On "The multiplicity of eigenvalues", by Peter Lax, *Bulletin of Amer. Math. Soc.* 6 (1982), 213-214.

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Let M_n be the set of $n \times n$ real matrices. $A \in \mathrm{M}_n$ is called *nondegenerate* if it has n distinct eigenvalues. Let $\mathcal{U}, \mathcal{H} \subset \mathrm{M}_n$ be subspaces. Then $\mathcal{U}$ is called *nondegenerate* if any nonzero matrix in $\mathcal{U}$ is nondegenerate. Otherwise $\mathcal{U}$ is called *degenerate*. $\mathcal{H}$ is called *hyperbolic* if any $A \in \mathcal{H}$ has only real eigenvalues. The subspace $\mathrm{S}_n \subset \mathrm{M}_n$ of symmetric matrices is hyperbolic. In what follows we assume that $\mathcal{H}$ is hyperbolic with a basis $A_1, ..., A_m$. Then the system $\frac{\partial u}{\partial t} = \sum_{i=1}^m A_i \frac{\partial u}{\partial x_i}$ is called a first order $n \times n$ hyperbolic system in m-space variables. If $\mathcal{H}$ is nondegenerate then the corresponding PDE system is strictly hyperbolic. It is easy to show the existence of two dimensional nondegenerate hyperbolic subspaces for any $n \geq 2$. (For example A_1 is any irreducible tridiagonal matrix and A_2 a diagonal matrix with pairwise distinct diagonal entries.) Strictly hyperbolic systems have simple characteristics and have nice numerical schemes. In his seminal paper Lax showed that any hyperbolic subspace of M_n of dimension $m \geq 3$ is degenerated for $n \equiv 2 \pmod 4$.

Friedland, Robbin and Sylvester [5] characterized the degenerate hyperbolic subspaces for any $n \geq 2$ as follows. Write $n = (2a+1)2^{c+4d}$ where a, c, d are nonnegative integers with $c \in \{0, 1, 2, 3\}$. Let $\rho(n) = 2^c + 8d$ be the Radon-Hurwitz number. Let $\sigma(n) = 2$ for $n \not\equiv 0, \pm 1 \pmod 8$ and $\sigma(n) = \rho(4b)$ for $n = 8b, 8b \pm 1$. Then any hyperbolic subspace $\mathcal{H} \subset \mathrm{M}_n$ of dimension $\sigma(n) + 1$ is degenerate. Moreover there exist a nondegenerate hyperbolic subspace $\mathcal{H} \subset \mathrm{S}_n$ of dimension $\sigma(n)$. Similar results are shown in [5] for real degenerate subspaces of $n \times n$ complex valued matrices.

The main idea of the proof of these results are as follows. The n eigenspaces of a nondegenerate A induce a decomposition of $\mathbf{C}^n$ to a direct sum of n lines. If in addition A has real eigenvalues then one has the corresponding decomposition of $\mathbf{R}^n$. Let $\mathrm{S}^k \subset \mathbf{R}^{k+1}$ be the standard unit sphere. Then one has the continuous map $\Phi : \mathrm{S}^{m-1} \to \mathrm{M}_n$ given by $\mathbf{x} = (x_1, ..., x_m) \mapsto \sum_{i=1}^m x_i A_i$. Assume that $\mathcal{H}$ is nondegenerate. Then one has a trivial $\mathbf{R}^n$ bundle on S^{m-1} which is a sum of n lines bundles. Using the identity $\Phi(-\mathbf{x}) = -\Phi(\mathbf{x})$ one shows that the existence of p copies of the canonical line bundle on the real projective space of dimension $m-1$, whose Whitney sum is trivial, for some $p \in \{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil\}$. I.e. there exists a continuous odd map $\Psi : \mathrm{S}^{m-1} \to \mathrm{GL}(p, \mathbf{R}), \Psi(-\mathbf{x}) = -\Psi(\mathbf{x})$. The fundamental theorem of Adams [1] yields the results in [5].

Let $A \in \mathcal{H}$ and arrange the eigenvalues of A in a decreasing order $\lambda_1(A) \geq ... \geq \lambda_n(A)$. If A is degenerate then $\lambda_i(A) = \lambda_{i+1}(A)$ for some i, and we say that $\lambda_i(A)$ is a multiple eigenvalue. Assume that $\mathcal{H}$ is hyperbolic of dimension $\sigma(n) + 1$ at least. Then $\mathcal{H}$ contains a nonzero A with a multiple $\lambda_i(A)$. Which i one should expect? From the proof of the Corollary in §5 of [5] it follows that for an even n any $\rho(n) + 1$ dimensional hyperbolic subspace $\mathcal{H} \subset \mathrm{M}_n$ contains a nonzero matrix A such that $\lambda_{\frac{n}{2}}(A) = \lambda_{\frac{n}{2}+1}(A)$. If n is even and $\mathcal{H} \subset \mathrm{S}_n$ then there exists $0 \neq A \in \mathcal{H}$ with $\lambda_{\frac{n}{2}}(A) = \lambda_{\frac{n}{2}+1}(A)$ if $\dim \mathcal{H} \geq \rho(\frac{n}{2}) + 2$. The proof of this result uses a nonlinear generalization of theorem of Adams-Lax-Philips [2]. Recall that for $n > 2$ any n-dimensional subspace of S_n contains a nonzero matrix with a multiple first eigenvalue ($i = 1$) and this result is sharp [4].

We conclude with another generalization of Lax's theorem due to Falikman, Friedland and Loewy [3]. Let $q \in [2, n]$ be an integer and $n \equiv \pm q \pmod{2^{\lceil \log_2 2q \rceil}}$. Then any $\binom{q+1}{2}$ dimensional subspace of S_n contains a nonzero matrix A with $\lambda_i(A) = ... = \lambda_{i+q-1}(A)$ for some $i \in [1, n-q+1]$ and this result is sharp. The case $q = 2$ is Lax's theorem.

References

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