - 1)^2 - 2n\binom{r}{2}(nr - 2r - (r - 2))$
Exact values for small matchings

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4. $\phi(4, G) = p_1(n, r) + c_4(G)$

\[ p_1(n, r) = \frac{n^4r^4}{24} + \frac{n^3r^3}{4}(1-2r) + \frac{n^2r^2}{24}(19 - 60r + 52r^2) + nr\left(\frac{5}{4} - 5r + 7r^2 - \frac{7r^3}{2}\right) \]
Exact values for small matchings

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\]

Notation:

\[
f(x) = \sum_{i=0}^{N} a_i x^i \preceq g(x) = \sum_{i=0}^{N} b_i x^i \iff a_i \leq b_i \text{ for } i = 1, \ldots, N.
\]
2-regular graphs
2-regular graphs

- $\Gamma(r, n)$ the set of $r$-regular graphs on $n$-vertices
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  \]
  \[
  \Phi_G(x) \preceq \Phi_{\frac{n-6}{4}K_{2,2} \cup C_6}(x) = \Phi_{C_4}(x)^{\frac{n-6}{4}} \Phi_{C_6}(x) \quad \text{if} \quad 4 \mid n - 2
  \]
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  \]
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  \]
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  \[ \Phi_G(x) \succeq \Phi \left( \frac{n}{3} K_3 \right)(x) = \Phi C_3(x)^{\frac{n}{3}} \text{ if } 3|n \]
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  \[ \Phi_G(x) \succeq \Phi \left( \frac{n-5}{3} K_3 \cup C_5 \right)(x) = \Phi C_3(x)^{\frac{n-5}{3}} \Phi C_5(x) \text{ if } 3|n - 2 \]
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  \]
  \[
  \Phi_G(x) \succeq \Phi_{\frac{n}{3}K_3}(x) = \Phi_{C_3}(x)^{\frac{n}{3}} \text{ if } 3|n
  \]
  \[
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  \]
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  \Phi_G(x) \succeq \Phi_{\frac{n-5}{3}K_3 \cup C_5}(x) = \Phi_{C_3}(x)^{\frac{n-5}{3}} \Phi_{C_5}(x) \text{ if } 3|n - 2
  \]
- If $n$ even $G$ multi-bipartite 2-regular graph then $\Phi_G(x) \succeq \Phi_{C_n}(x)$.
Relations between matching polynomials

- For $0 \leq i \leq j$
  \[ \Phi_{C_i}(x)\Phi_{C_j}(x) - \Phi_{C_{i+j}}(x) = (-1)^i x^i \Phi_{C_{j-i}}(x) \]

- $P_n$ path $1 \rightarrow 2 \rightarrow \ldots \rightarrow n$.

- $p_n(x) := \Phi_{P_n}(x)$, $q_n(x) := \Phi_{C_n}(x)$

- $p_k(x) = p_{k-1}(x) + xp_{k-2}(x)$

- $q_k(x) = p_k(x) + xp_{k-2}(x)$

- If $n = 0, 1 \mod 4$
  \[
  p_{n-1} = p_1 p_{n-1} \prec p_3 p_{n-3} \prec \cdots \prec p_{2 \lfloor \frac{n}{4} \rfloor} p_{n-2 \lfloor \frac{n}{4} \rfloor + 1} \prec \\
  p_{2 \lfloor \frac{n}{4} \rfloor} p_{n-2 \lfloor \frac{n}{4} \rfloor} \prec p_{2 \lfloor \frac{n}{4} \rfloor - 2} p_{n-2 \lfloor \frac{n}{4} \rfloor + 2} \prec \cdots \prec p_{2} p_{n-2} \prec p_0 p_n = p_n
  
  q_{n-1} = q_1 q_{n-1} \prec q_3 q_{n-3} \prec \cdots \prec q_{2 \lfloor \frac{n}{4} \rfloor} q_{n-2 \lfloor \frac{n}{4} \rfloor + 1} \prec \\
  q_{2 \lfloor \frac{n}{4} \rfloor} q_{n-2 \lfloor \frac{n}{4} \rfloor} \prec q_{2 \lfloor \frac{n}{4} \rfloor - 2} q_{n-2 \lfloor \frac{n}{4} \rfloor + 2} \prec \cdots \prec q_2 q_{n-2} \prec q_{n+1}
  
- Characterization of maximal and minimal matching polynomial graphs in family of graphs with given number of vertices of degrees one and two
Cubic bipartite graphs
Cubic bipartite graphs

\[ G(3, 6) = \{ K_{3,3} \} \]
Cubic bipartite graphs

- $G(3, 6) = \{K_{3,3}\}$
- $G(3, 8) = \{Q_3\}$ three dimensional cube
Cubic bipartite graphs

- $G(3, 6) = \{K_{3,3}\}$
- $G(3, 8) = \{Q_3\}$ three dimensional cube
- $G(3, 10) = \{G_1, M_{10}\}$ have incomparable matching polynomials
  
  \[
  \psi(x, G_1) := 1 + 15x + 75x^2 + 145x^3 + 96x^4 + 12x^5
  
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- For $2n$ from 12 to 24 the extremal graphs, with the maximal $\phi(l, G)$:
  \[
  \frac{2n}{6} K_{3,3} \quad \text{if } 6|2n
  \]
  \[
  \frac{2n-8}{6} K_{3,3} \cup Q_3 \quad \text{if } 6|(2n - 2)
  \]
  \[
  \frac{2n-10}{6} K_{3,3} \cup (G_1 \text{ or } M_{10}) \quad \text{if } 6|(2n - 4)
  \]
Two bipartite 3-regular graphs on 10 vertices

\[ M_{10} \]

\[ G_1 \]
Expected values of $k$-matchings
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- Permutation $\sigma : \langle nr \rangle \rightarrow \langle nr \rangle$ induces $G(\sigma) \in G_{\text{mult}}(r, 2n)$ and vice versa.

$$G(\sigma) = \{ (i, \lceil \frac{\sigma((i-1)r+j)}{r} \rceil) , \ j = 1, \ldots , r , \ i = 1, \ldots , n \} \subset \langle n \rangle \times \langle n \rangle$$

- The number of different $\sigma$ inducing the same simple $G$ is $(r!)^n$. 
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  $k = 1, \ldots, n$

- $1 \leq k_l \leq n_l, l = 1, \ldots,$ increasing sequences of integers s.t.
  
  \[ \lim_{l \to \infty} \frac{k_l}{n_l} = p \in [0, 1]. \]

Then

\[ \lim_{l \to \infty} \frac{\log E(k_l, n_l, r)}{2n_k} = f(p, r) \]

\[ f(p, r) := \frac{1}{2}(p \log r - p \log p - 2(1-p) \log(1-p) + (r-p) \log(1 - \frac{p}{r})) \]
$p$-matching and total matching entropies
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- Friedland-Peled confirmed Baxter’s computations to be published
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- $h_3 \geq 0.7849602275$ Friedland-Krop-Lundow-Markström accepted JOSS 2008
Asymptotic Lower and Upper Matching conjectures
FKLM 06:

\[ G_l = (E_l, V_l) \in \mathcal{G}(r, \# V_l), \; l = 1, 2, \ldots, \; \text{and} \; \lim_{l \to \infty} \frac{2k_l}{\# V_l} = p. \]
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\[ P_r(t) := \frac{\log \sum_{k=0}^{r} \binom{r}{k}^2 k! e^{2kt}}{2r}, \ t \in \mathbb{R}, \]

\[ p(t) := P'_r(t) \in (0, 1), \ h_{K(r)}(p(t)) := P_r(t) - tp(t) \]
Counting matchings in graphs with applications to the monomer-dimer models

Shmuel Friedland
Univ. Illinois at Chicago & Berlin Mathematical School

KTH, 16 April, 2008 26 / 38
Counting matchings in graphs with applications to the monomer-dimer models
Thm: $r \geq 3, s \geq 1$ integers, 

$B_n \in \Omega_n, n = 1, 2, \ldots$ each column of $B_n$ has at most $r$-nonzero entries. 

$k_n \in [0, n] \cap \mathbb{N}, n = 1, 2, \ldots, \lim_{n \to \infty} \frac{k_n}{n} = p \in (0, 1]$ then 

$$\liminf_{n \to \infty} \frac{\log \text{perm}_{k_n} B_n}{2n} \geq \frac{1}{2} (-p \log p - 2(1 - p) \log(1 - p)) + \frac{1}{2} (r + s - 1) \log(1 - \frac{1}{r + s}) - \frac{1}{2} (s - 1 + p) \log(1 - \frac{1 - p}{s})$$
Lower asymptotic bounds Friedland-Gurvits 2008

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**Prf** combines properties positive hyperbolic polynomials, capacity and the measure on $\mathcal{G}(r, 2n)$
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Prf combines properties positive hyperbolic polynomials, capacity and the measure on $\mathcal{G}(r, 2n)$

• Cor: $r$-ALMC holds for $p_s = \frac{r}{r+s}, s = 0, 1, \ldots$, 
Thm: $r \geq 3, s \geq 1$ integers,  
$B_n \in \Omega_n, n = 1, 2, \ldots$ each column of $B_n$ has at most $r$-nonzero entries.  
$k_n \in [0, n] \cap \mathbb{N}, n = 1, 2, \ldots, \lim_{n \to \infty} \frac{k_n}{n} = p \in (0, 1]$ then 
\[
\liminf_{n \to \infty} \frac{\log \text{perm}_{k_n} B_n}{2n} \geq \frac{1}{2} (-p \log p - 2(1 - p) \log(1 - p)) + \\
\frac{1}{2}(r + s - 1) \log(1 - \frac{1}{r + s}) - \frac{1}{2}(s - 1 + p) \log(1 - \frac{1 - p}{s})
\]
Prf combines properties positive hyperbolic polynomials, capacity and the measure on $G(r, 2n)$  
• Cor: $r$-ALMC holds for $p_s = \frac{r}{r+s}, s = 0, 1, \ldots,$  
• Con: under Thm assumptions 
\[
\liminf_{n \to \infty} \frac{\log \text{perm}_{k_n} B_n}{2n} \geq f(r, p) - \frac{p}{2} \log r
\]
Lower asymptotic bounds Friedland-Gurvits 2008

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Prf combines properties positive hyperbolic polynomials, capacity and the measure on $\mathcal{G}(r, 2n)$

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For $p_s = \frac{r}{r+s}, s = 0, 1, \ldots$, conjecture holds
Known lower and upper bounds for $p$-matchings

FKLM accepted JOSS 08:

\[
\text{low}_r(p) \geq \max(\text{low}_{r,1}(p), \text{low}_{r,2}(p))
\]
\[
\text{upp}_r(p) \leq \min(\text{upp}_{r,1}(p), \text{upp}_{r,2}(p))
\]

Lower estimates are based on F-G inequalities and Newton inequalities:

\[
f(x) = x^n + \sum_{i=1}^{n} a_i x^{n-i}
\]

have nonpositive roots then \((\frac{n}{k})^{-1} a_k\) log concave sequence

Upper estimates are based on Bregman inequalities:

\[
\phi(k, G) \leq \binom{n}{k} \frac{(r!)^{\frac{k}{n}} (n!)^{\frac{n-k}{n}}}{(n-k)!}
\]

and

\[
\max_{G \in G_{\text{mult}}(r, 2n)} \phi(k, G) = \binom{n}{k} r^k
\]
Concavity results

\[ h_d(p) + \frac{1}{2}(p \log p + (1 - p) \log(1 - p)) \]

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Prf: Newton inequalities
$r = 4$ lower bounds

Figure: $f_l(p, 4)$-red, $\text{low}_{4,1}(p)$-blue, $f(p, 4)$-green
$r = 4$ lower bounds differences

Figure: $\text{low}_{4,1}(p) - f(p, 4)$-black, $\text{low}_{4,2}(p) - f(p, 4)$-blue
$r = 4$ upper bounds

Figure: $h_{K(4)}$-green, $\text{upp}_{4,1}$-blue, $\text{upp}_{4,2}$-orange
Summary
Matches in graphs and their number is a basic concept in graphs.
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Estimating matchings in regular bipartite graphs fuses combinatorics and analysis.
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Generalizing matching concepts to infinite graphs brings in the elements of statistical physics: entropies, the grand partition function, pressure and probability.
Matches in graphs and their number is a basic concept in graphs. Counting $k$-matching in bipartite graphs is equivalent to computing permanents of $0 - 1$ matrices. Estimating matchings in regular bipartite graphs fuses combinatorics and analysis. Generalizing matching concepts to infinite graphs brings in the elements of statistical physics: entropies, the grand partition function, pressure and probability. Computation of these entropies to a good precision needs massive memory and huge computational power.
Open problems
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- Lower and upper matching conjectures for regular bipartite graphs.
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- Other lattices as Bethe lattices, i.e. infinite regular trees.
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- Closed formula or high precision values for $d \geq 3$ dimer and monomer-dimer entropies in $\mathbb{Z}^d$.
- Other lattices as Bethe lattices, i.e. infinite regular trees
- Non-bipartite graphs
References


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References


