Linear Algebra I - Lectures Notes - Spring 2013

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Preface: Linear Algebra is used in many areas as: engineering, biology, medicine, business, statistics, physics, mathematics, numerical analysis and humanities. The **reason**: many real world systems consist of many parts which interact linearly. Analysis of such systems involves the notions and the tools from Linear Algebra. These notes of linear algebra course emphasize the mathematical rigour over the applications, contrary to many books on linear algebra for engineers. My main goal in writing these notes was to give to the student a concise overview of the main concepts,ideas and results that usually are covered in the first course on linear algebra for mathematicians. These notes should be viewed as a supplementary notes to a regular book for linear algebra, as for example [1].

Main Topics of the Course

- SYSTEMS OF EQUATIONS
- VECTOR SPACES
- LINEAR TRANSFORMATIONS
- DETERMINANTS
- INNER PRODUCT SPACES
- EIGENVALUES
- JORDAN CANONICAL FORM-RUDIMENTS

Text: Jim Hefferon, *Linear Algebra*, and *Solutions*
Available for free download
ftp://joshua.smcvt.edu/pub/hefferon/book/book.pdf
ftp://joshua.smcvt.edu/pub/hefferon/book/jhanswer.pdf
Software: MatLab,Maple, Matematica

Contents

1 Linear equations and matrices

The object of this section is to study a set of m linear equations in n real variables $x_1, \ldots, x_n$. It is convenient to group n variable in one quantity: $(x_1, x_2, \ldots, x_n)$, which is called a *row vector*. For reasons that will be seen later we will consider a *column* vector denoted by $\mathbf{x}$, for which we either the round brackets ( ) or the straight brackets [ ]:

$$\mathbf{x} := (x_1, x_2, \ldots, x_n)^\top = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}. \tag{1.1}$$

We will denote by $\mathbf{x}^\top$ the row vector $(x_1, x_2, \ldots, x_n)$. We denote the set of all column vectors $\mathbf{x}$ by $\mathbb{R}^n$. So $\mathbb{R}^1 = \mathbb{R}$ all the points on the real line; $\mathbb{R}^2$ are all points in the plane; $\mathbb{R}^3$ all points in 3-dimensional space. $\mathbb{R}^n$ is called *n-dimensional* space. It is hard to visualize $\mathbb{R}^n$ for $n \geq 4$, but we ca study it efficiently using mathematical tools.

We will learn how to determine when a given system is linear equations in $\mathbb{R}^n$ is unsolvable or solvable, when the solution is unique or not unique, and how to express compactly all the solutions of the given system.

1.1 System of Linear Equation

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + & ... & + & a_{1n}x_n = b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & ... & + & a_{2n}x_n = b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & ... & + & a_{mn}x_n = b_m \end{array} \tag{1.2}$$

1.1.1 Examples

A trivial equation in n variables is

$$0x_1 + 0x_2 + \ldots + 0x_n = 0. \tag{1.3}$$

Clearly, every vector $\mathbf{x} \in \mathbb{R}^n$ satisfies this equation. Sometimes, when we see such an equation in our system of equations, we delete this equations from the given system. This will not change the form of the solutions.

The following system of n equations in n unknowns

$$x_1 = b_1,\ x_2 = b_2, \quad x_n = b_n. \tag{1.4}$$

clearly has a unique solution $\mathbf{x} = (b_1, \ldots, b_n)^\top$.

Next consider the equation

$$0x_1 + 0x_2 + \ldots + 0x_n = 1. \tag{1.5}$$

Clearly, this equation does not have any solution, i.e. *none*.

Consider m linear equations in $n = 2$ unknowns we assume that none of this equations is a trivial equation, i.e. of the form (1.3) for $n = 2$. Assume first that $m = 1$, i.e. we have one equation in two variables. Then the set of solutions is a line in $\mathbb{R}^2$. So the number of solutions is infinite, *many*, and can be parametrized by one real parameter.

Suppose next that $m = 2$. Then if the two lines are not parallel the system of two equations in two unknowns has a unique solution. This is the *generic* case, i.e. if we let the coefficients $a_{11}, a_{12}, a_{21}, a_{22}$ to be chosen at random by computer. (The values of b_1, b_2 are not relevant.) If the two lines are parallel but not the identical we do not have a solution. If the two lines are identical, the system of solutions is a line.

If $m \geq 3$, then in general, (*generically*), we would not have a solution, since usually three pairwise distinct lines would not intersect in one point. However, if all the lines chosen were passing through a given point then this system is solvable.

For more specific examples see [1, Chapter One, Section I]

1.1.2 Equivalent systems of equations

Suppose we have two systems of linear equation in the same number of variables, say n, but perhaps a different numbers of equations say m and m' respectively. Then these two systems are called *equivalent* if the two systems have the same set of solutions. I.e. a vector $\mathbf{x} \in \mathbb{R}^n$ is a solution to one system if and only if it is a solution to the second system.

Our solution of a given system boils down to find an equivalent system for which we can easy determine if the system is solvable or not, and if solvable we can describe easily the set of all solutions of this system.

A trivial example of two equivalent systems is as follows.

Example 1.1 *Consider the system* (1.2). *Then the following new system is equivalent to* (1.2)*:*

1. *Add to the system* (1.2) *a finite number of trivial equations of the form* (1.3).

2. *Assume that the system* (1.2) *has a finite number of trivial equations of the form* (1.3). *Delete some of these trivial equations*

The main result of this Chapter is that two systems of linear equations are equivalent if and and only if each of the system is equivalent to another system, where the final two systems are related by Example 1.1.

1.1.3 Elementary operations

Definition 1.2 *The following three operations on the given system of linear equations are called elementary.*

1. *Change the order of the equations.*

2. *Multiply an equation by a nonzero number.*

3. Add (subtract) from one equation a multiple of another equation.

Lemma 1.3 *Suppose that a system of linear equations, named system II, was obtained from a system of linear equations, named system I, by using a sequence of elementary operations. Then we can perform a sequence of elementary operations on system II to obtain system I. In particular, the systems I and II are equivalent.*

Proof. It is enough to show the lemma when perform. We now observe that all elementary operations are *reversible*. Namely performing one elementary operation and then the second one of the same type will give back the original system: change twice the order of the equations i and j;, multiply equation i by $a \neq 0$ and then multiply equation i by $\frac{1}{a}$; add equation j times a to equation $i \neq j$ and subtract equation j times a to equation $i \neq j$.

Hence if $\mathbf{x}$ satisfies system I it satisfies system II and vice versa. $\square$

1.1.4 Triangular systems

An example

$$\begin{array}{rcrl} x_1 & + & 2x_2 & = 5 \\ 2x_1 & + & 3x_2 & = 8 \end{array}$$

Subtract 2 times row 1 from row 2. Hefferon notation: $\rho_2 - 2\rho_1 \to \rho_2$. My notations: $R_2 - 2R_1 \to R_2$, or $R_2 \leftarrow R_2 - 2R_1$, or $R_2 \to R_2 - 2R_1$. Obtain a new system

$$\begin{array}{rcrl} x_1 & + & 2x_2 & = 5 \\ & - & x_2 & = -2 \end{array}$$

Find first the solution of the second equation: $x_2 = 2$. Substitute x_2 to the first equation: $x_1 + 2 \times 2 = 5 \Rightarrow x_1 = 5 - 4 = 1$. Unique solution $(x_1, x_2) = (1, 2)$.

A general triangular system of linear equations is the following system of n equations in n unknowns:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & ... & + & a_{1n}x_n & = b_1 \\ & + & a_{22}x_2 & + & ... & + & a_{2n}x_n & = b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ & & & & ... & & a_{nn}x_n & = b_n \end{array} \tag{1.6}$$

with n pivots: $a_{11} \neq 0, \ a_{22} \neq 0, \ldots a_{nn} \neq 0$.

Solve the system by back substitution from down to up:

$$\begin{aligned} & x_n = \frac{b_n}{a_{nn}}, \\ & x_{n-1} = \frac{-a_{(n-1)n}x_n + b_{n-1}}{a_{(n-1)(n-1)}}, \\ & x_i = \frac{-a_{i(i+1)}x_{i+1} - ... - a_{in}x_n + b_i}{a_{ii}}, \\ & i = n-2, ..., 1. \end{aligned} \tag{1.7}$$

1.2 Matrix formalism for solving systems of linear equations

In this subsection we introduce matrix formalism which allows us to find an equivalent form of the system (1.2) from which we can find if the system is solvable or not, and if solvable, to find a compact way to describe the sets of its solution. The main advantage of this notation is that it does not uses the variables $x_1, \ldots, x_n$ and easily adopted for programming on a computer.

1.2.1 The Coefficient Matrix of the system

Consider the system (1.2). The information given in the left-hand side of this system can be neatly written in terms on $m \times n$ *coefficient matrix*

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \tag{1.8}$$

We denote the set of all $m \times n$ matrices with real entries as $\mathbb{R}^{m\times n}$. Note that the right-hand side of (1.2) is given by a column vector $\mathbf{b} := (b_1, b_2, \ldots, b_m)^\top$. Hence the matrix that describes completely the system (1.2) is called the *augmented matrix* and denoted by $[A|\mathbf{b}]$, and sometimes $(A|\mathbf{b})$.

$$[A|\mathbf{b}] = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & | & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & | & b_2 \\ \vdots & \vdots & \vdots & \vdots & | & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & | & b_m \end{bmatrix} \tag{1.9}$$

One can view an augmented matrix $[A|\mathbf{b}]$ as an $m \times (n+1)$ matrix C. Sometimes we will use the notation A for any $m \times p$ matrix. So in this context A can be either a coefficient matrix or an augmented matrix, and no ambiguity should arise.

1.2.2 Elementary row operations

It is easy to see that elementary operations on a system of linear equations discussed in §1.1.3 are equivalent to the following *elementary row operations* on the augmented matrix corresponding to the system (1.2).

Definition 1.4 *(Elementary Row Operations-ERO) Let C be given $m \times p$ matrix. Then the following three operations are called ERO and denoted as follows.*

1. *Interchange the rows i and j, where $i \neq j$*

$$R_i \longleftrightarrow R_j, \quad (\rho_i \longleftrightarrow \rho_j).$$

2. Multiply the row i by a nonzero number $a \neq 0$

$$a \times R_i \longrightarrow R_i, \quad (R_i \longrightarrow a \times R_i, \quad a\rho_i \longrightarrow \rho_j).$$

3. Replace the row i by its sum with a multiple the row $j \neq i$

$$R_i + a \times R_j \longrightarrow R_i, \quad (R_i \longrightarrow R_i + a \times R_j, \quad \rho_i + a\rho_j \longrightarrow \rho_i).$$

The following lemma is the analog of Lemma 1.3 and its proof is similar.

Lemma 1.5 *The elementary row operations are reversible. More precisely*

1. *$R_i \longleftrightarrow R_j$ is the inverse of $R_i \longleftrightarrow R_j$,*
2. *$\frac{1}{a} \times R_i \longrightarrow R_i$ is the inverse of $a \times R_i \longrightarrow R_i$*
3. *$R_i - a \times R_j \longrightarrow R_i$ is the inverse of $R_i + a \times R_j \longrightarrow R_i$.*

1.2.3 Row Echelon Form

Definition 1.6 *A matrix C is in a row echelon form (REF) if it satisfies the following conditions*

1. *The first nonzero entry in each row is 1. This entry is called a pivot.*
2. *If row k does not consists entirely of zeros, then the number of leading zero entries in row $k+1$ is greater than the number of leading zeros in row k.*
3. *Zero rows appear below the rows having nonzero entries.*

Lemma 1.7 *Every matrix can be brought to a REF using ERO.*

Constructive proof of existence of REF. Let $C = [c_{ij}]_{i=j=1}^{m,p}$.

When we modify the entries of C by ERO we rename call this matrix C!

0. Set $\ell_1 = 1$ and $i = 1$.
1. If $c_{i\ell_i} = 0$ GOTO 3.
2. Divide row i by $c_{i\ell_i}$: $\frac{1}{c_{i\ell_i}} R_i \to R_i$, (Note: $c_{i\ell_i} = 1$.)
a. Subtract $c_{j\ell_i}$ times row i from row $j > i$: $-c_{j\ell_i} R_i + R_j \to R_j$ for $j = i+1, \dots, p$.
c. If $\ell_i = p$ or $i = m$ GOTO END
d. Set $i = i+1$, i.e. $i + 1 \to i$ and $\ell_i = \ell_{i-1} + 1$.
f. GOTO 1.
3. If $c_{i\ell_i} = \ldots = c_{(k-1)\ell_i} = 0$, and $c_{k\ell_i} \neq 0$ for some k, $i < k \leq m$: $R_i \leftrightarrow R_k$ GOTO 2.
4. $\ell_i + 1 \to \ell_i$. GOTO 1.

END

□

The steps in the constructive proof of Lemma 1.7 is called *Gauss Elimination*

Here are a few examples of REF.

$$\begin{bmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.10}$$

Note the REF of a matrix is not unique in general. For example by using elementary row operation of the form $R_i - t_{ij}R_j$ for $1 \le i < j$ one can always bring the above matrix in the row echelon form to the following matrix in a row echelon form.

$$\begin{bmatrix} 1 & 0 & f & 0 \\ 0 & 1 & g & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{1.11}$$

$$\begin{bmatrix} 0 & 1 & a & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.12}$$

Five possible REF of $(a\ b\ c\ d)$ $(1 \times 4$ matrix):

$$\begin{aligned} &(1\ u\ v\ w) \quad \texttt{if}\ \ a \neq 0, \\ &(0\ 1\ p\ q) \quad \texttt{if}\ \ a = 0,\ b \neq 0, \\ &(0\ 0\ 1\ r) \quad \texttt{if}\ \ a = b = 0,\ c \neq 0, \\ &(0\ 0\ 0\ 1) \quad \texttt{if}\ \ a = b = c = 0,\ d \neq 0, \\ &(0\ 0\ 0\ 0) \quad \texttt{if}\ \ a = b = c = d = 0. \end{aligned}$$

Definition 1.8 *Let $U = [u_{ij}]$ be an $m \times p$ matrix in a REF. Then the number of pivots is called the rank of U, and denoted by* $\operatorname{rank} U$.

Lemma 1.9 *Let $U = [u_{ij}]$ be an $m \times p$ matrix in a REF. Then*

1. $\operatorname{rank} U = 0$ *if and only if* $U = 0$.
2. $\operatorname{rank} U \le \min(m, p)$.
3. *If $m > \operatorname{rank} U$ then the last $m - \operatorname{rank} U$ rows of U are zero rows.*

Proof. Clearly, U does not have pivots if and only if $U = 0$. There are no two pivots on the same row or column. Hence $\operatorname{rank} U \le m$ and $\operatorname{rank} U \le p$.

Assume that $r := \operatorname{rank} U \ge 1$. Then the pivot number j is located on the row j in the column ℓ_j. So

$$1 \le \ell_1 < \ldots < \ell_r \le p \text{ is the column location of pivots, } r = \operatorname{rank} U. \tag{1.13}$$

Hence the last $m - \operatorname{rank} U$ rows of U are zero rows. □

Lemma 1.10 *Let B be an $m \times p$ matrix and assume that B can be brought to a row echelon matrix U. Then* $\operatorname{rank} U$ *and the location of the pivots in* (1.13) *do not depend on a particular choice of U.*

Proof. We prove the lemma by induction on p where m is fixed. For $p = 1$ we have two choices. If $A = 0$ then $U = 0$ and $r = 0$. If $A \neq 0$ then $U = (\underbrace{1, 0, \ldots, 0}_{m})^\top, r = 1, \ell_1 = 1$. So the lemma hods for $p = 1$. Suppose that the lemma holds for $p = q$ and let $p = q + 1$. So $B = [C|\mathbf{c}]$ where B is $m \times p$ matrix and $\mathbf{c} \in \mathbb{R}^m$. Let $U = [V|\mathbf{v}]$ be a REF of B. Hence V is a row echelon form of C. By the induction assumption, the number of pivots r' of V and their column location depend only on C, hence on B. If the last $m - r'$ rows of U are zero, i.e. $\operatorname{rank} U = \operatorname{rank} V = r'$ then U has r' pivots located in the columns $\{1, \ldots, p-1\}$ and the induction hypothesis implies that the location of the pivots and their number depends only on B. Otherwise C must have an additional pivot on the column p located at the row $r' + 1 = \operatorname{rank} U$. Again the number of the pivots of U and their location depends only on C. $\square$

Definition 1.11 *Let B be an $m \times p$ matrix. Then the rank of B, denoted by* $\operatorname{rank} B$*, is the number of pivots of a REF of B.*

1.2.4 Solution of linear systems

Definition 1.12 *Let $\hat{A} := [A|\mathbf{b}]$ be the augmented $m \times (n + 1)$ matrix representing the system* (1.2)*. Suppose that $\hat{C} = [C|\mathbf{c}]$ is a REF of $\hat{A}$. Assume that C has k-pivots in the columns $1 \leq \ell_1 < \ldots < \ell_k \leq n$. Then the variable $x_{\ell_1}, \ldots, x_{\ell_k}$ corresponding to these pivots are called the lead variables. The other variables are called free variables.*

Recall that $f : \mathbb{R}^n \to \mathbb{R}$ is called an affine function if $f(\mathbf{x}) = a_1x_1 + \ldots + a_nx_n + b$. f is called a *linear function* if $b = 0$. The following theorem describes exactly the set of all solutions of (1.2).

Theorem 1.13 *Let $\hat{A} := [A|\mathbf{b}]$ be the augmented $m \times (n + 1)$ matrix representing the system* (1.2)*. Suppose that $\hat{C} = [C|\mathbf{c}]$ be a REF of $\hat{A}$. Then the system* (1.2) *is solvable if and only if $\hat{C}$ does not have a pivot in the last column $n + 1$.*

Assume that (1.2) *is solvable. Then each lead variable is a unique affine function in free variables. These affine functions can be determined as follows.*

1. *Consider the linear system corresponding to $\hat{C}$. Move all the free variables to the right-hand side of the system. Then one obtains a triangular system in lead variables, where the right-had side are affine functions in free variables.*

2. *Solve this triangular system by back substitution.*

In particular, for the solvable system we have the following alternative.

1. *The system has a unique solution if and only if there are no free variables.*
2. *The system has many solutions, (infinite number), if and only if there is at least one free variable.*

Proof. We consider the linear system equations corresponding to $\hat{C}$. As ERO on $\hat{A}$ correspond to EO on the system (1.2) it follows that the system represented by $\hat{C}$ is equivalent to (1.2). Suppose first that $\hat{C}$ has a pivot in the last column. So the corresponding row of $\hat{C}$ which contains the pivot on the column $n+1$ is $(0, 1 \ldots, 0, 1)^\top$. The corresponding linear equation is of the form (1.5). This equation is unsolvable, hence the whole system corresponding to $\hat{C}$ is unsolvable. Therefore the system (1.2) is unsolvable.

Assume now that $\hat{C}$ does not have a pivot in the last column. Move all the free variables to the right-hand side of the system given by $\hat{C}$. It is a triangular system in the lead variables where the right-hand side of each equation is an affine function in the free variables. Now use back substitution to express each lead variable as an affine function of the free variables.

Each solution of the system is determined by the value of the free variables which can be chosen arbitrary. Hence, the system has a unique solution if and only if it has no free variables. The system has many solutions if and only if it has at least one free variable. $\square$

Consider the following example of $\hat{C}$:

$$\begin{bmatrix} 1 & -2 & 3 & -1 & | & 0 \\ 0 & 1 & 3 & 1 & | & 4 \\ 0 & 0 & 0 & 1 & | & 5 \end{bmatrix} \tag{1.14}$$

x_1, x_2, x_4 are lead variables, x_3 is a free variable.

$$\begin{aligned}
&x_4 = 5, \\
&x_2 + 3x_3 + x_4 = 4 \Rightarrow x_2 = -3x_3 - x_4 + 4 \Rightarrow \\
&x_2 = -3x_3 - 1, \\
&x_1 - 2x_2 + 3x_3 + -x_4 = 0 \Rightarrow x_1 = 2x_2 - 3x_3 + x_4 = 2(-3x_3 - 1) - 3x_3 + 5 \Rightarrow \\
&x_1 = -9x_3 + 3.
\end{aligned}$$

1.2.5 Reduced row echelon form

Among all row echelon forms U of a given matrix C there is one special REF which is called *reduced row echelon form* denoted by RREF.

Definition 1.14 *Let U be a matrix in a row echelon form. Then U is an RREF if 1 is a pivot on the column k of U then all other elements on the column k of U are zero.*

Here are two examples of 3×4 matrices in RREF

$$\begin{bmatrix} 1 & 0 & b & 0 \\ 0 & 1 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Bringing a matrix to RREF is called *Gauss-Jordan reduction*.
Here is a constructive algorithm to find a RREF of C.

Gauss-Jordan algorithm for RREF.

Let $C = [c_{ij}]_{i=j=1}^{m,p}$.
When we modify the entries of C by ERO we rename call this matrix C!

0. Set $\ell_1 = 1$ and $i = 1$.
1. If $c_{i\ell_i} = 0$ GOTO 3.
2. Divide row i by $c_{i\ell_i}$: $\frac{1}{c_{i\ell_i}} R_i \to R_i$, (Note: $c_{i\ell_i} = 1$.)
a. Subtract $c_{j\ell_i}$ times row i from row $j \neq i$: $-c_{j\ell_i} R_i + R_j \to R_j$ for $j = 1, \ldots, i-1, i+1, \ldots, p$.
c. If $\ell_i = p$ or $i = m$ GOTO END
d. Set $i = i + 1$, i.e. $i + 1 \to i$ and $\ell_i = \ell_{i-1} + 1$.
f. GOTO 1.
3. If $c_{i\ell_i} = \ldots = c_{(k-1)\ell_i} = 0$, and $c_{k\ell_i} \neq 0$ for some k, $i < k \leq m$: $R_i \leftrightarrow R_k$ GOTO 2.
4. $\ell_i + 1 \to \ell_i$. GOTO 1.
END

The advantage in bringing the augmented matrix $\hat{A} = [A|\mathbf{b}]$ to RREF $\hat{C}$ is that if (1.2) is solvable then its solution is given quite straightforward using $\hat{C}$. We need to use the following notation.

Notation 1.15 *Let S and T be a subsets of a set X. Then the set $T \setminus S$ is the set of elements of T which are not S. ($T \setminus S$ may be empty set.)*

Theorem 1.16 *Let $\hat{A} := [A|\mathbf{b}]$ be the augmented $m \times (n+1)$ matrix representing the system* (1.2). *Suppose that $\hat{C} = [C|\mathbf{c}]$ be a RREF of $\hat{A}$. Then the system* (1.2) *is solvable if and only if $\hat{C}$ does not have a pivot in the last column $n+1$.*

Assume that (1.2) *is solvable. Then each lead variable is a unique affine function in free variables determined as follows. The leading variable x_{ℓ_i} appears only in the equation i, for $1 \leq i \leq r = rank A$. Shift all other variables in the equation, (which are free variables), to the other side of equation to obtain x_{ℓ_i} as an affine function in free variables.*

Proof. Since RREF is a row echelon form, Theorem 1.13 yields that (1.2) is solvable if and only if $\hat{C}$ does not have a pivot in the last column $n+1$.

Assume that $\hat{C}$ does not have a pivot in the last column $n+1$. So all the pivots of $\hat{C}$ appear in C. Hence $rank\hat{A} = \operatorname{rank}\hat{C} = \operatorname{rank}C = \operatorname{rank}A(=r)$. The pivots of $C = [c_{ij}] \in \mathbb{R}^{m\times n}$ are located at row i and the column ℓ_i, denote as (i, ℓ_i), for $i = 1, \ldots, r$. Since C is also in RREF, in the column ℓ_i there is only one nonzero element which is equal 1 and is located in the row i.

Consider the system of linear equations corresponding to $\hat{C}$, which is equivalent to (1.2). Hence the lead variable ℓ_i appears only in the $i-th$ equation. Left hand-side of this equation is of the form x_{ℓ_i} plus an linear function in free variables whose indices are greater than x_{ℓ_i}. The right-hand is c_i, where $\mathbf{c} = (c_1, \ldots, c_m)^\top$. Hence by moving the free variables to the right-hand side we obtain the exact form of x_{ℓ_i}.

$$x_{\ell_i} = c_i - \sum_{j\in\{\ell_i,\ell_i+1,\ldots,m\}\backslash\{j_{\ell_1},j_{\ell_2},\ldots,j_{\ell_r}\}} c_{ij}x_j, \text{ for } i = 1, \ldots, r. \tag{1.15}$$

(So $\{\ell_i, \ell_i + 1, \ldots, m\}$ consist of $m - \ell_i + 1$ integers from j_{ℓ_i} to m, while $\{j_{\ell_1}, j_{\ell_2}, \ldots, j_{\ell_r}\}$ consist of the columns of the pivots in C.) $\square$

Example

$$\begin{bmatrix} 1 & 0 & b & 0 & | & u \\ 0 & 1 & d & 0 & | & v \\ 0 & 0 & 0 & 1 & | & w \end{bmatrix}$$

x_1, x_2, x_4 lead variables x_3 free variable

$$\begin{aligned} &x_1 + bx_3 = u \Rightarrow x_1 = -bx_3 + u, \\ &x_2 + dx_3 = v \Rightarrow x_2 = -dx_3 + v, \\ &x_4 = w. \end{aligned}$$

Definition 1.17 *The system* (1.2) *is called homogeneous if* $\mathbf{b} = \mathbf{0}$*, i.e.* $b_1 = \ldots = b_m = 0$*. A homogeneous system of linear equations has a solution* $\mathbf{x} = \mathbf{0}$*, which is called a trivial solution.*

Let A be a square $n \times n$. A is called nonsingular if the corresponding homogeneous system of n equations in n unknowns has only solution $\mathbf{x} = \mathbf{0}$*. Otherwise A is called singular.*

Theorem 1.18 *Let A be an $m \times n$ matrix. Then it RREF is unique.*

Proof. Let U be a RRREF of A. Consider the augmented matrix $\hat{A} := [A|\mathbf{0}]$ corresponding to the homogeneous system of equations. Clearly $\hat{U} = [U|\mathbf{0}]$ is a RREF of $\hat{A}$. Put the free variables on the other side of the homogeneous system corresponding to $\hat{U}$ to find the solution of the homogeneous system corresponding to $\hat{A}$, where each lead variable is a linear function of the free variables. Note that the exact formulas for the lead variables determine uniquely the columns of the RREF which correspond to the free variables.

Assume that U_1 is another RREF of A. Lemma 1.10 yields that U and U_1 have the same pivots. Hence U and U_1 have the same columns which correspond to pivots. By considering the homogeneous system of linear equations corresponding to $\hat{U}_1 = [U_1|\mathbf{0}]$ we find also the solution of the homogeneous system $\hat{A}$, by writing down the lead variables as linear functions in free variables. Since $\hat{U}$ and $\hat{U}_1$ give rise to the same lead and free variables, we deduce that the each linear function in free variables corresponding to a lead variable x_{ℓ_i} corresponding to $\hat{U}$ and $\hat{U}_1$ are equal. That is the matrices U and U_1 have the same row i for $i = 1, \ldots, \operatorname{rank} A$. All other rows of U and U_1 are zero rows. Hence $U = U_1$. □

Corollary 1.19 *$A \in \mathbb{R}^{n\times n}$ is nonsingular if and only if $\operatorname{rank} A = n$, i.e. the RREF of A is the identity matrix*

$$I_n = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & \ldots & 0 & 1 \end{bmatrix} \in \mathbb{R}^{n\times n}. \tag{1.16}$$

Proof. A is nonsingular if and only if no free variables. So $\operatorname{rank} A = n$ and the RREF is I_n. □

1.3 Operations on vectors and matrices

1.3.1 Operations on vectors

Vectors: Row Vector $(x_1, x_2, ..., x_n)$ is $1 \times n$ matrix. Column Vector $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$ is $m \times 1$ matrix. For convenience of notation we denote column vector $\mathbf{u}$ as $\mathbf{u} = (u_1, u_2, \ldots, u_m)^\top$.

The coordinates of a vector and real numbers are called scalars. In these notes scalars are denoted by small Latin letters, while vector are in a different font: $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors, while $a, b, c, d, x, y, z, u, v, w$ are scalars.

The rules for multiplications of vector by scalars and additions of vectors are:

$$\begin{aligned} & a(x_1, ..., x_n) := (ax_1, ..., ax_n), \\ & (x_1, ..., x_n) + (y_1, ..., y_n) := \\ & (x_1 + y_1, ..., x_n + y_n). \end{aligned}$$

The set of all vectors with n coordinates is denoted by $\mathbb{R}^n$, usually we will view the vectors in $\mathbb{R}^n$ as column vectors. The operations with column vectors are similar as with the row vectors.

$$a\mathbf{u} = a \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix},$$

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} := \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_m + v_m \end{pmatrix}$$

The zero vector $\mathbf{0}$ has all its coordinate 0. $-\mathbf{x} := (-1)\mathbf{x} := (-x_1, ..., -x_n)$, $\mathbf{x} + (-\mathbf{x}) = \mathbf{x} - \mathbf{x} = \mathbf{0}$.

1.3.2 Application to solutions of linear systems

Theorem 1.20 *Consider the system* (1.2) *of m equations with n unknowns. Then the following hold.*

1. *Assume that* (1.2) *is homogeneous, i.e.* $\mathbf{b} = \mathbf{0}$. *If* $\mathbf{x}$ *and* $\mathbf{y}$ *are solutions of a homogeneous system then* $s\mathbf{x}, \mathbf{x}+\mathbf{y}, s\mathbf{x}+t\mathbf{y}$ *are solutions of this homogeneous system for any scalars s, t. Suppose furthermore that $n > m$. Then this homogeneous system has a nontrivial solution, i.e. a solution* $\mathbf{x} \neq \mathbf{0}$.

2. *Assume that* (1.2) *is solvable. Let* $\mathbf{x}_0$ *is a solution of nonhomogeneous* (1.2). *Then all solutions of* (1.2) *are of the form* $\mathbf{x}_0 + \mathbf{x}$ *where* $\mathbf{x}$ *is the general solution of the homogeneous system corresponding to* (1.2) *with* $\mathbf{b} = \mathbf{0}$.

   In particular the general solution of (1.2) *obtained by reducing $\hat{A}$ to a REF is always of the following form*

   $$\mathbf{x}_0 + \sum_{j=1}^{n-\operatorname{rank} A} t_j \mathbf{c}_j. \tag{1.17}$$

   $q := n - \operatorname{rank} A$ *is the number of free variables. (So $q \geq 0$.) Here* $\mathbf{x}_0$ *is obtained by letting all free variables to be zero, (if $q > 0$). Rename the free variables of* (1.2) *as $t_1, \ldots, t_q$. Then* $\mathbf{c}_j$ *is the solution of* (1.2) *obtained by solving the homogeneous system corresponding to* $[A|\mathbf{0}]$, *by setting the free variable* t_j *to be* 1 *and all other free variables be* 0 *for* $j = 1, \ldots, q$.

Hence (1.2) *has a unique solution if and only if* $q = 0$, *which implies that* $m \le n$.

Proof. Consider the homogeneous system corresponding to $[A|\mathbf{0}]$. Assume that $\mathbf{x}, \mathbf{y}$ are solutions. Then it is straightforward to see that $s\mathbf{x}$ and $\mathbf{x} + \mathbf{y}$ are also solutions. Hence $s\mathbf{x} + t\mathbf{y}$ is also a solution.

Suppose now that (1.2) is solvable and $\mathbf{x}_0$ is it solution. Assume that $\mathbf{y}$ is a solution of (1.2). Then it is straightforward to show that $\mathbf{x} := \mathbf{y} - \mathbf{x}_0$ is a solution to homogeneous system corresponding to $[A|\mathbf{0}]$. Vice versa if $\mathbf{x}$ is a solution to homogeneous system corresponding to $[A|\mathbf{0}]$ then it is straightforward to see that $\mathbf{x}_0 + \mathbf{x}$ is a solution to (1.2).

To find a general solution of (1.2) bring the augmented matrix $[A\mathbf{b}]$ to a REF $[U|\mathbf{d}]$. The assumption that the system is solvable means that d does not have a pivot. Set all free variables to be 0. Solve the triangular system in lead variables to obtain the solution $\mathbf{x}_0$. The general solution of the homogeneous system corresponding to $[A|\mathbf{0}]$ is the general solution of the homogeneous system corresponding to $[U|\mathbf{0}]$. Now find the general solution by shifting all free variables to the right hand-side and solve the lead variables as linear functions in free variables $t_1, \ldots, t_q$. This solution is of the form $\sum_{j=1}^{q} t_j \mathbf{c}_j$. It is straightforward to see that $\mathbf{c}_j$ can be obtained by finding the unique values of lead variables when we let $t_j = 1$ and all other free variables are set to 0.

Finally, $\mathbf{x}_0$ is a unique solution if and only if there are no free variables. So we must have that $m \ge n$. $\square$

1.3.3 Products of matrices with vectors

Recall the definition of the scalar product in $\mathbb{R}^3$:

$$(u_1, u_2, u_3) \cdot (x_1, x_2, x_3) = u_1x_1 + u_2x_2 + u_3x_3.$$

We now define the product of a row vector with a column vector with the same number of coordinates:

$$\mathbf{u}^\top \mathbf{x} = (u_1 \; u_2 ... u_n) \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = u_1x_1 + u_2x_2 + ... + u_nx_n.$$

Next we define the product of $m \times n$ A and column vector $\mathbf{x} \in \mathbb{R}^n$:

$$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} =$$

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{nn}x_n \end{bmatrix} \in \mathbb{R}^m.$$

The system of m equations in n unknowns (1.2)

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n = b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n = b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n = b_m \end{array}$$

can be compactly written as

$$A\mathbf{x} = \mathbf{b}. \tag{1.18}$$

A is an $m \times n$ coefficient matrix, $\mathbf{x} \in \mathbb{R}^n$ is the columns vector of unknowns and $\mathbf{b} \in \mathbb{R}^m$ is the given column vector.

It is straightforward to see that for $A \in \mathbb{R}^{m\times n}, \mathbf{x}, \mathbf{y} \in \mathbb{R}^n, s, t \in \mathbb{R}$ we have

$$A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}, \tag{1.19}$$

$$A(s\mathbf{x}) = s(A\mathbf{x}), \tag{1.20}$$

$$A(s\mathbf{x} + t\mathbf{y}) = s(A\mathbf{x}) + t(A\mathbf{y}). \tag{1.21}$$

With these notations it is straightforward to see that any solution of (1.18) is of the form $\mathbf{x} = \mathbf{x}_0 + \mathbf{y}$ where $A\mathbf{x}_0 = \mathbf{b}$ and $A\mathbf{y} = \mathbf{0}$. Moreover if $\mathbf{y}, \mathbf{z}$ satisfy the homogeneous system $A\mathbf{y} = A\mathbf{z} = \mathbf{0}$ then

$$A(s\mathbf{y} + t\mathbf{z}) = s(A\mathbf{y}) + t(A\mathbf{z}) = s\mathbf{0} + t\mathbf{0} = \mathbf{0}.$$

This observation gives a simple proof of Theorem 1.20.

1.3.4 Row equivalence of matrices

Definition 1.21 *Let $A, B \in \mathbb{R}^{m\times n}$. B iss called row equivalent to A, denoted by $B \sim A$, if B can be obtained from A using ERO.*

Theorem 1.22 *Let $A, B \in \mathbb{R}^{m\times n}$. Then*

1. *$B \sim A \iff A \sim B$.*

2. *$B \sim A$ if and only if A and B have the same RREF.*

Proof. Since ERO are reversible we deduce 1. Suppose that $B \sim A$. Use ERO to bring B to A. Now use ERO to bring A to RREF, which is a matrix C. Since RREF is unique it follows that B has C as is RREF.

Vice versa suppose that A and B have the same RREF C. First use ERO on B to bring it to C. Now bring C using ERO to A. So we obtained A from B using ERO. □

2 Vector Spaces

2.1 Definition of a vector space

A set $\mathbf{V}$ is called a vector space if:
I. For each $\mathbf{x}, \mathbf{y} \in \mathbf{V}$, $\mathbf{x} + \mathbf{y}$ is an element of $\mathbf{V}$. (Addition)
II. For each $\mathbf{x} \in \mathbf{V}$ and $a \in \mathbb{R}$, $a\mathbf{x}$ is an element of $\mathbf{V}$. (Multiplication by scalar)

The two operations satisfy the following laws:
1. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$, commutative law.
2. $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$, associative law.
3. $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for each $\mathbf{x}$, neutral element $\mathbf{0}$.
4. $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$, unique anti element.
5. $a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y}$ for each $\mathbf{x}, \mathbf{y}$, distributive law.
6. $(a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$, distributive law.
7. $(ab)\mathbf{x} = a(b\mathbf{x})$, distributive law.
8. $1\mathbf{x} = \mathbf{x}$.

Lemma 2.1 *Let* $\mathbf{V}$ *be a vector space. Then* $0\mathbf{x} = \mathbf{0}$.

Proof. $0\mathbf{x} = (0 + 0)\mathbf{x} = 0\mathbf{x} + 0\mathbf{x} \Rightarrow \mathbf{0} = 0\mathbf{x} - 0\mathbf{x} = 0\mathbf{x}$. □

2.2 Examples of vector spaces

1. $\mathbb{R}$ - Real Line.

2. $\mathbb{R}^2$ = Plane.

3. $\mathbb{R}^3$ - Three dimensional space.

4. $\mathbb{R}^n$ - n-dimensional space.

5. $\mathbb{R}^{m \times n}$ - The space of $m \times n$ matrices.

Here we add two matrices $A = [a_{ij}], B = [b_{ij}] \in \mathbb{R}^{m \times n}$ coordinatewise.

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{bmatrix}$$

Also $bA := [ba_{ij}]$. The zero matrix $0_{m \times n}$, also simply denoted as 0, is an $m \times n$ matrix whose all entries are equal to 0:

$$0_{m \times n} = 0 = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

So $-A = -(a_{ij}) := (-a_{ij}) = (-1)A$ and $A + (-A) = A - A = 0, \ A - B = A + (-B)$.

6. $\mathcal{P}_n$ - Space of polynomials of degree at most n: $\mathcal{P}_n := \{p(x) = a_n x^n + a_{n-2} x^{n-2} + \dots + a_1 x + a_0\}$. The neutral element is the zero polynomial.

7. $C[a, b]$ - Space of continuous functions on the interval $[a, b]$. The neutral function is the zero function.

Note. The examples 1 - 6 are finite dimensional vector spaces. 7 - is infinite dimensional vector space.

2.3 Subspaces

Let $\mathbf{V}$ be a vector space. A subset $\mathbf{W}$ of $\mathbf{V}$ is called a subspace of $\mathbf{V}$ if the following two conditions hold:

a. For any $\mathbf{x}, \mathbf{y} \in \mathbf{W} \Rightarrow \mathbf{x} + \mathbf{y} \in \mathbf{W}$,

b. For any $\mathbf{x} \in \mathbf{W}, \ a \in \mathbb{R} \Rightarrow a\mathbf{x} \in \mathbf{W}$.

Note: The zero vector $\mathbf{0} \in \mathbf{W}$ since by the condition a: for any $\mathbf{x} \in \mathbf{W}$ one has $\mathbf{0} = 0\mathbf{x} \in \mathbf{W}$.

Equivalently: $\mathbf{W} \subseteq \mathbf{V}$ is a subspace $\iff$ $\mathbf{W}$ is a vector space with respect to the addition and the multiplication by a scalar defined in. $\mathbf{V}$. The following result is strightforward.

Proposition 2.2 *The above conditions a. and b. are equivalent to one condition: If $\mathbf{x}, \mathbf{y} \in \mathbf{U}$ then $a\mathbf{x} + b\mathbf{y} \in \mathbf{U}$ for any scalars a, b.*

Every vector space $\mathbf{V}$ has the following two subspaces:

1. $\mathbf{V}$.

2. The trivial subspace consisting of the zero element: $\mathbf{W} = \{0\}$.

2.4 Examples of subspaces

1. $\mathbb{R}^2$ - Plane: the whole space, lines through the origin, the trivial subspace.

2. $\mathbb{R}^3$ 3-dimensional space: the whole space, planes through the origin, lines through the origin, the trivial subspace.

3. For $A \in \mathbb{R}^{m\times n}$ the null space of A, denoted by $N(A)$, is a subspace of $\mathbb{R}^n$ consisting of all vectors $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = 0$.
Note: $N(A)$ is also called the kernel of A, and denoted by $\ker A$. (See below the explanation for this term.)

4. For $A \in \mathbb{R}^{m\times n}$ the range of A, denoted by $R(A)$, is a subspace of $\mathbb{R}^m$ consisting of all vectors $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{y} = A\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$. Equivalently $R(A) = A\mathbb{R}^n$.

Note: In 3. and 4. A is viewed as a transformation $A : \mathbb{R}^n \to \mathbb{R}^m$: The vector $\mathbf{x} \in \mathbb{R}^n$ is mapped to the vector $A\mathbf{x} \in \mathbb{R}^m$ ($\mathbf{x} \mapsto A\mathbf{x}$.) So $R(A)$ is the range of the transformation induced by A and $N(A)$ is the set of vectors mapped to zero vector in R^m.

2.5 Linear combination & span

For $\mathbf{v}_1, ..., \mathbf{v}_k \in \mathbf{V}$ and $a_1, ..., a_k \in \mathbb{R}$ the vector $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + ... + a_k\mathbf{v}_k$ is called a linear combination of $\mathbf{v}_1, ..., \mathbf{v}_k$. The set of all linear combinations of $\mathbf{v}_1, ..., \mathbf{v}_k$ is called the span of $\mathbf{v}_1, ..., \mathbf{v}_k$ and denoted by $\mathrm{span}(\mathbf{v}_1, ..., \mathbf{v}_k)$.

Proposition 2.3 $\mathrm{span}(\mathbf{v}_1, ..., \mathbf{v}_k)$ *is a linear subspace of* $\mathbf{V}$.

Proof. Assume that $\mathbf{x}, \mathbf{y} \in \mathrm{span}(\mathbf{v}_1, ..., \mathbf{v}_k)$. Then

$$\mathbf{x} = a_1\mathbf{v}_1 + \ldots + a_k\mathbf{x}_k,\ \mathbf{y} = b_1\mathbf{v}_1 + \ldots + b_k\mathbf{v}_k.$$

Hence $s\mathbf{x} + t\mathbf{y} = (sa_1 + tb_1)\mathbf{v}_1 + \ldots + (sa_k + tb_k)\mathbf{v}_k$. Thus $s\mathbf{x} + t\mathbf{y} \in \mathrm{span}(\mathbf{v}_1, ..., \mathbf{v}_k)$. $\square$

We will show that all subspaces in a finite dimensional vector spaces are always given as $\mathrm{span}(\mathbf{v}_1, ..., \mathbf{v}_k)$ for some corresponding vectors $\mathbf{v}_1, ..., \mathbf{v}_k$.

Examples.

1. Any line through the origin in $1, 2, 3$ dimensional space is spanned by any nonzero vector on the line.

2. Any plane through the origin in $2, 3$ dimensional space is spanned by any two nonzero vectors not lying on a line, i.e. non collinear vectors.
3. $\mathbb{R}^3$ spanned by any 3 non planar vectors.

In the following examples $A \in \mathbb{R}^{m \times n}$.

4. Consider the null space $\mathrm{N}(A)$. Let $B \in \mathbb{R}^{m \times n}$ be the RREF of A. B has p pivots and $k := n - p$ free variables. Let $\mathbf{v}_i \in \mathbb{R}^n$ be the following solution of $A\mathbf{x} = 0$. Let the $i - th$ free variable be equal to 1 while all other free variables are equal to 0. Then $\mathrm{N}(A) = \mathrm{span}(\mathbf{v}_1, ..., \mathbf{v}_k)$.

5. Consider the range $\mathrm{R}(A)$, which is a subspace of $\mathbb{R}^m$. View $A = [\mathbf{c}_1 ... \mathbf{c}_n]$ as a matrix composed of n columns $\mathbf{c}_1, ..., \mathbf{c}_n \in \mathbb{R}^m$. Then $\mathrm{R}(A) = \mathrm{span}(\mathbf{c}_1, ..., \mathbf{c}_n)$.

Proof. . Observe that for $\mathbf{x} = (x_1, ..., x_n)^{\mathrm{T}}$ one has $A\mathbf{x} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + ... + x_n\mathbf{c}_n$. □

Corollary 2.4 *The system $A\mathbf{x} = \mathbf{b}$ is solvable $\iff$ $\mathbf{b}$ is a linear combination of the columns of A.*

Problem. Let $\mathbf{v}_1, ..., \mathbf{v}_k \in \mathbb{R}^n$. When $\mathbf{b} \in \mathbb{R}^n$ is a linear combination of $\mathbf{v}_1, ..., \mathbf{v}_k$?

Answer. Let $C := [\mathbf{v}_1 \ \mathbf{v}_2 ... \ \mathbf{v}_k] \in \mathbb{R}^{n \times k}$. Then $\mathbf{b} \in \mathrm{span}(\mathbf{v}_1, ..., \mathbf{v}_k) \iff$ the system $A\mathbf{y} = \mathbf{b}$ is solvable.

Example. $\mathbf{v}_1 = (1, 1, 0)^{\mathrm{T}}, \mathbf{v}_2 = (2, 3, -1)^{\mathrm{T}}, \mathbf{v}_3 = (3, 1, 2)^{\mathrm{T}}, \mathbf{x} = (2, 1, 1)^{\mathrm{T}}, \mathbf{y} = (2, 1, 0)^{\mathrm{T}} \in \mathbb{R}^3$. Show $\mathbf{x} \in \mathbf{W} := \mathrm{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3), \mathbf{y} \notin \mathbf{W}$.

2.6 Spanning sets of vector spaces

$\mathbf{v}_1, ..., \mathbf{v}_k$ is called a spanning set of $\mathbf{V} \iff \mathbf{V} = \mathrm{span}(\mathbf{v}_1, ..., \mathbf{v}_k)$.

Example: Let $\mathbf{V}_{even}, \mathbf{V}_{odd} \subset \mathcal{P}_4$ be the subspaces of even and odd polynomials of degree 4 at most. Then $\mathbf{V}_{even} = \mathrm{span}(1, x^2, x^4)$, $\mathbf{V}_{odd} = \mathrm{span}(x, x^3)$.

Example: Which of these sets is a spanning set of R^3?
a. $[(1, 1, 0)^{\mathrm{T}}, (1, 0, 1)^{\mathrm{T}}]$,
b. $[(1, 1, 0)^{\mathrm{T}}, (1, 0, 1)^{\mathrm{T}}, (0, 1, -1)^{\mathrm{T}}]$,
c. $[(1, 1, 0)^{\mathrm{T}}, (1, 0, 1)^{\mathrm{T}}, (0, 1, -1)^{\mathrm{T}}, (0, 1, 0)^{\mathrm{T}}]$.

Theorem 2.5 $\mathbf{v}_1, ..., \mathbf{v}_k$ *is a spanning set of* $R^n \iff k \geq n$ *and REF of* $A = [\mathbf{v}_1\ \mathbf{v}_2 ... \mathbf{v}_k] \in \mathbb{R}^{n \times k}$ *has* n *pivots.*

Proof. Suppose that U is REF of A. If U has n pivots than REF of $[A|\mathbf{b}]$ is $[U|\mathbf{c}]$. Since U has n pivots, one in each row, $[U|\mathbf{c}]$ does not have a pivot in the last column. So the system $A\mathbf{x} = \mathbf{b}$ is solvable for each $\mathbf{b}$. Hence each $\mathbf{b}$ is a linear combination of the columns of A, i.e. $\mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \mathbb{R}^n$. Vice versa, suppose U does not have n pivots. So there exists $\mathbf{c} \in \mathbb{R}^n$ such that $[U|\mathbf{c}]$ has a pivot in the last column. Hence the system $U\mathbf{x} = \mathbf{c}$ is not solvable. Since U is row equivalent to A, we can use ERO to bring U to A. Use these elementary operations to bring $[U|\mathbf{c}]$ to $[A|\mathbf{b}]$. Hence the system $A\mathbf{x} = \mathbf{b}$ is not solvable, i.e. $\mathbf{b}$ is not a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$.

Finally observe that if U has n pivots than $k \geq n$. So necessary condition that $\mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \mathbb{R}^n$ is $k \geq n$. □

Lemma 2.6 *Let* $\mathbf{v}_1, ..., \mathbf{v}_k \in \mathbf{V}$ *and assume* $\mathbf{v}_i \in W := \mathrm{span}(\mathbf{v}_1, ..., \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, ..., \mathbf{v}_k)$. *Then* $\mathrm{span}(\mathbf{v}_1, ..., \mathbf{v}_k) = W$.

Proof. Assume for simplicity that $i = k$. Suppose that $\mathbf{v}_k = a_1\mathbf{v}_1 + \ldots + a_k\mathbf{v}_{k-1}$ Then

$$b_1\mathbf{v}_1 + \ldots + b_{k-1}\mathbf{v}_{k-1} + b_k\mathbf{v}_k = (b_1 + b_k a_1)\mathbf{v}_1 + \ldots (b_{k-1} + b_k a_{k-1})\mathbf{v}_{k-1}.$$

□

Corollary 2.7 *Let* $\mathbf{v}_1, ..., \mathbf{v}_n \in \mathbb{R}^m$. *Form* $A = [\mathbf{v}_1\ \mathbf{v}_2 ... \mathbf{v}_n] \in \mathbb{R}^{m \times n}$. *Let* $B \in \mathbb{R}^{m \times n}$ *be REF of* A. *Then* $\mathrm{span}(\mathbf{v}_1, ..., \mathbf{v}_n)$ *is spanned by* $\mathbf{v}_{j_1}, ..., \mathbf{v}_{j_r}$ *corresponding to the columns of* B *at which the pivots are located.*

Proof. Assume that x_i is a free variable. Set $x_i = 1$ and all other free variables are zero. We obtain a nontrivial solution $\mathbf{a} = (a_1, \ldots, a_n)^\top$ such that $a_i = 1$ and $a_k = 0$ if x_k is another free variable. $A\mathbf{a} = 0$ implies that $\mathbf{v}_i \in \mathrm{span}(\mathbf{v}_{j_1}, \ldots, \mathbf{v}_{j_r})$. □

Corollary 2.8 *Let* $A \in \mathbb{R}^{m \times n}$ *and assume that* $B \in \mathbb{R}^{m \times n}$ *be REF of* A. *Then* $\mathrm{R}(A)$, *the column space of* A, *is spanned by the columns of* A *corresponding to the columns of* B *at which the pivots are located.*

2.7 Linear Independence

$\mathbf{v}_1, ..., \mathbf{v}_n \in \mathbf{V}$ are linearly independent $\iff$ the equality $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + ... + a_n\mathbf{v}_n 0$ implies that $a_1 = a_2 = ... = a_n = 0$.

Equivalently $\mathbf{v}_1, ..., \mathbf{v}_n \in \mathbf{V}$ are linearly independent $\iff$ every vector in $\mathrm{span}(\mathbf{v}_1, ..., \mathbf{v}_n)$ can be written as a linear combination of $\mathbf{v}_1, ..., \mathbf{v}_n$ in a unique (one) way.

$\mathbf{v}_1, ..., \mathbf{v}_n \in \mathbf{V}$ are linearly dependent $\iff$ $\mathbf{v}_1, ..., \mathbf{v}_n \in \mathbf{V}$ are not linearly independent.

Equivalently $\mathbf{v}_1, ..., \mathbf{v}_n \in \mathbf{V}$ are linearly dependent $\iff$ there exists a nontrivial linear combination of $\mathbf{v}_1, ..., \mathbf{v}_n$ which equals to zero vector: $a_1\mathbf{v}_1 + ... + a_n\mathbf{v}_n = \mathbf{0}$ and $|a_1| + ... + |a_n| > 0$.

Lemma 2.9 *The following statements are equivalent*

1. $\mathbf{v}_1, \ldots, \mathbf{v}_n$ *lin.dep.*

2. $\mathbf{v}_i \in W := \mathrm{span}(\mathbf{v}_1, ..., \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, ..., \mathbf{v}_n)$ *for some* i.

Proof. , 1. $\Rightarrow$(ii). $a_1\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n = \mathbf{0}$ for some $(a_1, \ldots, a_n)^\top \neq 0$. Hence $a_i \neq 0$ for some i. So $\mathbf{v}_i = \frac{-1}{a_i}(a_1\mathbf{v}_1 + ... + a_{i-1}\mathbf{v}_{i-1} + a_{i+1}\mathbf{v}_{i+1} + \ldots + a_n\mathbf{v}_n)$.

2. $\Rightarrow$(i) $\mathbf{v}_i = a_1\mathbf{v}_1 + ... + a_{i-1}\mathbf{v}_{i-1} + a_{i+1}\mathbf{v}_{i+1} + \ldots + a_n\mathbf{v}_n$. So $a_1\mathbf{v}_1 + ... + a_{i-1}\mathbf{v}_{i-1} + (-1)\mathbf{v}_i + a_{i+1}\mathbf{v}_{i+1} + \ldots + a_n\mathbf{v}_n = 0$ □

Proposition 2.10 *Let* $\mathbf{v}_1, ..., \mathbf{v}_n \in \mathbb{R}^m$. *Form* $A = [\mathbf{v}_1 ... \mathbf{v}_n] \in \mathbb{R}^{m\times n}$. *Then* $\mathbf{v}_1, ..., \mathbf{v}_n$ *are linearly independent. (I.e* $\iff$ $A\mathbf{x} = 0$ *has only the trivial solution.* $\iff$ *REF of* A *has* n *pivots).*

Proof. Observe that $x_1\mathbf{v}_1 + \ldots + x_n\mathbf{v}_n = \mathbf{0} \iff A\mathbf{x} = \mathbf{0}$, where $\mathbf{x} = (x_1, \ldots, x_n)^\top$. The second part of the proposition follows from the theorem on solution of the homogeneous system of equations. □

2.8 Basis and dimension

Definition 2.11 $\mathbf{v}_1, ..., \mathbf{v}_n$ *form a basis in* $\mathbf{V}$ *if* $\mathbf{v}_1, ..., \mathbf{v}_n$ *are linearly independent and span* $\mathbf{V}$.

Equivalently: $\mathbf{v}_1, ..., \mathbf{v}_n$ form a basis in $\mathbf{V}$ if and only if ny vector in $\mathbf{V}$ can be expressed as a linear combination of $\mathbf{v}_1, ..., \mathbf{v}_n$ in a unique way.

Theorem 2.12 *Assume that* $\mathbf{v}_1, ..., \mathbf{v}_n$ *spans* $\mathbf{V}$. *Then any collection of* m *vectors* $\mathbf{u}_1, \ldots, \mathbf{u}_m \in \mathbf{V}$, *such that* $m > n$ *is linearly dependent.*

Proof. Let $\mathbf{u}_j = a_{1j}\mathbf{v}_1 + \ldots + a_{nj}\mathbf{v}_n, j = 1, \ldots, m$ Let $A = (a_{ij}) \in \mathbb{R}^{n\times m}$. Homogeneous system $A\mathbf{x} = 0$ has more variables than equations. It has a free variable, hence a nontrivial solution $\mathbf{x} = (x_1, \ldots, x_m)^\top \neq 0$. It follows $x_1\mathbf{u}_1 + ... + x_m\mathbf{u}_m = 0$. □

Corollary 2.13 *If* $[\mathbf{v}_1, ..., \mathbf{v}_n]$ *and* $[\mathbf{u}_1, ..., \mathbf{u}_m]$ *are bases in* $\mathbf{V}$ *then* $m = n$.

Definition 2.14 $\mathbf{V}$ *is called* n*-dimensional, if* $\mathbf{V}$ *has a basis consisting of* n *-elements. The dimension of* $\mathbf{V}$ *is* n, *which is denoted by* $\dim \mathbf{V}$.

The dimension of the trivial space $\{\mathbf{0}\}$ *is* 0.

Theorem 2.15 *Let* $\dim \mathbf{V} = n$. *Then*

1. *Any set of* n *linearly independent vectors* $\mathbf{v}_1, \ldots, \mathbf{v}_n$ *is a basis in* $\mathbf{V}$.

2. *Any set of* n *vectors* $\mathbf{v}_1, \ldots, \mathbf{v}_n$ *that span* $\mathbf{V}$ *is a basis in* $\mathbf{V}$.

Proof. 1. Let $\mathbf{v} \in \mathbf{V}$. Theorem 2.12 implies $\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{v}$ lin.dep.: $a_1\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n + a\mathbf{v} = \mathbf{0}, (a_1, \ldots, a_n, a)^\top \neq 0$. If $a = 0$ it follows $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are lin.dep. Contradiction! So $\mathbf{v} = \frac{-1}{a}(a_1\mathbf{v}_1 + ... + a_n\mathbf{v}_n)$.

2. Need to show $\mathbf{v}_1, \ldots, \mathbf{v}_n$ lin.ind. If not Lemmas 2.9 and 2.6 imply that $\mathbf{V}$ spanned by $n-1$ vectors. Theorem 2.12 yields that $v_1, \ldots, \mathbf{v}_n$ are linearly dependent. This contradicts that V has n lin. ind. vectors. □

Lemma 2.16 ***(Prunning Lemma)***. *Let* $\mathbf{v}_1, \ldots, \mathbf{v}_n$ *be vectors in a vector space* $\mathbf{V}$. *Let* $\mathbf{W} = \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$. *If* $\mathbf{v}_1 = \ldots = \mathbf{v}_n = \mathbf{0}$ *then* $\mathbf{W} = \{\mathbf{0}\}$ *and* $\dim \mathbf{W} = 0$. *Otherwise there exists* $k \in [n]$ *and integers* $1 \leq j_1 < \ldots < j_k \leq n$ *such that* $\mathbf{v}_{j_1}, \ldots, \mathbf{v}_{j_k}$ *is a basis in* $\mathbf{W}$.

Proof. It is enough to consider the case where not all $\mathbf{v}_i = \mathbf{0}$. If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent then $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a basis in $\mathbf{W}$. So $k = n$ and $j_i = i$ for $i \in [n]$.

Suppose that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent. by Lemma 2.9 there exists $i \in [n]$ such that $\mathbf{v}_i \in \mathbf{U} := \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_n)$. Hence $\mathbf{U} = \mathbf{W}$. Continue this process to conclude the lemma. □

Lemma 2.17 ***(Completion lemma)*** *Let* $\mathbf{V}$ *be a vector space of dimension* m. *Let* $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbf{V}$ *be* n *linearly independent vectors. (Hence* $m \geq n$.*) Then there exist* $m - n$ *vectors* $\mathbf{v}_{n+1}, \ldots, \mathbf{v}_m$ *such that* $\mathbf{v}_1, \ldots, \mathbf{v}_m$ *is a basis in* $\mathbf{V}$.

Proof. If $m = n$ then by Theorem 2.15 $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a basis. Assume that $m > n$. Hence by Thm 2.15 $\mathbf{W} := \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n) \neq \mathbf{V}$. Let $\mathbf{v}_{n+1} \in \mathbf{V}$ and $\mathbf{v}_{n+1} \notin \mathbf{W}$. We claim that $\mathbf{v}_1, \ldots, \mathbf{v}_{n+1}$ are linearly independent. Suppose that $a_1\mathbf{v}_1 + \ldots + a_{n+1}\mathbf{v}_{n+1} = \mathbf{0}$. If $a_{n+1} \neq 0$ then $\mathbf{v}_{+1} = -\frac{1}{a_{n+1}}(a_1\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n) \in \mathbf{W}$, which contradicts our assumption. So $a_{n+1} = 0$. Hence $a_1\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n = \mathbf{0}$. As $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent $a_1 = \ldots = a_n = 0$. So $\mathbf{v}_1, \ldots, \mathbf{v}_{n+1}$ are l.i. Continue in this manner to deduce the lemma. □

Theorem 2.18 *Let* $\dim \mathbf{V} = n$. *Then:*

1. *No set of less than* n *vectors can span* $\mathbf{V}$.

2. *Any spanning set of more than* n *vectors can be paired down to form a basis for* $\mathbf{V}$.

3. *Any subset of less than n linearly independent vectors can be extended to basis of $\mathbf{V}$.*

Proof.
1. If less than n vectors span $\mathbf{V}$, $\mathbf{V}$ can not have n lin. ind. vectors.
2. See Pruning Lemma.
3. See Completion Lemma. □

2.9 Row and column spaces of matrices

Definition 2.19 *Let $A \in \mathbb{R}^{m\times n}$.*

1. *Let $\mathbf{r}_1, \ldots, \mathbf{r}_m \in \mathbb{R}^{1\times n}$ be the m rows of A. Then the row space of A is $\mathrm{span}(\mathbf{r}_1, \ldots, \mathbf{r}_m)$, which is a subspace of $\mathbb{R}^{1\times n}$.*

2. *Let $\mathbf{c}_1, \ldots, \mathbf{c}_n \in \mathbb{R}^m$ be the n columns of A. Then the column space of A is $\mathrm{span}(\mathbf{c}_1, \ldots, \mathbf{c}_m)$, which is a subspace of $\mathbb{R}^m = \mathbb{R}^{m\times 1}$.*

Proposition 2.20 *Let $A, B \in \mathbb{R}^{m\times n}$ and assume that $A \sim B$. Then A and B have the same row spaces.*

Proof. We can obtain B from A

$$A \overset{ERO_1}{\rightarrow} A_1 \overset{ERO_2}{\rightarrow} A_2 \overset{ERO_3}{\rightarrow} \ldots A_{k-1} \overset{ERO_k}{\rightarrow} B$$

using a sequence of ERO. It is left to show that each elementary row operation doe not change the row space. Clearly, as each row of $C := EROA$ is a linear combination of the rows of A. Hence the row space of C is contained in the row space of A. On the other hand since $A = ERO^{-1}C$, where ERO^{-1} is the elementary row operation which is the inverse to ERO, it follows that the row space of C contains the row space of A. Hence C and A have the same row space. Therefore A and B have the same row space. □

Recall that the column space of A can be identified with the range of A, denoted by $R(A)$. The row space of A can be identified with $R(A^\top)$.

Let $A \in \mathbb{R}^{m\times n}$ and let B be its REF. Recall that the rank of A, denoted by $\mathrm{rank}\, A$ is the number of pivots in B, which is the number of nonzero rows in B.

Theorem 2.21 *Let $A \in \mathbb{R}^{m\times n}$. Then*

1. *A basis of the row space of A, which is a basis for $\mathrm{R}(A^\mathrm{T})$, consists of nonzero rows in B. $\dim \mathrm{R}(A^\mathrm{T}) = \mathrm{rank}\, A$ is number of lead variables.*

2. *A basis of column space of A consists of the columns of A in which the pivots of B are located. Hence $\dim \mathrm{R}(A) = \mathrm{rank}\, A$.*

3. A basis of the null space of A is obtained by letting each free variable to be equal 1 and all the other free variable equal to 0, and then finding the corresponding solution of $A\mathbf{x} = \mathbf{0}$. The dimension of $\mathrm{N}(A)$, called the nullity of A, is the number of free variables: $\mathrm{nul}\, A := \dim \mathrm{N}(A) = n - \mathrm{rank}\, A$.

Proof. 1. Two row equivalent matrices A and C have the same row space. (But not the same column space!)

2. Assume that the lead variables are $x_{j_1}, \ldots, x_{j_k}$ where $1 \le j_1 < \ldots < j_k \le n$ and $k = \mathrm{rank}\, A$. From Corollary 2.7 if follows the the column space of A is spanned by $\mathbf{c}_{j_1}, \ldots, \mathbf{c}_{j_k}$. Suppose that a $\sum_{i=1}^{k} x_{j_i}\mathbf{c}_{j_i} = \mathbf{0}$. This means that we have a solution to $A\mathbf{x} = \mathbf{0}$ where all free variables are 0. Hence $\mathbf{x} = 0$, i.e. $\mathbf{c}_{j_1}, \ldots, \mathbf{c}_{j_k}$ are linearly independent. Hence $\mathbf{c}_{j_1}, \ldots, \mathbf{c}_{j_k}$ is a basis in the column space of A.

3. Let $x_{i_1}, \ldots, x_{i_{n-k}}$, where $1 \le i_1 < \ldots < i_{n-k} \le n$, be all the free variables of the system $A\mathbf{x} = \mathbf{0}$. For each x_{i_j} let $\mathbf{x}_j$ be the solution of $A\mathbf{x} = \mathbf{0}$ where we let $x_{i_j} = 1$ and all other free variables be zero. Then the general solution of $A\mathbf{x} = 0$ is of the form $\sum_{i=1}^{n-k} x_{i_j}\mathbf{x}_j$. So $N(A) = \mathrm{span}(\mathbf{x}_1, \ldots, \mathbf{x}_{n-k})$. It is left to show that $\mathbf{x}_1, \ldots, \mathbf{x}_{n-k}$ are linearly independent. Suppose that $\mathbf{y} = (y_1, \ldots, y_n)^\top = \sum_{i=1}^{n-k} x_{i_j}\mathbf{x}_j = \mathbf{0}$. Note that in the i_j coordinate of $\mathbf{y}$ is x_{i_j}. Since $\mathbf{y} = \mathbf{0}$ we deduce that $x_{i_j} = 0$. Hence each free variable is zero, i.e. $\mathbf{x}_1, \ldots, \mathbf{x}_{n-k}$ are linearly independent. $\square$

2.10 An example for a basis of $N(A)$

Consider the homogeneous system $A\mathbf{x} = 0$ and assume that the RREF of A is given by $B = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -5 \end{bmatrix}$

$B\mathbf{x} = 0$ is the system

$$\begin{array}{rcrcrcrl} x_1 & + & 2x_2 & + & & & 3x_4 & = 0 \\ & & & & x_3 & - & 5x_4 & = 0 \end{array}$$

Note that x_1, x_3 are lead variables and x_2, x_3 are free variables. Express lead variables as functions of free variables: $x_1 = -2x_2 - 3x_4,\ x_3 = 5x_4$

First set $x_2 = 1, x_4 = 0$ to obtain $x_1 = -2, x_3 = 0$. So the whole solution is $\mathbf{u} = (-2, 1, 0, 0)^\top$ Second set $x_2 = 0, x_4 = 1$ to obtain $x_1 = -3, x_4 = 5$. So the whole solution is $\mathbf{v} = (-3, 0, 5, 1)^\top$. Hence $\mathbf{u}, \mathbf{v}$ is a basis in $N(A)$.

2.11 Useful facts

a. The column and the row space of A have the same dimension. Hence $\mathrm{rank}\, A^\top = \mathrm{rank}\, A$.

b. Standard basis in $\mathbb{R}^n$ are given by the n columns of $n \times n$ identity matrix I_n.

$\mathbf{e}_1 = (1,0)^\top, \mathbf{e}_2 = (0,1)^\top$ is a standard basis in $\mathbb{R}^2$. $\mathbf{e}_1 = (1,0,0)^\top, \mathbf{e}_2 = (0,1,0)^\top, \mathbf{e}_3 = (0,0,1)^\top$ is a standard basis in $\mathbb{R}^3$.

c. $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n \in \mathbb{R}^n$ form a basis in $\mathbb{R}^n \iff A := [\mathbf{v}_1\ \mathbf{v}_2 ... \mathbf{v}_n]$ has n pivots.

d. $\mathbf{v}_1, ..., \mathbf{v}_k \in \mathbb{R}^n$.

Question: Find the dimension and a basis of $\mathbf{V} := \mathrm{span}(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k)$.

Answer: Form a matrix $A = [\mathbf{v}_1\ \mathbf{v}_2 ... \mathbf{v}_k] \in \mathbb{R}^{n \times k}$. Then $\dim \mathbf{V} = \mathrm{rank}\, A$. Let B be a REF of A. Then the vectors $\mathbf{v}_j$ corresponding to the columns of B, where the pivots are located form a basis in $\mathbf{V}$.

2.12 The space $\mathcal{P}_n$

To find the dimension and a basis of a subspace in $\mathcal{P}_n$ One corresponds to each polynomial $p(x) = a_0 + a_1 x + ... + a_n x^n$ the vector $(a_0, a_1, \ldots, a_n) \in \mathbb{R}^{n+1}$ and treats these problems as corresponding problems in $\mathbb{R}^{n+1}$

2.13 Sum of two subspaces

Definition 2.22 *For any two subspaces* $\mathbf{U}, \mathbf{W} \subseteq \mathbf{V}$ *denote* $\mathbf{U} + \mathbf{W} := \{\mathbf{v} := \mathbf{u} + \mathbf{w},\ \mathbf{u} \in \mathbf{U}, \mathbf{w} \in \mathbf{W}\}$*, where we take all possible vectors* $\mathbf{u} \in \mathbf{U}, \mathbf{w} \in \mathbf{W}\}$.

Theorem 2.23 *Let* $\mathbf{V}$ *be a vector space and* $\mathbf{U}, \mathbf{W}$ *be subspaces in* $\mathbf{V}$*. Then*
(a) $\mathbf{U} + \mathbf{W}$ *and* $\mathbf{U} \cap \mathbf{W}$ *are subspace of* $\mathbf{V}$*.*
(b) Assume that $\mathbf{V}$ *is finite dimensional. Then*

1. $\mathbf{U}, \mathbf{W}, \mathbf{U} \cap \mathbf{W}$ *are finite dimensional Let* $l = \dim \mathbf{U} \cap \mathbf{W} \geq 0, p = \dim \mathbf{U} \geq 1, q = \dim \mathbf{W} \geq 1$

2. *There exists a basis in* $\mathbf{v}_1, \ldots, \mathbf{v}_m$ *in* $\mathbf{U} + \mathbf{W}$ *such that* $\mathbf{v}_1, \ldots, \mathbf{v}_l$ *is a basis in* $\mathbf{U} \cap \mathbf{W}$*,* $\mathbf{v}_1, \ldots, \mathbf{v}_p$ *a basis in* $\mathbf{U}$ *and* $\mathbf{v}_1, \ldots, \mathbf{v}_l, \mathbf{v}_{p+1}, \ldots, \mathbf{v}_{p+q-l}$ *is a basis in* $\mathbf{W}$*.*

3. $\dim(\mathbf{U} + \mathbf{W}) = \dim \mathbf{U} + \dim \mathbf{W} - \dim \mathbf{U} \cap \mathbf{W}$

Identity $\#(A \cup B) = \#A + \#B - \#(A \cap B)$ for finite sets A, B is analogous to 3. **Proof.**
(a) 1. Let $\mathbf{u}, \mathbf{w} \in \mathbf{U} \cap \mathbf{W}$. Since $\mathbf{u}, \mathbf{w} \in \mathbf{U}$ it follows $a\mathbf{u} + b\mathbf{w} \in \mathbf{U}$. Similarly $a\mathbf{u} + b\mathbf{w} \in \mathbf{W}$. Hence $a\mathbf{u} + b\mathbf{w} \in \mathbf{U} \cap \mathbf{W}$ and $\mathbf{U} \cap \mathbf{W}$ is a subspace.
(a) 2. Assume that $\mathbf{u}_1, \mathbf{u}_2 \in \mathbf{U}, \mathbf{w}_1, \mathbf{w}_2 \in \mathbf{W}$. Then $a(\mathbf{u}_1 + \mathbf{w}_1) + b(\mathbf{u}_2 + \mathbf{w}_2) = (a\mathbf{u}_1 + b\mathbf{u}_2) + (a\mathbf{w}_1 + b\mathbf{w}_2) \in \mathbf{U} + \mathbf{W}$. Hence $\mathbf{U} + \mathbf{W}$ is a subspace.
(b) 1. Any subspace of an m-dimensional space has dimension m at most.
(b) 2. Let $\mathbf{v}_1, \ldots, \mathbf{v}_l$ be a basis in $\mathbf{U} \cap \mathbf{W}$. Complete this linearly independent set in $\mathbf{U}$ and $\mathbf{W}$ to a basis $\mathbf{v}_1, \ldots, \mathbf{v}_p$ in $\mathbf{U}$ and a basis $\mathbf{v}_1, \ldots, \mathbf{v}_l, \mathbf{v}_{p+1}, \ldots, \mathbf{v}_{p+q-l}$

in $\mathbf{W}$. Hence any for any $\mathbf{u} \in \mathbf{U}, \mathbf{w} \in \mathbf{W}$ $\mathbf{u}+\mathbf{w} \in \mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{p+q-l})$. Hence $\mathbf{U}+\mathbf{W} = \mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{p+q-l})$.

We show that $\mathbf{v}_1, \ldots, \mathbf{v}_{p+q-l}$ lin.ind. . Suppose that $a_1\mathbf{v}_1 + \ldots + a_{p+q-l}\mathbf{v}_{p+q-l} = 0$. So $\mathbf{u} := a_1\mathbf{v}_1 + \ldots a_p\mathbf{v}_p = -a_{p+1}\mathbf{v}_{p+1} + \ldots - a_{p+q-l}\mathbf{v}_{p+q-l} := \mathbf{w}$. Note $\mathbf{u} \in \mathbf{U}, \mathbf{w} \in \mathbf{W}$. So $\mathbf{w} \in \mathbf{U} \cap \mathbf{W}$. Hence $\mathbf{w} = b_1\mathbf{v}_1 + \ldots + \mathbf{b}_l\mathbf{v}_l$. Since $\mathbf{v}_1, \ldots, \mathbf{v}_l, \mathbf{v}_{p+1}, \ldots, \mathbf{v}_{p+q-l}$ lin.ind. $a_{p+1} = \ldots a_{p+q-l} = b_1 = \ldots = b_l = 0$. So $\mathbf{w} = 0 = \mathbf{u}$. Since $\mathbf{v}_1, \ldots, \mathbf{v}_p$ l.i. $a_1 = \ldots = a_p = 0$. Hence $\mathbf{v}_1, \ldots, \mathbf{v}_{p+q-l}$ lin.ind.
(b) 3. Note that from (b) 2 $\dim(\mathbf{U}+\mathbf{W}) = p+q-l$. □

Observe $\mathbf{U}+\mathbf{W} = \mathbf{W}+\mathbf{U}$.

Definition 2.24 . *The subspace* $\mathbf{X} := \mathbf{U}+\mathbf{W}$ *is called a direct sum of* $\mathbf{U}$ *and* $\mathbf{W}$*, if any vector* $\mathbf{v} \in \mathbf{U}+\mathbf{W}$ *has a unique representation of the form* $\mathbf{v} = \mathbf{u}+\mathbf{w}$*, where* $\mathbf{u} \in \mathbf{U}, \mathbf{w} \in \mathbf{W}$*. Equivalently, if* $\mathbf{u}_1 + \mathbf{w}_1 = \mathbf{u}_2 + \mathbf{w}_2$*, where* $\mathbf{u}_1, \mathbf{u}_2 \in \mathbf{U}, \mathbf{w}_1, \mathbf{w}_2 \in \mathbf{W}$*, then* $\mathbf{u}_1 = \mathbf{u}_2, \mathbf{v}_1 = \mathbf{v}_2$*.*

A direct sum of $\mathbf{U}$ *and* $\mathbf{W}$ *is denoted by* $\mathbf{U} \oplus \mathbf{W}$

Proposition 2.25 *For two finite dimensional vectors subspaces* $\mathbf{U}, \mathbf{W} \subseteq \mathbf{V}$ *TFAE (the following are equivalent):*
(a) $\mathbf{U}+\mathbf{W} = \mathbf{U} \oplus \mathbf{W}$
(b) $\mathbf{U} \cap \mathbf{W} = \{0\}$
(c) $\dim \mathbf{U} \cap \mathbf{W} = 0$
(d) $\dim(\mathbf{U}+\mathbf{W}) = \dim \mathbf{U} + \dim \mathbf{W}$
(e) For any bases $\mathbf{u}_1, \ldots, \mathbf{u}_p$, $\mathbf{w}_1, \ldots, \mathbf{w}_q$ *in* $\mathbf{U}, \mathbf{W}$ *respectively* $\mathbf{u}_1, \ldots, \mathbf{u}_p, \mathbf{w}_1, \ldots, \mathbf{w}_q$ *is a basis in* $\mathbf{U}+\mathbf{W}$*.*

Proof. Straightforward. □

Example 1. Let $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{l\times n}$. Then $N(A) \cap N(B) = N(\begin{pmatrix} A \\ B \end{pmatrix})$

Note $\mathbf{x} \in N(A) \cap N(B) \iff A\mathbf{x} = 0 = B\mathbf{x}$.

Example 2. Let $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{m\times l}$. Then $R(A) + R(B) = R((A\ B))$.

Note $R(A) + R(B)$ is the span of the columns of A and B

2.14 Sums of many subspaces

Definition 2.26 *Let* $\mathbf{U}_1, \ldots, \mathbf{U}_k$ *be* k *subspaces of* $\mathbf{V}$*. Then* $\mathbf{X} := \mathbf{U}_1 + \ldots + \mathbf{U}_k$ *is the subspace consisting all vectors of the form* $\mathbf{u}_1 + \mathbf{u}_2 + \ldots + \mathbf{u}_k$*, where* $\mathbf{u}_i \in \mathbf{U}_i, i = 1, \ldots, k$*.* $\mathbf{U}_1 + \ldots + \mathbf{U}_k$ *is called a direct sum of* $\mathbf{U}_1, \ldots, \mathbf{U}_k$*, and denoted by* $\oplus_{i=1}^k \mathbf{U}_i := \mathbf{U}_1 \oplus \ldots \oplus \mathbf{U}_k$ *if any vector in* $\mathbf{X}$ *can be represented in a unique way as* $\mathbf{u}_1 + \mathbf{u}_2 + \ldots + \mathbf{u}_k$*, where* $\mathbf{u}_i \in \mathbf{U}_i, i = 1, \ldots, k$*.*

Proposition 2.27 *For finite dimensional vectors subspaces* $\mathbf{U}_i \subseteq \mathbf{V}, i = 1, \ldots, k$ *TFAE (the following are equivalent):*
(a) $\mathbf{U}_1 + \ldots + \mathbf{U}_k = \oplus_{i=1}^k \mathbf{U}_i$,
(b) $\dim(\mathbf{U}_1 + \ldots + \mathbf{U}_k) = \sum_{i=1}^k \dim \mathbf{U}_i$
(c) For any bases $\mathbf{u}_{1,i}, \ldots, \mathbf{u}_{p_i,i}$ *in* $\mathbf{U}_i, i = 1, \ldots, k$, *the vectors* $\mathbf{u}_{j,i}, j = 1, \ldots, p_i, i = 1, \ldots, k$ *form a basis in* $\mathbf{U}_1 + \ldots + \mathbf{U}_k$.

Proof. (a) $\Rightarrow$ (c). Choose a basis $\mathbf{u}_{1,i}, \ldots, \mathbf{u}_{p_i,i}$ in $\mathbf{U}_i, i = 1, \ldots, k$. Since every vector in $\oplus_{i=1}^k \mathbf{U}_i$ has a unique representation as $\mathbf{u}_1 + \ldots + \mathbf{u}_k$, where $\mathbf{u}_i \in \mathbf{U}_i$ for $i = 1, \ldots, k$ it follows that $\mathbf{0}$ be written in the unique form as a trivial linear combination of all $\mathbf{u}_{1,i}, \ldots, \mathbf{u}_{p_i,i}$ for $i \in [k]$. So all these vectors are linearly independent and span $\oplus_{i=1}^k \mathbf{U}_i$. Hence.

Similarly, (c) $\Rightarrow (a)$.

Clearly (c) $\Rightarrow$ (b).

(b) $\Rightarrow$ (c). As $\mathbf{u}_{1,i}, \ldots, \mathbf{u}_{p_i,i}$ for $i \in [k]$ we deduce that $\dim(\mathbf{U}_1 + \ldots + \mathbf{U}_k) \leq \sum_{i=1}^k \dim \mathbf{U}_i$. Equality holds if and only if (c) holds. $\square$

2.15 Fields

Definition 2.28 *A set* $\mathbb{F}$ *is called a field if for any two elements* $a, b \in \mathbb{F}$ *one has two operations* $a+b$, ab, *such that* $a+b, ab \in \mathbb{F}$ *and these two operations satisfy the following properties.*
A. The addition operation has the same properties as the addition operation of vector spaces
1. $a + b = b + a$, *commutative law;*
2. $(a + b) + c = a + (b + c)$, *associative law;*
3. There exists unique neutral element 0 *such that*
$a + 0 = a$ *for each* a,
4. For each a *there exists a unique anti element* $a + (-a) = 0$.
B. The multiplication operation has similar properties as the addition operation.
5. $ab = ba$, *commutative law;*
6. $(ab)c = a(bc)$, *associative law;*
7. There exists unique identity element 1 *such that*
$a1 = a$ *for each* a;
8. For each $a \neq 0$ *there exists a unique inverse* $aa^{-1} = 1$;
C. The distributive law:
9. a(b+c)=ab+ac
Note*: The commutativity implies* $(b + c)a = ba + ca$.
$0a = a0 = 0$ *for all* $a \in \mathbb{F}$*:*
$0a = (0 + 0)a = 0a + 0a \Rightarrow 0a = 0$.

Examples of Fields
1. Real numbers $\mathbb{R}$

2. Rational numbers $\mathbb{Q}$
3. Complex numbers $\mathbb{C}$

2.16 Finite Fields

Definition 2.29 *Denote by* $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$ *the set of positive integers and the set of whole integers respectively. Let* $m \in \mathbb{N}$. $i, j \in \mathbb{Z}$ *are called equivalent modulo* m, *denoted as* $i \equiv j \bmod m$, *if* $i - j$ *is divisible by* m. $\bmod m$ *is an equivalence relation. (Easy to show.) Denote by* $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ *the set of equivalence classes, usually identified with* $\{0, \ldots, m-1\}$.

(Any integer $i \in \mathbb{Z}$ *induces a unique element* $a \in \{0, \ldots, m-1\}$ *such that* $i - a$ *is divisible by* m.*)*

In $\mathbb{Z}_m$ *define* $a + b, ab$ *by taking representatives* $i, j \in \mathbb{Z}$.

Proposition 2.30 *For any* $m \in \mathbb{N}$, $\mathbb{Z}_m$ *satisfies all the properties of the field 2.28, except 8 for some* m. *Property 8 holds, i.e.* $\mathbb{Z}_m$ *is a field, if and only if* m *is a prime number. (*$p \in \mathbb{N}$ *is a prime number if* p *is divisible by* 1 *and* p *only.)*

Proof. . Note that $\mathbb{Z}$ satisfies all the properties of the field, except 8. ($0, 1$ are the zero and the identity element of $\mathbb{Z}$.) Hence $\mathbb{Z}_m$ satisfies all the properties of the field except 8.

Suppose m is composite $m = ln, l, n \in \mathbb{N}, l, n > 1$. Then $l, n \in 2, \ldots, m-2$ and ln is zero element in $\mathbb{Z}_m$. So l and n can not have inverses.

Suppose $m = p$ prime. Take $i \in \{1, \ldots, m-1\}$. Look at $S := \{i, 2i, \ldots, (m-1)i\} \subset \mathbb{Z}_m$. Consider $ki - ji = (k-j)i$ for $1 \le j < k \le m-1$. So $(k-j)i$ is not divisible by p. Hence $S = \{1, \ldots, m-1\}$ as a subset of $\mathbb{Z}_m$. So there is exactly one integer $j \in [1, m-1]$ such that $ji = 1$. i.e. j is the inverse of $i \in \mathbb{Z}_m$. $\square$

Theorem 2.31 *The number of elements in a finite field* $\mathbb{F}$ *is* p^k, *where* p *is prime and* $k \in \mathbb{N}$. *For each prime* $p > 1$ *and* $k \in \mathbb{N}$ *there exists a finite field* $\mathbb{F}$ *with* p^k *elements. Such* $\mathbb{F}$ *is unique up to an isomorphism, and denoted by* $\mathbb{F}_{p^k}$.

2.17 Vector spaces over fields

Definition 2.32 *Let* $\mathbb{F}$ *be a field. Then* $\mathbf{V}$ *is called vector field over* $\mathbb{F}$ *if* $\mathbf{V}$ *satisfies all the properties stated on in §2.1, where the scalars are the elements of* $\mathbb{F}$.

Example For any $n \in \mathbb{N}$ $\mathbb{F}^n := \{\mathbf{x} = (x_1, \ldots, x_n)^\top : x_1, \ldots, x_n \in \mathbb{F}\}$ is a vector space over $\mathbb{F}$.

We can repeat all the notions that we developed for vector spaces over $\mathbb{R}$ for a general field $\mathbb{F}$.

For example $\dim \mathbb{F}^n = n$

If $\mathbb{F}$ is a finite field with $\#\mathbb{F}$ elements, then $\mathbb{F}^n$ is a finite vector space with $(\#\mathbb{F})^n$ elements.

Finite vector spaces are very useful in coding theory.

3 Linear transformations

3.1 One-to-one and onto maps

Definition 3.1 *T is called a transformation, or map, from the source space V to the target space W, if to each element $v \in V$ the transformation T corresponds an element $w \in W$. We denote $w = T(v)$, and $T : V \to W$. (In other books T is called a map.)*

Example 1: A function $f(x)$ on the real line $\mathbb{R}$ can be regarded as a transformation $f : \mathbb{R} \to \mathbb{R}$.
Example 2: A function $f(x, y)$ on the plane $\mathbb{R}^2$ can be regarded as a transformation $f : \mathbb{R}^2 \to \mathbb{R}$.
Example 3: A transformation $f : \mathbf{V} \to \mathbb{R}$ is called a real valued function on $\mathbf{V}$.
Example 4: Let V be a map of USA, where at each point we plot the vector of the wind blowing at this point. Then we get a transformation $T : V \to \mathbb{R}^2$.

Definition 3.2 *$T : X \to Y$ is called one-to-one, or injective, denoted by $1-1$, if for any $x, y \in X, x \neq y$ one has $Tx \neq Ty$. I.e. the image of two different elements of X by T are different.*

T is called onto, or surjective if $TX = Y \iff$ Range $(T) = Y$, i.e, for each $y \in Y$ there exists $\mathbf{x} \in X$ so that $Tx = y$.

Example 1. $X = \mathbb{N}$, $T : \mathbb{N} \to \mathbb{N}$ given by $T(i) = 2i$. T is $1-1$ but not onto. However $T : \mathbb{N} \to$ Range T is one-to-one and onto.
Example 2. $Id : X \to X$ is defined as $Id(x) = x$ for all $x \in X$ is one-to-one and onto map of X onto itself. The following Proposition is demonstrated straightforward.

Proposition 3.3 *Let X, Y be two sets. Assume that $F : X \to Y$ is one-to-one and onto. Then there exists a one-to-one and onto map $G : Y \to X$ such that $F \circ G = Id_Y, G \circ F = Id_X$. G is the inverse of F denoted by F^{-1}. (Note $(F^{-1})^{-1} = F$.)*

3.2 Isomorphism of vector spaces

Definition 3.4 . *Two vector spaces* $\mathbf{U}, \mathbf{V}$ *over a field* $\mathbb{F}(= \mathbb{R})$ *are called isomorphic if there exists one-to-one and onto map* $L : \mathbf{U} \to \mathbf{V}$*, which preserves the linear structure on* $\mathbf{U}, \mathbf{V}$*:*
1. $L(\mathbf{u}_1 + u_2) = L(\mathbf{u}_1) + L(\mathbf{u}_2)$ *for all* $\mathbf{u}_1, \mathbf{u}_2 \in \mathbf{U}$*. (Note that the first addition is in* $\mathbf{U}$*, and the second addition is in* $\mathbf{V}$*.)*
2. $L(a\mathbf{u}) = aL(\mathbf{u})$ *for all* $\mathbf{u} \in \mathbf{U}, a \in \mathbb{F}(= \mathbb{R})$*.*

Note that the above two conditions are equivalent to one condition.

3. $L(a_1\mathbf{u}_1 + a_2\mathbf{u}_2) = a_1L(\mathbf{u}_1) + a_2L(\mathbf{u}_2)$ for all $\mathbf{u}_1, \mathbf{u}_2 \in \mathbf{U}, a_1, a_2 \in \mathbb{F}(= R)$. Intuitively $\mathbf{U}$ and $\mathbf{V}$ are isomorphic if they are the same spaces modulo renaming, where L is the renaming function.

If $L : \mathbf{U} \to \mathbf{V}$ is an isomorphism then $L(0_{\mathbf{U}}) = 0_{\mathbf{V}}$:
$0_{\mathbf{V}} = 0L(0_{\mathbf{U}}) = L(0\ 0_{\mathbf{U}}) = L(0_{\mathbf{U}})$.
It is straightfoward to show that.

Proposition 3.5 *The inverse of isomorphism is an isomorphism*

3.3 Iso. of fin. dim. vector spaces

Theorem 3.6 *Two finite dimensional vector spaces* $\mathbf{U}, \mathbf{V}$ *over* $F(= \mathbb{R})$ *are isomorphic if and only if they have the same dimension.*

Proof. . (a) $\dim \mathbf{U} = \dim \mathbf{V} = n$. Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}, \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be bases in $\mathbf{U}, \mathbf{V}$ respectively. Define $T : \mathbf{U} \to \mathbf{V}$ by $T(a_1\mathbf{u}_1 + \ldots + a_n\mathbf{u}_n) = a_1\mathbf{v}_1 + \ldots a_n\mathbf{v}_n$. Since any $\mathbf{u} \in \mathbf{U}$ is of the form $\mathbf{u} = a_1\mathbf{u}_1 + \ldots a_n\mathbf{u}_n$ T is a mapping from $\mathbf{U}$ to $\mathbf{V}$. It is straightforward to check that T is linear. As $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a basis in $\mathbf{V}$, it follows that T is onto. Furthermore $T\mathbf{u} = 0$ implies $a_1, \ldots, a_n = 0$. Hence $\mathbf{u} = 0$, i.e. $T^{-1}0 = 0$. Suppose that $T(\mathbf{x}) = T(\mathbf{y})$. Hence $0_{\mathbf{V}} = T(x) - T(y) = T(x - y)$. Since $T^{-1}0_{\mathbf{V}} = 0_{\mathbf{U}} \Rightarrow \mathbf{x} - \mathbf{y} = 0$, i.e. T is 1-1.
(b) Assume that $T : \mathbf{U} \to \mathbf{V}$ is an isomorphism. Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be a basis in $\mathbf{U}$. Denote $T(\mathbf{u}_i) = \mathbf{v}_i, i = 1, \ldots, n$. The linearity of T yields $T(a_1\mathbf{u}_1 + \ldots + a_n\mathbf{u}_n) = a_1\mathbf{v}_1 + \ldots a_n\mathbf{v}_n$. Assume that $a_1\mathbf{v}_1 + \ldots a_n\mathbf{v}_n = 0$. Then $a_1\mathbf{u}_1 + \ldots + a_n\mathbf{u}_n = 0$. Since $\mathbf{u}_1, \ldots, \mathbf{u}_n$ lin.ind. $a_1 = \ldots = a_n = 0$, i.e. $\mathbf{v}_1, \ldots, \mathbf{v}_n$ lin.ind.. For an $\mathbf{v} \in \mathbf{V}$, there exists $\mathbf{u} = a_1\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n \in \mathbf{U}$ s.t. $\mathbf{v} = T\mathbf{u} = T(a_1\mathbf{u}_1 + \ldots + a_n\mathbf{u}_n) = a_1\mathbf{v}_1 + \ldots a_n\mathbf{v}_n$. So $\mathbf{V} = \mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ and $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a basis. So $\dim \mathbf{U} = \dim \mathbf{V} = n$. □

Corollary 3.7 . *Any finite dimensional vector space is isomorphic to* $\mathbb{R}^n$ $(\mathbb{F}^n)$*.*

Example. $\mathcal{P}_n$- the set of polynomials of degree n at most isomorphic to $\mathbb{R}^{n+1}$: $T((a_0, \ldots, a_n)^\top) = a_0 + a_1x + \ldots + a_nx^n$.

3.4 Isomorphisms of $\mathbb{R}^n$

Recall that $A \in \mathbb{R}^{n\times n}$ is called nonsingular if any REF of A has n pivots, i.e. RREF of A is I_n, the $n \times n$ diagonal matrix which has all $1's$ on the main diagonal.

Note that the columns of I_n: $\mathbf{e}_1, \ldots, \mathbf{e}_n$ form a standard basis of $\mathbb{R}^n$.

Theorem 3.8 *$T : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism if and only if there exists a nonsingular matrix $A \in \mathbb{R}^{n\times n}$ such that $T(\mathbf{x}) = A\mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^n$.*

Proof. . (a) Suppose $A \in \mathbb{R}^{n\times n}$ is nonsingular. Let $T(\mathbf{x}) = A\mathbf{x}$. Clearly T linear. Since any system $A\mathbf{x} = \mathbf{b}$ has a unique solution T is onto and $1-1$.
(b) Assume $T : \mathbb{R}^n \to \mathbb{R}^n$ isomorphism. Let $T\mathbf{e}_i = \mathbf{c}_i, i = 1, \ldots, n$. Since $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are linearly independent and T is an isomorphism it follows that $\mathbf{c}_1, \ldots, \mathbf{c}_n$ are linearly independent. Let $A = [\mathbf{c}_1\ \mathbf{c}_2\ \ldots \mathbf{c}_n]$. $\operatorname{rank} A = n$ so A is nonsingular. Note $T((a_1, \ldots, a_n)^\top) = T(\sum_{i=1}^n a_i\mathbf{e}_i) = \sum_{i=1} a_iT(\mathbf{e}_i) = \sum_{i=1}^n a_i\mathbf{c}_i = A(a_1, \ldots, a_n)^\top$. $\square$

3.5 Examples

Definition 3.9 *The matrix A corresponding to the isomorphism $T : \mathbb{R}^n \to \mathbb{R}^n$ in Theorem (3.8) is called the representation matrix of T.*

Examples: (a) The identity isomorphism $Id : \mathbb{R}^n \to \mathbb{R}^n$, i.e. $Id(\mathbf{x}) = \mathbf{x}$, is represented by I_n, as $I_n\mathbf{x} = \mathbf{x}$. Hence I_n is called the identity matrix.
(b) The dilatation isomorphism $T(\mathbf{x}) = a\mathbf{x}, a \neq 0$ is represented by aI_n.
(c) The reflection of $\mathbb{R}^2$: $R((a, b)^\top) = (a, -b)^\top$ is represented by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
(d) A rotation by an angle θ in $\mathbb{R}^2$: $(a, b)^\top \mapsto (\cos\theta a+\sin\theta b, -\sin\theta a+\cos\theta b)^\top$ is represented by $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

3.6 Linear Transformations (Homomorphisms)

T is called a transformation or map from the source space $\mathbf{V}$ to the target space $\mathbf{W}$, if to each element $\mathbf{v} \in \mathbf{V}$ the transformation T corresponds an element $\mathbf{w} \in \mathbf{W}$. We denote $\mathbf{w} = T(\mathbf{v})$, and $T : \mathbf{V} \to \mathbf{W}$. (In other books T is called a map.)

Definition 3.10 *Let $\mathbf{V}$ and $\mathbf{W}$ be two vector spaces. A transformation $T : \mathbf{V} \to \mathbf{W}$ is called linear if*
1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$.
2. $T(a\mathbf{v}) = aT(\mathbf{v})$ for any scalar $a \in \mathbb{R}$.

The conditions 1. and 2. are equivalent to one condition $T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in \mathbf{V}$ and $a, b \in \mathbb{R}$.

Corollary 3.11 *If $T : \mathbf{V} \to \mathbf{W}$ is linear then $T(0_{\mathbf{V}}) = 0_{\mathbf{W}}$.*

Proof. $0_{\mathbf{W}} = 0T(\mathbf{v}) = T(0\mathbf{v}) = T(0_{\mathbf{V}})$. □

Linear transformation is also called *linear operator*.

Example: Let $A \in \mathbb{R}^{m\times n}$ and define $T : \mathbb{R}^n \to \mathbb{R}^m$ as $T(\mathbf{v}) = A\mathbf{v}$. Then T is a linear transformation:

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}, \quad A(a\mathbf{v}) = a(A\mathbf{v}).$$

$\mathrm{R}(T)$ - the range of T. $\mathrm{R}(T)$ is a subspace of $\mathbf{W}$. $\dim \mathrm{R}(T) = \operatorname{rank} T$ is called the rank of T.

$\ker T$ - the kernel of T, or the null space of T, is the set of all vectors in $\mathbf{V}$ mapped by T to a zero vector in $\mathbf{W}$. $\ker T$ is a subspace of $\mathbf{V}$. $\dim \ker T = \operatorname{nul} T$ is called the nullity of T.

Indeed $aT(\mathbf{u}) + bT(\mathbf{v}) = T(a\mathbf{u} + b\mathbf{v})$.
$T(\mathbf{u}) = T(\mathbf{v}) = 0 \Rightarrow T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v}) = a\mathbf{0} + b\mathbf{0} = \mathbf{0}$.

Theorem 3.12 *Any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is given by some $A \in \mathbb{R}^{m\times n}$: $T\mathbf{x} = A\mathbf{x}$ for each $\mathbf{x} \in \mathbb{R}^n$.*

Proof. Let $T(\mathbf{e}_i) = \mathbf{c}_i \in \mathbb{R}^m, i = 1, \ldots, n$. Then $A = [\mathbf{c}_1 \ \ldots \ c_n]$. □

Examples: (a) $C^k(a, b)$ all continuous functions on the interval (a, b) with k continuous derivatives. $C^0(a, b) = C(a, b)$ the set of continuous functions in (a, b). Let $p(x), q(x) \in C(a, b)$. Then $L : C^2(a, b) \to C(a, b)$ given by $L(f)(x) = f''(x) + p(x)f'(x) + q(x)f(x)$ is a linear operator. $\ker L$ is the subspace of all functions f satisfying the second order linear differential equation: $y''(x) + p(x)y'(x) + q(x)y(x) = 0$.

It is known that the above ODE has a unique solution satisfies the initial conditions, IC: $y(x_0) = a_1, y'(x_0) = a_2$ for any fixed $x_0 \in (a, b)$. Hence $\dim \ker L = 2$. Using the theory of ODE one can show that $\mathrm{R}(L) = C(a, b)$.

(b) $L : \mathcal{P}_n \to \mathcal{P}_{n-2}$ given by $L(f) = f''$ is a linear operator. L is onto and $\dim \ker L = 2$ if $n \geq 2$.

3.7 Rank-nullity theorem

Theorem 3.13 *For linear $T : \mathbf{V} \to \mathbf{W}$* $\operatorname{rank} T + \operatorname{nul} T = \dim \mathbf{V}$.

Remark. If $\mathbf{V} = \mathbb{R}^n, \mathbf{W} = \mathbb{R}^m$ then by Theorem (3.12) $T\mathbf{x} = A\mathbf{x}$. for some $A \in \mathbb{R}^{m\times n}$. $\operatorname{rank} T = \operatorname{rank} A$ is the number of lead variables, $\operatorname{nul} T = \operatorname{nul} A = \dim N(A)$ is the number of free variables, so the total number of variables is $n = \dim \mathbb{R}^n$.

Proof. (a) Suppose that $\operatorname{nul} T = 0$. Then T is $1-1$. So $T : \mathbf{V} \to \mathrm{R}(T)$ is isomorphism. $\dim \mathbf{V} = \operatorname{rank} T$.
(b) If $\ker T = \mathbf{V}$ then $\mathrm{R}(T) = \{0\}$ so $\operatorname{nul} T = \dim \mathbf{V}, \operatorname{rank} T = 0$.
(c) $0 < m := \operatorname{nul} T < n := \dim \mathbf{V}$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be a basis in $\ker T$. Complete these set of lin.ind. vectors to a basis of $\mathbf{V}$: $\mathbf{v}_1, \ldots, v_n$. Hence $T(\mathbf{v}_{m+1}), \ldots, T(\mathbf{v}_n)$ is a basis in $\mathrm{R}(T)$. Thus $n - m = \operatorname{rank} T$. So $\operatorname{rank} T + \operatorname{nul} T = m + (n - m) = \dim \mathbf{V}$. □

3.8 Matrix representations of linear transformations

Let $\mathbf{V}$ and $\mathbf{W}$ be finite dimensional vector spaces with the bases $[\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_n]$ and $[\mathbf{w}_1\ \mathbf{w}_2 \ldots \mathbf{w}_m]$. Let $T : \mathbf{V} \to \mathbf{W}$ be a linear transformation. Then T induces the representation matrix $A \in \mathbb{R}^{m \times n}$ as follows. The column j of A is the coordinate vector of $T(\mathbf{v}_j)$ in the basis $[\mathbf{w}_1\ \mathbf{w}_2 \ldots \mathbf{w}_m]$.

The definition of A can be formally stated as

$$[T(\mathbf{v}_1)\ T(\mathbf{v}_2) \ldots T(\mathbf{v}_n)] = [\mathbf{w}_1\ \mathbf{w}_2 \ldots \mathbf{w}_m]A. \tag{3.22}$$

A is called the representation matrix of T in the bases $[\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_n]$ and $[\mathbf{w}_1\ \mathbf{w}_2 \ldots \mathbf{w}_m]$.

Theorem 3.14 *Assume the above assumptions. Assume that $\mathbf{a} \in \mathbb{R}^n$ is the coordinate vector of $\mathbf{v} \in \mathbf{V}$ in the basis $[\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_n]$ and $\mathbf{b} \in \mathbb{R}^m$ is the coordinate vector of $T(\mathbf{v}) \in \mathbf{W}$ in the basis $[\mathbf{w}_1\ \mathbf{w}_2 \ldots \mathbf{w}_m]$. Then $\mathbf{b} = A\mathbf{a}$.*

3.9 Composition of maps

Definition 3.15 *Let $\mathbf{U}, \mathbf{V}, \mathbf{W}$ be three sets. Assume that we have two maps $S : \mathbf{U} \to \mathbf{V}, T : \mathbf{V} \to \mathbf{W}$. $T \circ S : \mathbf{U} \to \mathbf{W}$ is defined by $T \circ S(\mathbf{u}) = T(S(\mathbf{u}))$, called the composition map, and denoted TS.*

Example 1: $f : \mathbb{R} \to \mathbb{R}, g : \mathbb{R} \to \mathbb{R}$. Then $(f \circ g)(x) = f(g(x)), (g \circ f)(x) = g(f(x))$.
Example 2: $f : \mathbb{R}^2 \to \mathbb{R}$, i.e. $f = f(x, y)$, $g : \mathbb{R} \to \mathbb{R}$. Then $(g \circ f)(x, y) = g(f(x, y))$, while $f \circ g$ is not defined

Proposition 3.16 *Let $\mathbf{U}, \mathbf{V}, \mathbf{W}$ be vector spaces. Assume that the maps $S : \mathbf{U} \to \mathbf{V}, T : \mathbf{V} \to \mathbf{W}$ are linear. Then $T \circ S : \mathbf{U} \to \mathbf{W}$ is linear.*

Proof. $T(S(a\mathbf{u}_1 + b\mathbf{u}_2)) = T(aS(\mathbf{u}_1) + bS(\mathbf{u}_2)) = aT(S(\mathbf{u}_1)) + bT(S(\mathbf{u}_2)) = a(T \circ S)(\mathbf{u}_1) + b(T \circ S)(\mathbf{u}_2)$. □

3.10 Product of matrices

We can multiply matrix A times B if the number of columns in the matrix A is equal to the number of rows in B.

Equivalently, we can multiply A times B if A is an $m \times n$ matrix and B is an $n \times p$ matrix. The resulting matrix $C = AB$ is $m \times p$ matrix. The (i, k) entry of AB is obtained by multiplying $i - th$ row of A and $k - th$ column of B.

$$\begin{aligned}
&A = [a_{ij}]_{i=j=1}^{i=m,j=n}, \quad B = [b_{jk}]_{j=k=1}^{j=n,k=p}, \\
&C = (c_{ik})_{i=k=1}^{i=m,k=p}, \\
&c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + ... + a_{in}b_{nk} = \\
&\sum_{j=1}^{n} a_{ij}b_{jk}.
\end{aligned}$$

So A, B can be viewed as linear transformations $B : \mathbb{R}^p \to \mathbb{R}^n, B(\mathbf{u}) = B\mathbf{u}$, $A : \mathbb{R}^n \to \mathbb{R}^m, A(\mathbf{v}) = B\mathbf{v}$ Thus AB represents the composition map $AB : \mathbb{R}^p \to \mathbb{R}^m$.

Example

$$\begin{bmatrix} 1 & -2 \\ -3 & 4 \\ 0 & 2 \\ -7 & -1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} =$$
$$\begin{bmatrix} a - 2d & b - 2e & c - 2f \\ -3a + 4d & -3b + 4e & -3c + 4f \\ 2d & 2e & 2f \\ -7a - d & -7b - e & -7c - f \end{bmatrix}$$

Note that in general $AB \neq BA$ for several reasons.

1. AB may be defined but not BA, (as in the above example), or the other way around.
2. AB and BA defined $\iff A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{n \times m} \Rightarrow AB \in R^{m \times m}, BA \in \mathbb{R}^{n \times n}$
3. For $A, B \in \mathbb{R}^{n \times n}$ and $n > 1$, usually $AB \neq BA$.

Example $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

Rules involving products and additions of matrices. (Note: whenever we write additions and products of matrices we assume that they are all defined, i.e. the dimensions of corresponding matrices match.)

1. $(AB)C = A(BC)$, associative law.
2. $A(B + C) = AB + AC$, distributive law.
3. $(A + B)C = AC + BC$, distributive law.
4. $a(AB) = (aA)B = A(aB)$, algebra rule.

3.11 Transpose of a matrix $A^\top$

Let $A = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix}$

Then $A^\top = \begin{bmatrix} a_{11} & a_{21} & ... & a_{m1} \\ a_{12} & a_{22} & ... & a_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & ... & a_{mn} \end{bmatrix}$

$(A+B)^\top = A^\top + B^\top$
$(AB)^\top = B^\top A^\top$

Examples

$$\begin{bmatrix} -1 & 2 \\ a & b \\ e^{10} & \pi \end{bmatrix}^\top = \begin{bmatrix} -1 & a & e^{10} \\ 2 & b & \pi \end{bmatrix}.$$

$$\left(\begin{bmatrix} 2 & 3 & -4 \\ 5 & -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & -4 \\ 10 & 1 \end{bmatrix} \right)^\top =$$

$$\begin{bmatrix} -33 & -12 \\ -8 & 14 \end{bmatrix}^\top = \begin{bmatrix} -33 & -8 \\ -12 & 14 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 \\ 3 & -4 \\ 10 & 1 \end{bmatrix}^\top \begin{bmatrix} 2 & 3 & -4 \\ 5 & -1 & 0 \end{bmatrix}^\top =$$

$$\begin{bmatrix} -1 & 3 & 10 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 3 & -1 \\ -4 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} -33 & -8 \\ -12 & 14 \end{bmatrix}$$

Let $A \in \mathbb{R}^{m\times n}$. Then $A^\top \in \mathbb{R}^{n\times m}$ and $(A^\top)^\top = A$.

$$\left[\begin{pmatrix} -1 & 2 \\ a & b \\ e^{10} & \pi \end{pmatrix}^{\mathrm{T}} \right]^{\mathrm{T}} = \begin{bmatrix} -1 & a & e^{10} \\ 2 & b & \pi \end{bmatrix}^{\mathrm{T}} =$$

$$\begin{bmatrix} -1 & 2 \\ a & b \\ e^{10} & \pi \end{bmatrix}.$$

3.12 Symmetric Matrices

$A \in \mathbb{R}^{m\times m}$ is called symmetric if $A^\top = A$. The $i-th$ row of a symmetric matrix is equal to its $i-th$ column for $i = 1, ..., m$.

Equivalently: $A = [a_{ij}]_{i,j=1}^m$ is symmetric $\iff$ $a_{ij} = a_{ji}$ for all $i, j = 1, ..., m$.

Examples of 2×2 and 3×3 symmetric matrices:

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}, \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$$

Note: the symmetricity is with respect to the main diagonal.

Assume that $A \in \mathbb{R}^{n\times n} \Rightarrow$. Then $A^\top A \in \mathbb{R}^{n\times n}$ and $AA^\top \in \mathbb{R}^{m\times m}$ are symmetric.

Indeed $(AA^\top)^\top = (A^\top)^\top A^\top = AA^\top$, $(A^\top A)^\top = A^\top (A^\top)^\top = A^\top A$

Identity Matrix

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \in \mathbb{R}^{n\times n}.$$

I_n is in RREF with no zero rows. Clearly, I_n is a symmetric matrix.

Example $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Property of the identity matrix:

$I_m A = AI_n = A,$ `for all` $A \in \mathbb{R}^{m\times n}$.

Example: $I_2 A$, where $A \in \mathbb{R}^{2\times 3}$:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

3.13 Powers of square matrices and Markov chains

Let $A \in \mathbb{R}^{m\times m}$. Define

I. Positive Powers of Square Matrices: $A^2 := AA$, $A^3 := A(AA) = (AA)A = A^2A = AA^2$. For a positive integer k $A^k := A...A$ is product of A k times. For k, q positive integers $A^{k+q} = A^kA^q = A^qA^k$. Also we let $A^0 := I_m$.

Markov chains:

In one town people catch cold and recover every day at the following rate: 90% of healthy stay in the morning healthy the next morning; 60% of sick in the morning recover the next morning.

Find the transition matrix of this phenomenon after one day, two days, and after many days.

$$a_{HH} = 0.9, a_{SH} = 0.1, a_{HS} = 0.6, a_{SS} = 0.4,$$
$$A = \begin{bmatrix} 0.9 & 0.6 \\ 0.1 & 0.4 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_H \\ x_S \end{bmatrix}.$$

Note that if $\mathbf{x}^{\mathrm{T}} = (x_H, x_S)$ represents the number of healthy and sick in a given day, then the situation in the next day is given by $(0.9x_H + 0.6x_S, 0.1x_H + 0.4x_S)^{\mathrm{T}} = A\mathbf{x}$ Hence the number of healthy and sick after two days are given by $A(A\mathbf{x}) = A^2\mathbf{x}$, i.e. the transition matrix given by A^2:

$$\begin{bmatrix} 0.9 & 0.6 \\ 0.1 & 0.4 \end{bmatrix} \begin{bmatrix} 0.9 & 0.6 \\ 0.1 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.87 & 0.78 \\ 0.13 & 0.22 \end{bmatrix}$$

The transition matrix after k days is given by A^k. It can be shown that

$$\lim_{k\to\infty} A^k = \begin{bmatrix} \frac{6}{7} & \frac{6}{7} \\ \frac{1}{7} & \frac{1}{7} \end{bmatrix} \sim \begin{bmatrix} 0.857 & 0.857 \\ 0.143 & 0.143 \end{bmatrix}.$$

The reason for these numbers is the equilibrium state for which we have the equations

$$A\mathbf{x} = \mathbf{x} = I_2\mathbf{x} \Rightarrow (A - I_2)\mathbf{x} = 0 \Rightarrow 0.1x_H = 0.6x_S \Rightarrow x_H = 6x_S.$$

If $x_H + x_S = 1 \Rightarrow x_H = \frac{6}{7}, x_S = \frac{1}{7}$. In the equilibrium stage $\frac{6}{7}$ of all population: $x_H + x_S$ are healthy and $\frac{1}{7}$ of all population is sick.

3.14 Inverses of square matrices

II. Positive Powers of Square Matrices: A invertible if there exists A^{-1} such that $AA^{-1} = A^{-1}A = I_m$. For a positive integer k $A^{-k} := (A^{-1})^k$.

Theorem 3.17 *Let $A \in \mathbb{R}^{m\times m}$. View $A : \mathbb{R}^m \to \mathbb{R}^m$ as a linear transformation. Then the following are equivalent.*
a. A 1-1.
b. A onto.
c. $A : \mathbb{R}^m \to \mathbb{R}^m$ is isomorphism.
d. A is invertible.

Proof. $a \Rightarrow b$. Clearly $A\mathbf{0} = \mathbf{0}$. Since A 1-1 $A\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x} = 0$. Hence $A\mathbf{x} = \mathbf{0}$ has nor free variable. So the number of lead variables is m, i.e. rank $A = m$. Since the RREF of A is I_m. Hence the RREF of the augmented matirx $[A|\mathbf{b}] = [I_n|\mathbf{c}]$. That is, the system $A\mathbf{x} = \mathbf{b}$ has a unique solution ($\mathbf{c}$) for any $\mathbf{b}$. Therefore A is onto.

$b \Rightarrow c$. A is onto, which implies that for any $\mathbf{b}$ the system $A\mathbf{x} = \mathbf{b}$ is solvable. Hence rank $A = m$. So A is also $1-1$. Therefore A is an isomorphism.

$c \Rightarrow d$. Since $\mathbf{x} \mapsto A\mathbf{x}$ is an isomorphism, then the inverse map $\mathbf{y} \to B\mathbf{y}$ implies that $AB = BA = I_m$, i.e. B is the inverse matrix of A.

$d \Rightarrow a$. Suppose that $A\mathbf{x} = \mathbf{b}$. Then $A^{-1}\mathbf{b} = A^{-1}(A\mathbf{x}) = (A^{-1}A)\mathbf{x} = I_n\mathbf{x} = \mathbf{x}$. Hence A is $1-1$. □

Suppose $A \in \mathbb{R}^{n\times n}$ is invertible. Then the system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x} = (x_1\ x_2 ... x_n)^{\mathrm{T}}$, $\mathbf{b} = (b_1\ b_2 ... b_n)^{\mathrm{T}} \in \mathbb{R}^n$, i.e. the system of n equations and n unknowns has a unique solution: $\mathbf{x} = A^{-1}\mathbf{b}$. (See the proof of the above Theorem.)

Observe that if $ad - bc \neq 0$ then the inverse of 2×2 matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

On other hand if $ad - bc = 0$ then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d \\ -c \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -b \\ a \end{bmatrix} = 0$$

So A is not invertible.

Observe next that if $A_1, ..., A_k \in \mathbb{R}^{n\times n}$ are invertible then $A_1 ... A_k$ are invertible and $(A_1 ... A_k)^{-1} = A_k^{-1} ... A_1^{-1}$.

3.15 Elementary Matrices

Definition 3.18 *Elementary Matrix is a square matrix of order m which is obtained by applying one of the three Elementary Row Operations to the identity matrix I_m.*

- *Interchange two rows $R_i \longleftrightarrow R_j$.*

  Example: Apply $R_1 \longleftrightarrow R_3$ to I_3:

  $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to E_I = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

- *Multiply i-th row by $a \neq 0$: $aR_i \longrightarrow R_i$*

  Example: Apply $aR_2 \longrightarrow R_2$ to I_3:

  $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to E_{II} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- *Replace a row by its sum with a multiple of another row $R_i + a \times R_j \longrightarrow R_i$*

  Example: Apply $R_1 + a \times R_3 \longrightarrow R_1$:

  $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to E_{III} = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Proposition 3.19 *All elementary matrices are invertible. More precisely, The inverse of an elementary matrix is given by another elementary matrix of the same kind corresponding to reversing the first elementary operation:*

- *The inverse of E_I is E_I: $E_I E_I = E_I^2 = I_m$.*

  Example:
$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- *The inverse of E_{II} corresponding to $aR_i \longrightarrow R_i$ is E_{II}^{-1} corresponding to $\frac{1}{a}R_i \longrightarrow R_i$*

  Example:
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{a} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- *The inverse of E_{III} corresponds to $R_i + aR_j \longrightarrow R_i$ is E_{III}^{-1} corresponds to $R_i - aR_j \longrightarrow R_i$*

  Example:
$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let $A \in \mathbb{R}^{m \times n}$. Then performing an elementary row operation on A is equivalent to multiplying A by the corresponding elementary matrix E: $A \to EA$.

Example I: Apply $R_1 \leftrightarrow R_3$ to $A \in \mathbb{R}^{3\times 2}$:

$$\begin{bmatrix} u & v \\ w & x \\ y & z \end{bmatrix} \to \begin{bmatrix} y & z \\ w & x \\ u & v \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u & v \\ w & x \\ y & z \end{bmatrix}$$

Example II: Apply $aR_2 \to R_2$ to $A \in \mathbb{R}^{3\times 2}$:

$$\begin{bmatrix} u & v \\ w & x \\ y & z \end{bmatrix} \to \begin{bmatrix} u & v \\ aw & ax \\ y & z \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u & v \\ w & x \\ y & z \end{bmatrix}$$

Example III: Apply $R_1 + a \times R_3 \longrightarrow R_1$: to $A \in \mathbb{R}^{3\times 2}$:

$$\begin{bmatrix} u & v \\ w & x \\ y & z \end{bmatrix} \to \begin{bmatrix} u+ay & v+az \\ w & x \\ y & z \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u & v \\ w & x \\ y & z \end{bmatrix}$$

3.16 ERO in terms of Elementary Matrices

Let $B \in \mathbb{R}^{m\times p}$ and perform k ERO:

$$\begin{aligned}
&B \overset{ERO_1}{\to} B_1 \overset{ERO_2}{\to} B_2 \overset{ERO_3}{\to} \ldots B_{k-1} \overset{ERO_k}{\to} B_k, \\
&B_1 = E_1 B,\ B_2 = E_2 B_1 = E_2 E_1 B, \ldots B_k = E_k \ldots E_1 B \Rightarrow \\
&B_k = MB,\ M = E_k E_{k-1} ... E_2 E_1
\end{aligned}$$

M is invertible matrix since $M^{-1} = E_1^{-1} E_2^{-1} ... E_k^{-1}$.

The system $A\mathbf{x} = \mathbf{b}$, represented by the augmented matrix $B := [A|\mathbf{b}]$, after k ERO is given by $B_k = [A_k|\mathbf{b}_k] = MB = M[A|\mathbf{b}] = [MA, M\mathbf{b}]$ and represents the system $MA\mathbf{x} = M\mathbf{b}$. As M invertible $M^{-1}(MA\mathbf{x}) = A\mathbf{x} = M^{-1}(M\mathbf{b}) = \mathbf{b}$. Thus performing elementary row operations on a system results in equivalent system, i.e. the original and the new system of equations have the same solutions.

3.17 Matrix inverse as products of elementary matrices

Let A_k be the reduced row echelon form of A. Then $A_k = MA$. Assume that $A \in \mathbb{R}^{n\times n}$. As M invertible A invertible $\iff$ A_k invertible: $A = M^{-1} A_k \Rightarrow A^{-1} = A_k^{-1} M$.

If A invertible $A\mathbf{x} = 0$ has only the trivial solution, hence A_k has n pivots (no free variables). Thus $A_k = I_n$ and $A^{-1} = M$!

Proposition 3.20 *$A \in \mathbb{R}^{n\times n}$ is invertible $\iff$ its reduced row echelon form is the identity matrix. If A is invertible its inverse is given by the product of the elementary matrices: $A^{-1} = M = E_k \ldots E_1$.*

Proof. Straightforward. □

3.18 Gauss-Jordan algorithm for A^{-1}

- Form the matrix $B = [A|I_n]$.
- Perform the ERO to obtain RREF of B: $C = [D|F]$.

- A is invertible $\iff D = I_n$.
- If $D = I_n$ then $A^{-1} = F$.

Numerical Example: Let $A = \begin{bmatrix} 1 & 2 & -1 \\ -2 & -5 & 5 \\ 3 & 7 & -5 \end{bmatrix}$. Write $B = [A|I_3]$ and observe that the $(1,1)$ entry in B is a pivot:

$$B = \left[\begin{array}{rrrcrrr} 1 & 2 & -1 & | & 1 & 0 & 0 \\ -2 & -5 & 5 & | & 0 & 1 & 0 \\ 3 & 7 & -5 & | & 0 & 0 & 1 \end{array}\right]$$

Perform ERO: $R_2 + 2R_1 \to R_2,\ R_3 - 3R_1 \to R_3$:

$$B_1 = \left[\begin{array}{rrrcrrr} 1 & 2 & -1 & | & 1 & 0 & 0 \\ 0 & -1 & 3 & | & 2 & 1 & 0 \\ 0 & 1 & -2 & | & -3 & 0 & 1 \end{array}\right].$$

To make $(2,2)$ entry pivot do: $-R_2 \to R_2$:

$$B_2 = \left[\begin{array}{rrrcrrr} 1 & 2 & -1 & & 1 & 0 & 0 \\ 0 & 1 & -3 & | & -2 & -1 & 0 \\ 0 & 1 & -2 & | & -3 & 0 & 1 \end{array}\right].$$

To eliminate $(1,2), (1,3)$ entries do $R_1 - 2R_2 \to R_1,\ R_3 - R_2 \to R_3$

$$B_3 = \left[\begin{array}{rrrcrrr} 1 & 0 & 5 & | & 5 & 2 & 0 \\ 0 & 1 & -3 & | & -2 & -1 & 0 \\ 0 & 0 & 1 & | & -1 & 1 & 1 \end{array}\right].$$

$(3,3)$ is a pivot. To eliminate $(1,3), (2,3)$ entries do: $R_1 - 5R_3 \to R_1,\ R_2 + 3R_3 \to R_2$

$$B_4 = \left[\begin{array}{rrrcrrr} 1 & 0 & 0 & | & 10 & -3 & -5 \\ 0 & 1 & 0 & | & -5 & 2 & 3 \\ 0 & 0 & 1 & | & -1 & 1 & 1 \end{array}\right].$$

So $B_4 = [I_3|F]$ is RREF of B. Thus A has the inverse:

$$A^{-1} = \begin{bmatrix} 10 & -3 & -5 \\ -5 & 2 & 3 \\ -1 & 1 & 1 \end{bmatrix}.$$

Why Gauss-Jordan algorithm works: Perform ERO operations on $B = [A|I_n]$ to obtain RREF of B, which is given by

$$B_k = MB = M[A|I_n] = [MA|MI_n] = [MA|M].$$

$M \in \mathbb{R}^{n\times n}$ is an invertible matrix, which is a product of elementary matrices. A is invertible $\iff$ RREF of A is $I_n \iff$ the first n columns of B have n pivots $\iff MA = I_n \iff M = A^{-1} \iff B_k = [I_n|A^{-1}]$.

Proposition 3.21 *$A \in \mathbb{R}^{n\times n}$ is invertible if and only if $A^\top$ is invertible. Furthermore $(A^\top)^{-1} = (A^{-1})^\top$.*

Proof. The first part of the Propostiton follows from $\operatorname{rank} A = \operatorname{rank} A^\top$. (Recall A invertible $\iff$ $\operatorname{rank} A = n$.) The second part follows from the identity $I_n = I_n^\top = (AA^{-1})^\top = (A^{-1})^\top A^\top$. $\square$

3.19 Change of basis

Assume that $\mathbf{V}$ is an n-dimensional vector space. Let $\mathbf{v} = \mathbf{v}_1, ..., \mathbf{v}_n$ be a basis in $\mathbf{V}$. **Notation**: $[\mathbf{v}_1\ \mathbf{v}_2 \dots \mathbf{v}_n]$. Then any vector $\mathbf{x} \in \mathbf{V}$ can be uniquely presented as $\mathbf{x} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n$.

There is one to one correspondence between $\mathbf{x} \in \mathbf{V}$ and the coordinate vector of $\mathbf{x}$ in the basis $[\mathbf{v}_1\ \mathbf{v}_2 \dots \mathbf{v}_n]$: $\mathbf{a} = (a_1, a_2, \ldots, a_n)^\top \in \mathbb{R}^n$. Thus if $\mathbf{y} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \ldots b_n\mathbf{v}_n$, so $\mathbf{y} \leftrightarrow \mathbf{b} = (b_1, b_2, ..., b_n)^{\mathrm{T}} \in \mathbb{R}^n$ then $r\mathbf{x} \leftrightarrow r\mathbf{a}$ and $\mathbf{x} + \mathbf{y} \leftrightarrow \mathbf{a} + \mathbf{b}$. Thus $\mathbf{V}$ is isomorphic $\mathbb{R}^n$.

Denote: $\mathbf{x} = [\mathbf{v}_1\ \mathbf{v}_2 \dots \mathbf{v}_n] \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$

Let $\mathbf{u}_1\ \mathbf{u}_2 \dots \mathbf{u}_n$ be n vectors in $\mathbf{V}$. Write

$$\mathbf{u}_j = u_{1j}\mathbf{v}_1 + u_{2j}\mathbf{v}_2 + \ldots + u_{nj}\mathbf{v}_j, j = 1, ..., n. \tag{3.23}$$

Define $U = \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ u_{21} & u_{22} & \dots & u_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ u_{n1} & u_{n2} & \dots & u_{nn} \end{bmatrix}$. In short we write (3.23) as

$$[\mathbf{u}_1\ \mathbf{u}_2 \dots \mathbf{u}_n] = [\mathbf{v}_1\ \mathbf{v}_2 \dots \mathbf{v}_n]U \tag{3.24}$$

Proposition 3.22 *$\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n$ is a basis in $\mathbf{V}$ $\iff$ U is invertible.*

Proof. Let $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n$ be is a basis in $\mathbf{V}$. Then

$$[\mathbf{v}_1\ \mathbf{v}_2 \dots \mathbf{v}_n] = [\mathbf{u}_1\ \mathbf{u}_2 \dots \mathbf{u}_n]V. \tag{3.25}$$

Substituting this expression to (3.24) we deduce that $[\mathbf{u}_1\ \mathbf{u}_2 \dots \mathbf{u}_n] = [\mathbf{u}_1\ \mathbf{u}_2 \dots \mathbf{u}_n](VU)$. Since $[\mathbf{u}_1\ \mathbf{u}_2 \dots \mathbf{u}_n]$ is a basis $\mathbf{u}_i$ can be expressed only in a unique combination of $\mathbf{u}_1 \dots, \mathbf{u}_n$, namely $\mathbf{u}_i = 1\mathbf{u}_i$. Hence $VU = I_n$, so U is invertible. Vice versa, if U is invertible, and $V = U^{-1}$ a straightforward calculation shows that (3.25) hold. $\square$

If $[\mathbf{u}_1\ \mathbf{u}_2 \dots \mathbf{u}_n]$ and $[\mathbf{v}_1\ \mathbf{v}_2 \dots \mathbf{v}_n]$ are bases in $\mathbf{V}$ then the matrix U U is called the *transition matrix* from basis $[\mathbf{u}_1\ \mathbf{u}_2 \dots \mathbf{u}_n]$ to basis $[\mathbf{v}_1\ \mathbf{v}_2 \dots \mathbf{v}_n]$. Denoted as $[\mathbf{u}_1\ \mathbf{u}_2 \dots \mathbf{u}_n] \xrightarrow{U} [\mathbf{v}_1\ \mathbf{v}_2 \dots \mathbf{v}_n]$. The proof of Proposition 3.22 yields.

Corollary 3.23 *Let* $[\mathbf{u}_1\ \mathbf{u}_2 \ldots \mathbf{u}_n]$ *and* $[\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_n]$ *be two bases in* $\mathbf{V}$. *Assume that* (3.24) *holds. Then* U^{-1} *is the transition matrix from basis* $[\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_n]$ *to basis* $[\mathbf{u}_1\ \mathbf{u}_2 \ldots \mathbf{u}_n]$: $[\mathbf{u}_1\ \mathbf{u}_2 \ldots \mathbf{u}_n] \xleftarrow{U^{-1}} [\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_n]$.

Let $\mathbf{x} = [\mathbf{u}_1\ \mathbf{u}_2 \ldots \mathbf{u}_n](b_1, b_2, ..., b_n)^\top \iff \mathbf{x} = b_1\mathbf{u}_1 + \ldots b_n\mathbf{u}_n$, i.e. the vector coordinates of $\mathbf{x}$ in the basis $[\mathbf{u}_1\ \mathbf{u}_2 \ldots \mathbf{u}_n]$ is $\mathbf{b} := (b_1, b_2, ..., b_n)^\top$. Then the coordinate vector of $\mathbf{x}$ in the basis $[\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_n]$ is $\mathbf{a} = U\mathbf{b}$.

Indeed, let $\mathbf{x} = [\mathbf{u}_1\ \mathbf{u}_2 \ldots \mathbf{u}_n]\mathbf{b} = [\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_n]U\mathbf{a}$. If $\mathbf{a} \in \mathbb{R}^n$ is the coordinate vector of $\mathbf{x}$ in the basis $[\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_n]$ then $U^{-1}\mathbf{a}$ is the coordinate vector of $\mathbf{x}$ in the basis $[\mathbf{u}_1\ \mathbf{u}_2 \ldots \mathbf{u}_n]$.

Theorem 3.24 *Let* $[\mathbf{u}_1\ \mathbf{u}_2 \ldots \mathbf{u}_n] \xrightarrow{U} [\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_n]$ *and* $[\mathbf{w}_1\ \mathbf{w}_2 \ldots \mathbf{w}_n] \xrightarrow{W} [\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_n]$. *Then* $[\mathbf{w}_1\ \mathbf{w}_2 \ldots \mathbf{w}_n] \xrightarrow{U^{-1}W} [\mathbf{u}_1\ \mathbf{u}_2 \ldots \mathbf{u}_n]$.

Proof. $[\mathbf{w}_1\ \mathbf{w}_2 \ldots \mathbf{w}_n] = [\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_n]W = ([\mathbf{u}_1\ \mathbf{u}_2 \ldots \mathbf{u}_n]U^{-1})W$. □

Note: To obtain $U^{-1}W$ take $A := [U\ W] \in R^{n\times(2n)}$ and bring it to RREF $B = [I\ C]$. Then $C = U^{-1}W$.

3.20 An example

Let $\mathbf{u} = [\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}], \mathbf{w} = [\begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \end{bmatrix}]$

Find the transition matrix from the basis $\mathbf{w}$ to basis $\mathbf{u}$.

Solution: Introduce the standard basis $\mathbf{v} = [\mathbf{e}_1, \mathbf{e}_2]$ in $\mathbb{R}^2$. So $\mathbf{u} = [\mathbf{e}_1, \mathbf{e}_2]\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}, \mathbf{w} = [\mathbf{e}_1, \mathbf{e}_2]\begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix}$ Hence the transition matrix is $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}^{-1}\begin{pmatrix} 3 & 4 \\ 4 & 5 \end{pmatrix}$.

To find this matrix get the RREF of $\begin{bmatrix} 1 & 1 & | & 3 & 4 \\ 2 & 3 & | & 4 & 5 \end{bmatrix}$ which is $\begin{bmatrix} 1 & 0 & | & 5 & 7 \\ 0 & 1 & | & -2 & -3 \end{bmatrix}$

Answer $\begin{bmatrix} 5 & 7 \\ -2 & -3 \end{bmatrix}$

3.21 Change of the representation matrix under the change of bases

Proposition 3.25 $T : \mathbf{V} \to \mathbf{W}$ *linear transformation.* T *is represented by* A *in* $\mathbf{v}, \mathbf{w}$ *bases:* $[T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)] = [\mathbf{w}_1, \ldots, \mathbf{w}_m]A$. *Change basis in* $\mathbf{W}$ $[\mathbf{w}_1\ \mathbf{w}_2 \ldots \mathbf{w}_m] \xrightarrow{P} [\mathbf{x}_1\ \mathbf{x}_2 \ldots \mathbf{x}_m]$ *and in* $\mathbf{V}$ $[\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_n] \xrightarrow{Q} [\mathbf{u}_1\ \mathbf{u}_2 \ldots \mathbf{u}_n]$. *Then*

1. *The representation matrix of* T *in bases* $[\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_n]$ *and* $[\mathbf{x}_1\ \mathbf{x}_2 \ldots \mathbf{x}_m]$ *is given by the matrix* PA, *where* P *invertible.*

2. *The representation matrix of* T *in bases* $[\mathbf{u}_1\ \mathbf{u}_2 \ldots \mathbf{u}_n]$ *and* $[\mathbf{w}_1\ \mathbf{w}_2 \ldots \mathbf{w}_m]$ *is given by the matrix* AQ^{-1}.

3. *The representation matrix of T in bases $[\mathbf{u}_1\ \mathbf{u}_2 \ldots \mathbf{u}_n]$ and $[\mathbf{x}_1\ \mathbf{x}_2 \ldots \mathbf{x}_m]$ is given by the matrix PAQ^{-1}.*

Proof. *1.* $[T(\mathbf{v}_1)\ T(\mathbf{v}_2) \ldots T(\mathbf{v}_n)] = [\mathbf{w}_1\ \mathbf{w}_2 \ldots \mathbf{w}_m]A = [\mathbf{x}_1\ \mathbf{x}_2 \ldots \mathbf{x}_m]PA$.

2. $[T(\mathbf{v}_1)\ T(\mathbf{v}_2) \ldots T(\mathbf{v}_n)] = [T(\mathbf{u}_1)\ T(\mathbf{u}_2) \ldots T(\mathbf{u}_n)]Q = [\mathbf{w}_1\ \mathbf{w}_2 \ldots \mathbf{w}_m]A$ Hence $[T(\mathbf{u}_1)\ T(\mathbf{u}_2) \ldots T(\mathbf{u}_n)] = [\mathbf{w}_1\ \mathbf{w}_2 \ldots \mathbf{w}_m]AQ^{-1}$.

3. Combine *1.* and *2.*. □

3.22 Example

$D : \mathcal{P}_2 \to \mathcal{P}_1, D(p) = p'$. Choose bases $[1, x, x^2], [1, x]$ in $\mathcal{P}_2, \mathcal{P}_1$ respectively. $D(1) = 0 = 0 \cdot 1 + 0 \cdot x, D(x) = 1 = 1 \cdot 1 + 0 \cdot x, D(x^2) = 2x = 0 \cdot 1 + 2 \cdot x$. Representation matrix of T in this basis is $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ Change the basis to $[1+2x, x-x^2, 1-x+x^2]$ in $\mathcal{P}_2$. One can find the new representation matrix A_1 in 2 ways. First

$$D(1+2x) = 2, \quad D(x-x^2) = 1-2x, \quad D(1-x+x^2) = -1+2x,$$

Hence $A_1 = \begin{bmatrix} 2 & 1 & -1 \\ 0 & -2 & 2 \end{bmatrix}$

Second way

$$[1+2x, x-x^2, 1-x+x^2] = [1, x, x^2] \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

So

$$A_1 = \begin{bmatrix} 2 & 1 & -1 \\ 0 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Now choose a new basis in $\mathcal{P}_1$: $[1+x, 2+3x]$. Then

$$[1+x, 2+3x] = [1, x] \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}.$$

Hence the representation matrix of D in bases $[1+2x, x-x^2, 1-x+x^2]$ and $[1+x, 2+3x]$ is

$$A_2 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 1 & -1 \\ 0 & -2 & 2 \end{bmatrix} =$$

$$\frac{1}{1 \cdot 3 - 1 \cdot 2} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 0 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 7 & -7 \\ -2 & -3 & 3 \end{bmatrix}.$$

So

$$D(1+2x) = 2 = 6(1+x) - 2(2+3x), \quad D(x-x^2) = 1-2x = 7(1+x) - 3(2+3x),$$
$$D(1-x+x^2) = -1+2x = -7(1+x) + 3(2+3x).$$

3.23 Equivalence of matrices

Definition: $A, B \in \mathbb{R}^{m\times n}$ are called equivalent if there exist two invertible matrices $P \in \mathbb{R}^{m\times m}, R \in \mathbb{R}^{n\times n}$ such that $B = PAR$.

It is straightforward to show.

Proposition 3.26 *Equivalence of matrices is an equivalence relation.*

Theorem 3.27 *$A, B \in \mathbb{R}^{m\times n}$ are equivalent if and only if they have the same rank.*

Proof. Let $E_{k,m,n} = [e_{ij}]_{i,j=1}^{m,n} \in \mathbb{R}^{m\times n}$ be a matrix such that $e_{11} = e_{22} = \ldots = e_{kk} = 1$ and all other entries of $E_{k,m,n}$ are equal to zero. We claim that A is equivalent to $E_{k,m,n}$, where $\operatorname{rank} A = k$.

Let $SA = C$, where C is RREF of A and S invertible. Then RREF of $C^\top$ is $E_{k,n,m}$! (Prove it!). So $UC^\top = E_{k,n,m} \Rightarrow CU^\top = E_{k,m,n} = SAU^\top$, where U is invertible. □

Corollary 3.28 *$A, B \in \mathbb{R}^{m\times n}$ are equivalent iff they represent the same linear transformation $T : \mathbf{V} \to \mathbf{W}, \dim \mathbf{V} = n, \dim \mathbf{W} = m$ in different bases.*

We now give an alternative proof of Theorem 3.27 using a particular choice of bases in the vector spaces $\mathbf{V}, \mathbf{W}$ for a given linear transformation $T : \mathbf{V} \to \mathbf{W}$.

Theorem 3.29 *Let $\mathbf{V}, \mathbf{W}$ be two vector spaces of dimensions n and m respectively. Let $T : \mathbf{V} \to \mathbf{W}$ be a linear transformation of rank $k = \dim T(\mathbf{V})$. Then there exists bases $[\mathbf{v}_1, \ldots, \mathbf{v}_n], [\mathbf{w}_1, \ldots, \mathbf{w}_m]$ in $\mathbf{V}, \mathbf{W}$ respectively with the following properties.*

$$T(\mathbf{v}_i) = \mathbf{w}_i \text{ for } i = 1, \ldots, k, \quad T(\mathbf{v}_i) = \mathbf{0} \text{ for } i = k+1, \ldots, m. \tag{3.26}$$

In particular, T is represented in these bases as the matrix $E_{k,m,n}$.

Proof. If $k = 0$ then $T = 0$ and any choice of bases in $\mathbf{V}$ and $\mathbf{W}$ is fine. Assume that $k \geq 1$. Choose a basis $\mathbf{w}_1, \ldots, \mathbf{w}_k$ in the range of T ($T(\mathbf{V})$). Complete these k linearly independent vectors to a basis $[\mathbf{w}_1, \ldots, \mathbf{w}_m]$ in $\mathbf{W}$. Let $\mathbf{v}_i$ be a T preimage of $\mathbf{w}_i$, i.e. $T(\mathbf{v}_i) = \mathbf{w}_i$ for $i = 1, \ldots, k$. Suppose that $\sum_{i=1}^k a_i \mathbf{v}_i = \mathbf{0}$. Apply T to both sides to deduce that $\mathbf{0} = \sum_{i=1}^k T(\mathbf{v}_i) = \sum_{i=1}^k a_i \mathbf{w}_i$. Since $\mathbf{w}_1, \ldots, \mathbf{w}_k$ are linearly independent it follows that $a_1 = \ldots = a_k = 0$. So $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent. Let $\mathbf{U} = \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$. Let $N(T) = \{\mathbf{v} \in \mathbf{V},\ T(\mathbf{v}) = \mathbf{0}\}$ be the null space of T. The previous argument shows that $\mathbf{U} \cap N(T) = \{\mathbf{0}\}$. Recall that $\dim N(T)$, the nullity of T, is $n - k$. So

$$\dim(\mathbf{U} + N(T)) = \dim \mathbf{U} + \dim N(T) - \dim(\mathbf{U} \cap N(T)) = k + n - k - 0 = n.$$

Hence $\mathbf{V} = \mathbf{U} \oplus N(T)$. Let $\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n$ be a basis in $N(T)$. Then $[\mathbf{v}_1, \ldots, \mathbf{v}_n]$ is a basis of $\mathbf{V}$. Clearly, (3.26) holds. Hence T is represented in these bases by $E_{k,m,n}$. □

4 Inner product spaces

4.1 Scalar Product in $\mathbb{R}^n$

In $\mathbb{R}^2$ a scalar or dot product is defined for $\mathbf{x} = (x_1, x_2)^\top, \mathbf{y} = (y_1, y_2)^\top \in \mathbb{R}^2$ as: $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 = \mathbf{y}^\top \mathbf{x}$.

In $\mathbb{R}^3$ a scalar or dot product is defined for $\mathbf{x} = (x_1, x_2, x_3)^\top, \mathbf{y} = (y_1, y_2, y_3)^\top \in \mathbb{R}^3$ as: $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + x_3 y_3 = \mathbf{y}^\top \mathbf{x}$. In $\mathbb{R}^n$ a scalar or dot product is defined for $\mathbf{x} = (x_1, \ldots, x_n)^\top, \mathbf{y} = (y_1, \ldots, y_n)^\top \in \mathbb{R}^n$ as: $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \ldots + x_n y_n = \mathbf{y}^{\mathrm{T}} \mathbf{x}$. Note that $\mathbf{x} \cdot \mathbf{y}$ is bilinear and symmetric

$$(a\mathbf{x} + b\mathbf{z}) \cdot \mathbf{y} = a(\mathbf{x} \cdot \mathbf{y}) + b(\mathbf{z} \cdot \mathbf{y}),\ \mathbf{x} \cdot (a\mathbf{y} + b\mathbf{z}) = a(\mathbf{x} \cdot \mathbf{y}) + b(\mathbf{x} \cdot \mathbf{z}), \tag{4.27}$$

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x} \text{ for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n. \tag{4.28}$$

The above equalities hold for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$. Recall that (4.27) is the *bilinearity condition* and (4.28) is called the symmetricity condition.

The length of $\mathbf{x} = (x_1, \ldots, x_n)^\top \in \mathbb{R}^n$ is $||\mathbf{x}|| := \sqrt{\mathbf{x}^\top \mathbf{x}} = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$. $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are called orthogonal if $\mathbf{y}^\top \mathbf{x} = \mathbf{x}^\top \mathbf{y} = 0$.

4.2 Cauchy-Schwarz inequality

Proposition 4.1 *For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ the following inequality holds: $|\mathbf{x}^\top \mathbf{y}| \le ||\mathbf{x}||\ ||\mathbf{y}||$. (This is the Cauchy-Schwarz inequality, abbreviated as CSI.) Equality holds iff $\mathbf{x}, \mathbf{y}$ are linearly dependent, equivalently if $\mathbf{y} \neq 0$ then $\mathbf{x} = a\mathbf{y}$ for some $a \in \mathbb{R}$.*

Proof. If either $\mathbf{x}$ or $\mathbf{y}$ are zero vectors then equality holds in CSI. Suppose that $\mathbf{y} \neq 0$. Then for $t \in \mathbb{R}$ define

$$f(t) := (\mathbf{x} - t\mathbf{y})^\top (\mathbf{x} - t\mathbf{y}) = ||\mathbf{y}||^2 t^2 - 2(\mathbf{x}^{\mathrm{T}} \mathbf{y})t + ||\mathbf{x}||^2 \ge 0.$$

The equation $f(t) = 0$ is either unsolvable, in the case $f(t)$ is always positive, or has one solution. Hence CSI holds. Equality holds if $\mathbf{x} - a\mathbf{y} = 0$. $\square$

The cosine of the angle between two nonzero vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is defined as $\cos\theta = \frac{\mathbf{y}^\top \mathbf{x}}{||\mathbf{x}||\ ||y||}$. Note that $\cos\theta \in [-1, 1]$ in view of the CSI.

By expanding $\|\mathbf{y} - \mathbf{x}\|^2 = (\mathbf{y} - \mathbf{x}) \cdot (\mathbf{y} - \mathbf{x})$ using (4.27)-(4.28) and using the definition of the cosine of the angle between nonzero $\mathbf{x}$ and $\mathbf{y}$ we deduce the **Cosine Law**:

$$||\mathbf{y} - \mathbf{x}||^2 = ||\mathbf{y}||^2 + ||\mathbf{x}||^2 - 2||\mathbf{y}||\ ||\mathbf{x}|| \cos\theta \tag{4.29}$$

The above formula still holds if either $\mathbf{x}$ or $\mathbf{y}$ are zero vectors, since the value of θ does not matter in this case. So if $\mathbf{x} \perp \mathbf{y}$ Pithagoras theorem holds: $||\mathbf{x} - \mathbf{y}||^2 = ||\mathbf{x}||^2 + ||\mathbf{y}||^2 = ||\mathbf{x} + \mathbf{y}||^2$.

4.3 Scalar and vector projection

The scalar projection of $\mathbf{x} \in \mathbb{R}^n$ on nonzero $\mathbf{y} \in \mathbb{R}^n$ is given by $\frac{\mathbf{x}^\top \mathbf{y}}{||\mathbf{y}||} = \cos\theta ||\mathbf{x}||$.
The vector projection of $\mathbf{x} \in \mathbb{R}^n$ on nonzero $\mathbf{y} \in \mathbb{R}^n$ is given by $\frac{\mathbf{x}^\top \mathbf{y}}{||\mathbf{y}||^2}\mathbf{y} = \frac{\mathbf{x}^\top \mathbf{y}}{\mathbf{y}^\top \mathbf{y}}\mathbf{y}$.
Example. Let $\mathbf{x} = (2, 1, 3, 4)^\top, \mathbf{y} = (1, -1, -1, 1)^\top$.
a. Find the cosine of angle between $\mathbf{x}, \mathbf{y}$.
b. Find the scalar and vector projection of $\mathbf{x}$ on $\mathbf{y}$.

Solution: $||\mathbf{y}|| = \sqrt{1^2 + (-1)^2 + (-1)^2 + 1^2} = \sqrt{4} = 2, ||\mathbf{x}|| = \sqrt{2^2 + 1^2 + 3^2 + 4^2} = \sqrt{30}$, $\mathbf{x}^\top \mathbf{y} = 2 - 1 - 3 + 4 = 2, \cos\theta = \frac{2}{2\sqrt{30}} = \frac{1}{\sqrt{30}}$. **Scalar projection**: $\frac{2}{2} = 1$.
Vector projection: $\frac{2}{4}\mathbf{y} = (.5, -.5, -.5, .5)^\top$.

4.4 Orthogonal subspaces

Definitions: Two subspaces $\mathbf{U}$ and $\mathbf{V}$ in $\mathbb{R}^n$ are called orthogonal if any $\mathbf{u} \in \mathbf{U}$ is orthogonal to any $\mathbf{v} \in \mathbf{V}$: $\mathbf{v}^\top \mathbf{u} = 0$. This is denoted by $\mathbf{U} \perp \mathbf{V}$.

For a subspace $\mathbf{U}$ of $\mathbb{R}^n$ $\mathbf{U}^\perp$ denotes all vectors in $\mathbb{R}^n$ orthogonal to $\mathbf{U}$.

Example: In $\mathbb{R}^3$: $\mathbf{V}^\perp$ is an orthogonal line to the plane $\mathbf{V}$, which intersect at the origin.

Proposition 4.2 *Let* $\mathbf{u}_1, \ldots, \mathbf{u}_k$ *span* $\mathbf{U} \subseteq \mathbb{R}^n$. *Form a matrix* $A = [\mathbf{u}_1\ \mathbf{u}_2 \ldots \mathbf{u}_k] \in \mathbb{R}^{n\times k}$. *Then*
(a): $N(A^\top) = \mathbf{U}^\perp$.
(b): $\dim \mathbf{U}^\perp = n - \dim \mathbf{U}$.
(c): $(\mathbf{U}^\perp)^\perp = \mathbf{U}$.
(Note: (b-c) holds for any subspace $\mathbf{U} \subseteq \mathbb{R}^n$*.)*

Proof. . (a) Follows from definition of $\mathbf{U}^\perp$.
(b) Follows from $\dim \mathbf{U} = \operatorname{rank} A$, and $\operatorname{nul} A^\top = n - \operatorname{rank} A^\top = n - \operatorname{rank} A$.
(c) Follows from the observations $(\mathbf{U}^\perp)^\perp \supseteq \mathbf{U}$, $\dim(\mathbf{U}^\perp)^\perp = n - \dim \mathbf{U}^\perp = n - (n - \dim \mathbf{U}) = \dim \mathbf{U}$. □

Corollary 4.3 $\mathbb{R}^n = \mathbf{U} \oplus \mathbf{U}^\perp$.

Proof. Observe that if $\mathbf{x} \in \mathbf{U} \cap \mathbf{U}^\perp$ then $\mathbf{x}^\top \mathbf{x} = 0 \Rightarrow \mathbf{x} = 0 \Rightarrow \mathbf{U} \cap \mathbf{U}^\perp = \{0\}$. □

Observe that part (b) of Proposition 4.2 yields $\dim \mathbf{U} + \dim \mathbf{U}^\perp = n$.

Proposition 4.4 *For* $A \in \mathbb{R}^{n\times m}$*:*
(a): $\mathrm{N}(A^\top) = \mathrm{R}(A)^\perp$.
(b): $\mathrm{N}(A^\top)^\perp = \mathrm{R}(A)$.

Proof. Any vector in $N(A^\top)$ satisfies $A^\top \mathbf{y} = 0 \iff \mathbf{y}^\top A = 0$. Any vector $\mathbf{z} \in \mathrm{R}(A)$ is of the form $\mathbf{z} = A\mathbf{x}$. So $\mathbf{y}^\top \mathbf{z} = \mathbf{y}^\top A\mathbf{x} = (\mathbf{y}^\top A)\mathbf{x} = 0^\top \mathbf{x} = 0$. Hence $\mathrm{N}(A^\top) \subseteq \mathrm{R}(A)^\perp$. Recall $\dim \mathrm{N}(A^\top) = n - \mathrm{rank}\, A^\top = n - \mathrm{rank}\, A$ Proposition 4.2 yields $\dim \mathrm{R}(A)^\perp = n - \dim \mathrm{R}(A) = n - \mathrm{rank}\, A$. Hence (a) follows. Apply $\perp$ operation to (a) and use part (c) of Proposition 4.2 to deduce (b). $\square$

Proposition 4.5 *Let $A \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Then either $A\mathbf{x} = \mathbf{b}$ is solvable or there exists $\mathbf{y} \in \mathrm{N}(A^\top)$ such that $\mathbf{y}^\top \mathbf{b} \neq 0$.*

Proof. $A\mathbf{x} = \mathbf{b}$ solvable iff $\mathbf{b} \in \mathrm{R}(A)$. (a) of Proposition 4.4 yields $\mathrm{R}(A)^\perp = \mathrm{N}(A^\top)$. So $A\mathbf{x} = \mathbf{b}$ is not solvable iff $\mathrm{N}(A^\top)$ is not orthogonal to $\mathbf{b}$.

4.5 Example

Let $\mathbf{u} = (1,2,3,4)^\top, \mathbf{v} = (2,4,5,2)^\top, \mathbf{w} = (3,6,8,6)^\top$. Find a basis in $\mathrm{span}(\mathbf{u},\mathbf{v},\mathbf{w})^\perp$.
Solution: Set $A = [\mathbf{u}\ \mathbf{v}\ \mathbf{w}]$. Then

$$A^\top = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 5 & 2 \\ 3 & 6 & 8 & 6 \end{bmatrix}.$$

RREF of $A^\top$ is:

$$B = \begin{bmatrix} 1 & 2 & 0 & -14 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence a basis in $N(A^\top) = N(B)$ is $(-2,1,0,0,)^\top, (14,0,-6,1)^\top$.

Note that a basis of the row space of $A^\top$ is given by the nonzero rows of B. Hence a basis of $\mathrm{span}(\mathbf{u},\mathbf{v},\mathbf{w})$ is given by $(1,2,0,-14)^\top, (0,0,1,6)^\top$.

4.6 Projection on a subspace

Let $\mathbf{U}$ be a subspace of $\mathbb{R}^n$. Let $\mathbb{R}^m = \mathbf{U} \oplus \mathbf{U}^\perp$ and $\mathbf{b} \in \mathbb{R}^m$. Express $\mathbf{b} = \mathbf{u} + \mathbf{v}$ where $\mathbf{u} \in \mathbf{U}, \mathbf{v} \in \mathbf{U}^\perp$. Then $\mathbf{u}$ is called the projection of $\mathbf{b}$ on $\mathbf{U}$ and denoted by $P_{\mathbf{U}}(\mathbf{b})$: $(\mathbf{b} - P_{\mathbf{U}}(\mathbf{b})) \perp \mathbf{U}$.

Proposition 4.6 $P_{\mathbf{U}} : \mathbb{R}^n \to \mathbf{U}$ *is a linear transformation.*

Proof. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then $P_{\mathbf{U}}(\mathbf{x}), P_{\mathbf{U}}(\mathbf{y}) \in \mathbf{U}$ and $\mathbf{x} - P_{\mathbf{U}}(\mathbf{x}), \mathbf{y} - P_{\mathbf{U}}(\mathbf{y}) \in \mathbf{U}^\perp$. So $aP_{\mathbf{U}}(\mathbf{x}) + bP_{\mathbf{U}}(\mathbf{y}) \in \mathbf{U}$ and $a(\mathbf{x} - P_{\mathbf{U}}(\mathbf{x})) + b(\mathbf{y} - P_{\mathbf{U}}(\mathbf{y})) \in \mathbf{U}^\perp$. Hence $P_{\mathbf{U}}(a\mathbf{x} + \mathbf{by}) = aP_{\mathbf{U}}(\mathbf{x}) + bP_{\mathbf{U}}(\mathbf{y})$. $\square$

Proposition 4.7 $P_{\mathbf{U}}(\mathbf{b})$ *is the unique solution of the minimal problem* $\min_{\mathbf{x}\in\mathbf{U}} ||\mathbf{b} - \mathbf{x}|| = ||\mathbf{b} - P_{\mathbf{U}}(\mathbf{b})||$.

Proof. Since $\mathbf{b} - P_{\mathbf{U}}(\mathbf{b}) \in \mathbf{U}^{\perp}$ the Pithagoras theorem yields

$$\|\mathbf{b}-\mathbf{x}\|^2 = \|(\mathbf{b}-P_{\mathbf{U}}(\mathbf{b}))+(P_{\mathbf{U}}(\mathbf{b})-\mathbf{x})\|^2 = \|\mathbf{b}-P_{\mathbf{U}}(\mathbf{b})\|^2+\|P_{\mathbf{U}}(\mathbf{b})-\mathbf{x})\|^2 \geq \|\mathbf{b}-P_{\mathbf{U}}(\mathbf{b})\|^2.$$

Equality holds if and only if $\mathbf{x} = P_{\mathbf{U}}(\mathbf{b})$. □

Theorem 4.8 *(Least Square Theorem) Let $A \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$. Then the system*

$$A^{\top}A\mathbf{x} = A^{\top}\mathbf{b} \tag{4.30}$$

is always solvable. Any solution $\mathbf{z}$ to this system is called the least square solution of $A\mathbf{x} = \mathbf{b}$. Furthermore $P_{\mathrm{R}(A)}(\mathbf{b}) = A\mathbf{z}$.

Proof.

$$A^{\top}A\mathbf{x} = 0 \Rightarrow \mathbf{x}^{\top}A^{\top}A\mathbf{x} = 0 \iff ||A\mathbf{x}||^2 = 0 \Rightarrow \mathbf{x} \in \mathrm{N}(A) \Rightarrow \mathbf{x} \in \mathrm{N}(A^{\top}A).$$

Let $B := A^{\top}A$ and $B^{\top} = B$. If $\mathbf{y} \in \mathrm{N}(B^{\top})$ then $A\mathbf{y} = 0 \Rightarrow \mathbf{y}^{\top}A^{\top} = 0 \Rightarrow \mathbf{y}^{\top}A^{\top}\mathbf{b} = 0$. Proposition 4.5 yields that $A^{\mathrm{T}}A\mathbf{x} = A^{\mathrm{T}}\mathbf{b}$ is solvable.

Observe finally that

$$A^{\top}A\mathbf{z} = A^{\top}\mathbf{b} \iff A^{\top}\mathbf{b}-A^{\top}A\mathbf{z} = 0 \iff A^{\top}(\mathbf{b}-A\mathbf{z}) \iff (\mathbf{b}-A\mathbf{z}) \perp \mathrm{R}(A).$$

As $A\mathbf{z} \in \mathrm{R}(A)$ we deduce that $P_{\mathrm{R}(A)}(\mathbf{b}) = A\mathbf{z}$. □

4.7 Example

Consider the system of three equations in two variables.

$$\begin{array}{rcrcrcl} & x_1 & + & x_2 & = & 3 & \\ - & 2x_1 & + & 3x_2 & = & 1 & \Rightarrow A\mathbf{x} = \mathbf{b} \\ & 2x_1 & - & x_2 & = & 2 & \end{array}$$

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 3 \\ 2 & -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix},$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \hat{A} = \begin{bmatrix} 1 & 1 & | & 3 \\ -2 & 3 & | & 1 \\ 2 & -1 & | & 2 \end{bmatrix}.$$

The RREF of A: $B = \begin{bmatrix} 1 & 0 & | & 0 \\ 0 & 1 & | & 0 \\ 0 & 0 & | & 1 \end{bmatrix}$ Hence the original system is unsolvable!

The least square system $A^{\top}A\mathbf{x} = A^{\top}\mathbf{b} \iff C\mathbf{x} = \mathbf{c}$:

$$C = A^{\top}A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 9 & -7 \\ -7 & 11 \end{bmatrix},$$

$$\mathbf{c} = A^{\mathrm{T}}\mathbf{b} = \begin{bmatrix} 1 & -2 & 2 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

Since C is invertible the solution of the LSP is:

$$\mathbf{x} = C^{-1}\mathbf{c} = \frac{1}{9 \cdot 11 - (-7)^2} \begin{bmatrix} 11 & 7 \\ 7 & 9 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 1.66 \\ 1.42 \end{bmatrix}$$

Hence $A\mathbf{x} = \begin{bmatrix} 3.08 \\ 0.94 \\ 1.90 \end{bmatrix}$ is the projection of $\mathbf{b}$ on the column space of A.

4.8 Finding the projection on span

Proposition 4.9 *To find the projection of* $\mathbf{b} \in \mathbb{R}^m$ *on the subspace* $\mathrm{span}(\mathbf{u}_1, \ldots, \mathbf{u}_n) \subseteq \mathbb{R}^m$ *do the following:*
a. Form the matrix $A = [\mathbf{u}_1 \ \ldots \ \mathbf{u}_n] \in \mathbb{R}^{m \times n}$.
b. Solve the system $A^\top A\mathbf{x} = A^\top \mathbf{b}$.
c. For any solution $\mathbf{x}$ *of b.* $A\mathbf{x}$ *is the required projection.*

Proof. Since $\mathrm{R}(A) = \mathrm{span}(\mathbf{u}_1, \ldots, \mathbf{u}_n)$ the above proposition follows straightforward from the Least Squares Theorem. □

Proposition 4.10 *Let* $A \in \mathbb{R}^{m \times n}$. *Then* $\mathrm{rank}\, A = n \iff A^\top A$ *is invertible. In that case* $\mathbf{z} = (A^\top A)^{-1} A^\top \mathbf{b}$ *is the least square solution of* $A\mathbf{x} = \mathbf{b}$. *Also* $A(A^\top A)^{-1}\mathbf{b}$ *is the projection of* $\mathbf{b}$ *on the column space of* A.

Proof.

$$A\mathbf{x} = 0 \iff ||A\mathbf{x}|| = 0 \iff \mathbf{x}^\top A^\mathrm{T} A\mathbf{x} = 0 \iff A^\mathrm{T} A\mathbf{x} = 0.$$

So $N(A) = N(A^\mathrm{T} A)$. Hence $\mathrm{rank}\, A = n \iff N(A) = \{0\} = N(A^\mathrm{T} A) \iff A^\mathrm{T} A$ invertible. □

4.9 The best fit line

Problem: Fit a straight line $y = a + bx$ in the $X - Y$ plane through m given points $(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)$.
Solution: The line should satisfy m conditions:

$$\begin{array}{ccccc} 1 \cdot a & + & x_1 \cdot b & = & y_1 \\ 1 \cdot a & + & x_2 \cdot b & = & y_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 \cdot a & + & x_m \cdot b & = & y_m \end{array} \Rightarrow \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_m \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = \mathbf{y} = \mathbf{c}.$$

$$A \qquad \mathbf{z} \quad = \quad \mathbf{c}.$$

The least squares system $A^\top A\mathbf{z} = A^\top \mathbf{c}$:

$$\begin{bmatrix} m & x_1+x_2+\ldots+x_m \\ x_1+x_2+\ldots+x_m & x_1^2+x_2^2+\ldots+x_m^2 \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1+y_2+\ldots+y_m \\ x_1y_1+x_2y_2+\ldots+x_my_m \end{bmatrix},$$

$$\mathrm{det\ A^\top A = m(x_1^2+x_2^2+\ldots+x_m^2) - (x_1+x_2+\ldots+x_m)^2},$$

$$\mathrm{det\ A^T A = 0 \iff x_1 = x_2 = \ldots = x_m}.$$

If $\mathrm{det\ A^\top A} \neq 0$ then

$$a^\star = \frac{(\sum_{i=1}^m x_i^2)(\sum_{i=1}^m y_i) - (\sum_{i=1}^m x_i)(\sum_{i=1}^m x_iy_i)}{\mathrm{det\ A^T A}},$$

$$b^\star = \frac{-(\sum_{i=1}^m x_i)(\sum_{i=1}^m y_i) + m(\sum_{i=1}^m x_iy_i)}{\mathrm{det\ A^T A}}.$$

We now explain the solution for the best fit line. We are given m points in the plane $(x_1, y_1), \ldots, (x_m, y_m)$. We are trying to fit a line $y = bx + a$ through these m points. Suppose we chose the parameters $a, b \in \mathbb{R}$. Then this line passes through the point $(x_i, bx_i + a)$ for $i = 1, \ldots, m$. The square of the distance between the point (x_i, y_i) and $(x_i, bx_i + a)$ is $(y_i - (1 \cdot a + x_i \cdots b))^2$. The sum of the squares of distances is $\sum_{i=1}^m (y_i - (1 \cdot a + x_i \cdot b))^2$. Note that this sum is $\|\mathbf{y} - A\mathbf{z}\|^2$ where $A, \mathbf{z}, \mathbf{y}$ are as above. So $A\mathbf{z} \in R(A)$. Hence $\min_{\mathbf{z}\in\mathbb{R}^2} \|\mathbf{y} - A\mathbf{z}\|^2$ is achieved for the least square solution $\mathbf{z}^\star = (a^\star, b^\star)$ given as above, (if not all x_i are equal.) So the line $y = a^\star + b^\star x$ is the best fit line.

4.10 Example

Given three points in $\mathbb{R}^2$: $(0, 1), (3, 4), (6, 5)$. Find the best least square fit by a linear function $y = a + bx$ to these three points.
Solution.

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 3 \\ 1 & 6 \end{bmatrix}, \mathbf{z} = \begin{bmatrix} a \\ b \end{bmatrix}, \mathbf{c} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix},$$

$$\mathbf{z} = (A^\top A)^{-1} A^\top \mathbf{c} = \begin{bmatrix} 3 & 9 \\ 9 & 45 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 42 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

The best least square fit by a linear function is $y = \frac{4}{3} + \frac{2}{3}x$.

4.11 Orthonormal sets

$\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^m$ is called an orthogonal set (OS) if $\mathbf{v}_i^\top \mathbf{v}_j = 0$ if $i \neq j$, i.e. any two vectors in this set is an orthogonal pair.
Example 1: The standard basis $\mathbf{e}_1, \ldots, \mathbf{e}_m \in \mathbb{R}^m$ is an orthogonal set.
Example 2: The vectors $\mathbf{v}_1 = (3, 4, 1, 0)^\top, \mathbf{v}_2 = (4, -3, 0, 2)^\top, \mathbf{v}_3 = (0, 0, 0, 0)^\top$ are three orthogonal vectors in $\mathbb{R}^4$.

Theorem 4.11 *An orthogonal set of nonzero vectors is linearly independent.*

Proof. Suppose that $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n = 0$. Multiply by $\mathbf{v}_1^\top$:

$$0 = \mathbf{v}_1^\top 0 = \mathbf{v}_1^\top(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n) = a_1\mathbf{v}_1^\top\mathbf{v}_1 + a_2\mathbf{v}_1^\top\mathbf{v}_2 + \ldots + a_n\mathbf{v}_1^\top\mathbf{v}_n.$$

Since $\mathbf{v}_1^\top\mathbf{v}_i = 0$ for $i > 1$ we obtain: $0 = a_1(\mathbf{v}_1^\top\mathbf{v}_1) = a_1||\mathbf{v}_1||^2$. Since $||\mathbf{v}_1|| > 0$ we deduce $a_1 = 0$. Continue in the same manner to deduce that all $a_i = 0$. $\square$

$\mathbf{v}_1, ..., \mathbf{v}_n \in \mathbf{V}$ is called an orthonormal set (ONS) if $\mathbf{v}_1, ..., \mathbf{v}_n$ is an orthogonal set and each $\mathbf{v}_i$ has length 1. In Example 1 $\mathbf{e}_1, \ldots, \mathbf{e}_m$ is an ONS. In Example 2 the set $\{\frac{1}{\sqrt{26}}\mathbf{v}_1, \frac{1}{\sqrt{29}}\mathbf{v}_2\}$ is an ONS.

Notation: Let $I_n \in \mathbb{R}^{n\times n}$ be an identity matrix. Let $\delta_{ij},\ i,j = 1,\ldots,n$ be the entries of I_n. So $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ii} = 1$ for $i = 1,\ldots,n$.

Remark δ_{ij} are called the Kronecker's delta, in honor of Leopold Kronecker (1823-1891).

Normalization: A nonzero OS $\mathbf{u}_1, ..., \mathbf{u}_n$ can be normalized to an ONS by $\mathbf{v}_i := \frac{\mathbf{u}_i}{||\mathbf{u}_i||}$ for $i = 1,\ldots,n$.

Theorem 4.12 *Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be ONS in $\mathbb{R}^m$. Denote $\mathbf{U} := \mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$. Then*

1. *Any vector $\mathbf{u} \in \mathbf{U}$ can be written as a unique linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$: $\mathbf{u} = \sum_{i=1}^n (\mathbf{v}_i^\top\mathbf{u})\mathbf{v}_i$.*

2. *For any $\mathbf{v} \in \mathbb{R}^m$ the orthogonal projection $P_\mathbf{U}(\mathbf{v})$ on the subspace $\mathbf{U}$ is given by*

$$P_\mathbf{U}(\mathbf{v}) = \sum_{i=1}^n (\mathbf{v}_i^\top\mathbf{v})\mathbf{v}_i. \tag{4.31}$$

   In particular

$$||\mathbf{v}||^2 = \mathbf{v}^\top\mathbf{v} \geq \sum_{i=1}^n |\mathbf{v}_i^\top\mathbf{v}|^2. \tag{4.32}$$

   (This inequality is called Bessel's inequality.) Equality holds $\iff$ $\mathbf{v} \in \mathbf{U}$.

3. *If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is an orthonormal basis (ONB) in $\mathbf{V}$ then for any vector $\mathbf{v} \in \mathbf{V}$ one has:*

$$\mathbf{v} = \sum_{i=1}^n (\mathbf{v}_i^\top\mathbf{v})\mathbf{v}_i, \quad ||\mathbf{v}||^2 = \sum_{i=1}^n |\mathbf{v}_i^\top\mathbf{v}|^2. \tag{4.33}$$

   (The last equality is called Parseval's formula.)

Proof. Let $\mathbf{v} \in \mathbf{V}$. Define $\mathbf{w} := \sum_{j=1}^{n} (\mathbf{v}_j^\top \mathbf{v})\mathbf{v}_j$. Clearly, $\mathbf{w} \in \mathbf{U}$. Observe next that

$$(\mathbf{v}_i^\top (\mathbf{v} - \mathbf{w}) = \mathbf{v}_i^\top \mathbf{v} - \sum_{j=1}^{n} (\mathbf{v}_i^\top \mathbf{v})(\mathbf{v}_i^\top \mathbf{v}_j) = \mathbf{v}_i^\top \mathbf{v} - \mathbf{v}_i^\top \mathbf{v} = 0$$

So $\mathbf{v} - \mathbf{w} \in \mathbf{U}^\perp$. Hence $\mathbf{w} = P_{\mathbf{U}}(\mathbf{v})$. As $\mathbf{v} = \mathbf{w} + (\mathbf{v} - \mathbf{w})$. The Pithagoras identity implies that

$$\|\mathbf{v}\|^2 = \|\mathbf{w}\|^2 + \|\mathbf{v} - \mathbf{w}\|^2 \geq \|\mathbf{w}\|^2.$$

Equality holds if and only if $\mathbf{v} = \mathbf{w}$. Since $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is ONS the equality (4.31) yields the equality

$$\|\mathbf{w}\|^2 = \sum_{i=1}^{n} |\mathbf{v}_i^\top \mathbf{v}|^2.$$

This implies part *2*. Clearly, if $\mathbf{v} \in \mathbf{U}$ then $P_{\mathbf{U}}(\mathbf{v}) = \mathbf{v}$, which yields *1*. If $n = m$ then $\mathbf{w} = \mathbf{v}$ and we must have always equality (4.33). $\square$

4.12 Orthogonal Matrices

$Q \in \mathbb{R}^{n\times n}$ is called an orthogonal matrix if $Q^\top Q = I_n$. The following proposition is deduced strightforward.

Proposition 4.13 *Let $Q \in \mathbb{R}^{n\times n}$. TFAE*

1. *The columns of Q form an ONB in $\mathbb{R}^n$.*
2. *$Q^{-1} = Q^\top$, i.e. $QQ^\top = I_n$.*
3. *$(Q\mathbf{y})^\top (Q\mathbf{x}) = \mathbf{y}^\top \mathbf{x}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. I.e. $Q : \mathbb{R}^n \to \mathbb{R}^n$ preserves angles & lengths of vectors.*
4. *$||Q\mathbf{x}||^2 = ||\mathbf{x}||^2$ for all $\mathbf{x} \in \mathbb{R}^n$. I.e. $Q : \mathbb{R}^n \to \mathbb{R}^n$ preserves length.*

Example 1: I_n is an orthogonal matrix since $I_n I_n^\top = I_n I_n = I_n$. (Note $I_n = [\mathbf{e}_1\ \mathbf{e}_2\ \ldots\ \mathbf{e}_n]$)

Example 2: $Q = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. (Note $Q = [\mathbf{e}_3\ \mathbf{e}_1\ \mathbf{e}_2]$)

Example 3: $Q = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Example 4: Any 2×2 orthogonal matrix is either of the form $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, which is a rotation, or $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$ which is a reflection around the line forming an angle θ with the X axis which passes through the origin.

Definition $P \in \mathbb{R}^{n\times n}$ is called a permutation matrix if in each row and column of P there is one nonzero entry which equals to 1.

It is straightforward to see that the set of the columns of an permutation matrix consists of the standard basis vectors in $\mathbb{R}^n$ in some order. Hence a permutation matrix is orthogonal.

If P is a permutation matrix and $(y_1, ..., y_n)^\top = P(x_1, ..., x_n)^\top$ then the coordinates of $\mathbf{y}$ are permutation of the coordinates of $\mathbf{x}$, and this permutation does not depend on the coordinates of $\mathbf{x}$.

n columns of $A \in \mathbb{R}^{m\times n}$ form an ONB in the columns space $\mathrm{R}(A)$ of $A \iff A^\top A = I_n$. In that case the Least Square Solution of $A\mathbf{x} = \mathbf{b}$ is $\mathbf{z} = A^\top\mathbf{b}$, which is the projection of $\mathbf{b}$ the column space of A.

4.13 Gram-Schmidt orthogonolization process

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be linearly independent vectors in $\mathbb{R}^m$. Then the Gram-Schmidt (*orthogonalization*) process gives a recursive way to generate ONS $\mathbf{q}_1, \ldots, \mathbf{q}_n \in \mathbb{R}^m$ from $\mathbf{x}_1, \ldots, \mathbf{x}_n$, such that $\mathrm{span}(\mathbf{x}_1, \ldots, \mathbf{x}_k) = \mathrm{span}(\mathbf{q}_1, \ldots, \mathbf{q}_k)$ for $k = 1, \ldots, n$. If $m = n$, i.e. $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is a basis of $\mathbb{R}^n$ then $\mathbf{q}_1, \ldots, \mathbf{q}_n$ is an ONB of $\mathbb{R}^n$.

GS-algorithm:

$$r_{11} := ||\mathbf{x}_1||, \quad \mathbf{q}_1 := \frac{1}{r_{11}}\mathbf{x}_1,$$

$$r_{12} := \mathbf{q}_1^\top\mathbf{x}_2, \quad \mathbf{p}_1 := r_{12}\mathbf{q}_1, \quad r_{22} := ||\mathbf{x}_2 - \mathbf{p}_1||, \quad \mathbf{q}_2 := \frac{1}{r_{22}}(\mathbf{x}_2 - \mathbf{p}_1),$$

$$r_{13} := \mathbf{q}_1^\top\mathbf{x}_3, \quad r_{23} := \mathbf{q}_2^\top\mathbf{x}_3, \quad \mathbf{p}_2 := r_{13}\mathbf{q}_1 + r_{23}\mathbf{q}_2,$$

$$r_{33} := ||\mathbf{x}_3 - \mathbf{p}_2||, \quad \mathbf{q}_3 := \frac{1}{r_{33}}(\mathbf{x}_3 - \mathbf{p}_2).$$

Assume that $\mathbf{q}_1, \ldots, \mathbf{q}_k$ were computed. Then

$$r_{1(k+1)} := \mathbf{q}_1^\top\mathbf{x}_{k+1}, \ldots, r_{1(k+1)} := \mathbf{q}_k^\top\mathbf{x}_{k+1}, \quad \mathbf{p}_k := r_{1(k+1)}\mathbf{q}_1 + \ldots r_{k(k+1)}\mathbf{q}_k,$$

$$r_{(k+1)(k+1)} := ||\mathbf{x}_{k+1} - \mathbf{p}_k||, \quad \mathbf{q}_{k+1} := \frac{1}{r_{(k+1)(k+1)}}(\mathbf{x}_{k+1} - \mathbf{p}_k).$$

4.14 Explanation of G-S process

First observe that $r_{i(k+1)} := \mathbf{q}_i^\top\mathbf{x}_{k+1}$ is the scalar projection of $\mathbf{x}_{k+1}$ on $\mathbf{q}_i$. Next observe that $\mathbf{p}_k$ is the projection of $\mathbf{x}_{k+1}$ on $\mathrm{span}(\mathbf{q}_1, \ldots, \mathbf{q}_k) = \mathrm{span}(\mathbf{x}_1, \ldots, \mathbf{x}_k)$.

Hence $\mathbf{x}_{k+1} - \mathbf{p}_k \perp \operatorname{span}(\mathbf{q}_1, \ldots, \mathbf{q}_k)$. Thus $\mathbf{r}_{(k+1)(k+1)} = ||\mathbf{x}_{k+1} - \mathbf{p}_k||$ is the distance of $\mathbf{x}_{k+1}$ to $\operatorname{span}(\mathbf{q}_1, \ldots, \mathbf{q}_k) = \operatorname{span}(\mathbf{x}_1, \ldots, \mathbf{x}_k)$. The assumption that $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent yields that $r_{(k+1)(k+1)} > 0$. Hence $\mathbf{q}_{k+1} = r_{(k+1)(k+1)}^{-1}(\mathbf{x}_{k+1} - \mathbf{p}_k)$ is a vector of unit length orthogonal to $\mathbf{q}_1, \ldots, \mathbf{q}_k$.

4.15 An example of G-S process

Let $\mathbf{x}_1 = (1, 1, 1, 1)^\top, \mathbf{x}_2 = (-1, 4, 4, -1)^\top$, $\mathbf{x}_3 = (4, -2, 2, 0)^\top$. Then

$$
\begin{aligned}
&r_{11} = ||\mathbf{x}_1|| = 2, \mathbf{q}_1 = \frac{1}{r_{11}}\mathbf{x}_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})^\top, \\
&r_{12} = \mathbf{q}_1^\top \mathbf{x}_2 = 3, \mathbf{p}_1 = r_{12}\mathbf{q}_1 = 3\mathbf{q}_1 = (\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2})^\top, \\
&\mathbf{x}_2 - \mathbf{p}_1 = (-\frac{5}{2}, \frac{5}{2}, \frac{5}{2}, -\frac{5}{2})^\top, \mathbf{r}_{22} = ||\mathbf{x}_2 - \mathbf{p}_1|| = 5, \\
&\mathbf{q}_2 = \frac{1}{r_{22}}(\mathbf{x}_2 - \mathbf{p}_1) = (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})^\top, \\
&r_{13} = \mathbf{q}_1^\top \mathbf{x}_3 = 2, \mathbf{r}_{23} = \mathbf{q}_2^\top \mathbf{x}_3 = -2, \\
&\mathbf{p}_2 = r_{13}\mathbf{q}_1 + r_{23}\mathbf{q}_2 = (2, 0, 0, 2)^\top, \\
&\mathbf{x}_3 - \mathbf{p}_2 = (2, -2, 2, -2)^\top, r_{33} = ||\mathbf{x}_3 - \mathbf{p}_2|| = 4, \\
&\mathbf{q}_3 = \frac{1}{r_{33}}(\mathbf{x}_3 - \mathbf{p}_2) = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})^\top.
\end{aligned}
$$

4.16 QR Factorization

Let $A = [\mathbf{a}_1\ \mathbf{a}_2 \ldots \mathbf{a}_n] \in \mathbb{R}^{m\times n}$ and assume that $\operatorname{rank} A = n$. (I.e. the columns of A are linearly independent.) Perform G-S process with the book keeping as above:

- $r_{11} := ||\mathbf{a}_1||$, $\mathbf{q}_1 := \frac{1}{r_{11}}\mathbf{a}_1$.
- Assume that $\mathbf{q}_1, \ldots, \mathbf{q}_{k-1}$ were computed. Then $r_{ik} := \mathbf{q}_i^\mathrm{T}\mathbf{a}_k$ for $i = 1, \ldots, k-1$. $\mathbf{p}_{k-1} := r_{1k}\mathbf{q}_1 + r_{2k}\mathbf{q}_2 + \ldots r_{(k-1)k}\mathbf{q}_{k-1}$ and $r_{kk} := ||\mathbf{a}_k - \mathbf{p}_{k-1}||$, $\mathbf{q}_k := \frac{1}{r_{kk}}(\mathbf{a}_k - \mathbf{p}_{k-1})$ for $k = 2, ..., n$.

Let $Q = [\mathbf{q}_1\ \mathbf{q}_2 \ldots \mathbf{q}_n] \in \mathbb{R}^{m\times n}$ and $R = \begin{bmatrix} r_{11} & r_{12} & r_{13} & \ldots & r_{1n} \\ 0 & r_{22} & r_{23} & \ldots & r_{2n} \\ 0 & 0 & r_{33} & \ldots & r_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & r_{nn} \end{bmatrix}$

Then $A = QR$, $Q^\mathrm{T}Q = I_n$ and $A^\top A = R^\top R$. The Least Squares Solution of $A\mathbf{x} = \mathbf{b}$ is given by the upper triangular system $R\hat{\mathbf{x}} = Q^\top\mathbf{b}$ which can be solved by back substitution. Formally $\hat{\mathbf{x}} = R^{-1}Q^\top\mathbf{b}$.

Proof $A^\top A\mathbf{x} = R^\top Q^\top QR\mathbf{x} = R^\top R\mathbf{x} = A^\top\mathbf{b} = R^\top Q^\top\mathbf{b}$. Multiply from left by $(R^\top)^{-1}$ to get $R\hat{\mathbf{x}} = Q^\top\mathbf{b}$

Note: $QQ^\top \mathbf{b}$ is the projection of $\mathbf{b}$ on the columns space of A. The matrix $P := QQ^\top$ is called an orthogonal projection. It is symmetric and $P^2 = P$, as $(QQ^\top)(QQ^\top) = Q(Q^\top Q)Q^\top = Q(I)Q^\top = QQ^\top$. Note $QQ^\top : \mathbb{R}^m \to \mathbb{R}^m$ is the orthogonal projection.

Equivalently: The assumption that $\operatorname{rank} A = n$ is equivalent to the assumption that $A^\top A$ is invertible. So the LSS $A^\top A\hat{x} = A^\top \mathbf{b}$ has unique solution $\hat{\mathbf{x}} = (A^\top A)^{-1}\mathbf{b}$. Hence the projection of $\mathbf{b}$ on the column space of A is $P\mathbf{b} = A\hat{\mathbf{x}} = A(A^\top A)^{-1}A^\top\mathbf{b}$. Hence $P = A(A^\top A)^{-1}A^\top$.

4.17 An example of QR algorithm

Let $A = [\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3] = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$ be the matrix corresponding to the Example of G-S algorithm §4.15. Then

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{bmatrix},$$

$$Q = [\mathbf{q}_1\ \mathbf{q}_2\ \mathbf{q}_3] = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

(*Explain why in this example $A = QR$!*) **Note** $QQ^\top : \mathbb{R}^4 \to \mathbb{R}^4$ is the projection on $\operatorname{span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$.

4.18 Inner Product Spaces

Let $\mathbf{V}$ be a vector space over $\mathbb{R}$. Then the function $\langle\cdot,\cdot\rangle : \mathbf{V}\times\mathbf{V} \to \mathbb{R}$ is called an *inner product* on $\mathbf{V}$ if the following conditions hold:

- For each pair $\mathbf{x}, \mathbf{y} \in \mathbf{V}$ $\langle\mathbf{x}, \mathbf{y}\rangle$ is a real number.
- $\langle\mathbf{x}, \mathbf{y}\rangle = \langle\mathbf{y}, \mathbf{x}\rangle$. (*symmetricity.*)
- $\langle\mathbf{x} + \mathbf{z}, \mathbf{y}\rangle = \langle\mathbf{x}, \mathbf{y}\rangle + \langle\mathbf{z}, \mathbf{y}\rangle$. (*linearity I*)
- $\langle\alpha\mathbf{x}, \mathbf{y}\rangle = \alpha\langle\mathbf{x}, \mathbf{y}\rangle$ for any scalar $\alpha \in \mathbb{R}$. (*linearity II*)
- For any $0 \neq \mathbf{x} \in \mathbf{V}$ $\langle\mathbf{x}, \mathbf{x}\rangle > 0$. (*positivity*)

Note:

- The two linearity conditions can be put in one condition: $\langle\alpha\mathbf{x}+\beta\mathbf{z}, \mathbf{y}\rangle = \alpha\langle\mathbf{x}, \mathbf{y}\rangle + \beta\langle\mathbf{z}, \mathbf{y}\rangle$.

- The symmetricity condition yields linearity in the second variable: $\langle \mathbf{x}, \alpha\mathbf{y} + \beta\mathbf{z}\rangle = \alpha\langle \mathbf{x}, \mathbf{y}\rangle + \beta\langle \mathbf{x}, \mathbf{z}\rangle$.
- Each linearity condition implies $\langle \mathbf{0}, \mathbf{y}\rangle = \mathbf{0} \Rightarrow \langle \mathbf{0}, \mathbf{0}\rangle = 0$.
- $\langle \mathbf{x}, \mathbf{x}\rangle \geq 0$ For any $\mathbf{x} \in \mathbf{V}$.

4.19 Examples of IPS

- $\mathbf{V} = \mathbb{R}^n$, $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{y}^\top \mathbf{x}$.
- $\mathbf{V} = \mathbb{R}^n$, $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{y}^\top D\mathbf{x}$, $D = \mathrm{diag}(d_1, ..., d_n)$ is a diagonal matrix with positive diagonal entries. Then $\mathbf{y}^\mathrm{T} D\mathbf{x} = d_1x_1y_1 + \ldots + d_nx_ny_n$.
- $\mathbf{V} = \mathbb{R}^{m\times n}$, $\langle A, B\rangle = \sum_{i,j=1}^{m,n} a_{ij}b_{ij}$.
- $\mathbf{V} = \mathrm{C}[a, b]$, $\langle f, g\rangle = \int_a^b f(x)g(x)dx$.
- $\mathbf{V} = \mathrm{C}[a, b]$, $\langle f, g\rangle = \int_a^b f(x)g(x)p(x)dx$, where $p(x) \in \mathrm{C}[a, b]$, $p(x) \geq 0$ and $p(x) = 0$ at most at a finite number of points.
- $\mathbf{V} = P_n$: All polynomials of degree $n - 1$ at most. Let $t_1 < t_2 < \ldots < t_n$ be any n real numbers. $\langle p, q\rangle := \sum_{i=1}^n p(t_i)q(t_i) = p(t_1)q(t_1) + \ldots + p(t_n)q(t_n)$

4.20 Length and angle in IPS

- The norm (*length*) of the vector $\mathbf{x}$ is $||\mathbf{x}|| := \sqrt{\langle \mathbf{x}, \mathbf{x}\rangle}$.
- *Cauchy-Schwarz* inequality: $|\langle \mathbf{x}, \mathbf{y}\rangle| \leq ||\mathbf{x}||\ ||\mathbf{y}||$.
- The *cosine* of the angle between $\mathbf{x} \neq 0$ and $\mathbf{y} \neq 0$: $\cos\theta := \frac{\langle \mathbf{x}, \mathbf{y}\rangle}{||\mathbf{x}||\ ||\mathbf{y}||}$.
- $\mathbf{x}$ and $\mathbf{y}$ are *orthogonal* if: $\langle \mathbf{x}, \mathbf{y}\rangle = 0$.
- Two subspace $\mathbf{X}, \mathbf{Y}$ of $\mathbf{V}$ are *orthogonal* if any $\mathbf{x} \in \mathbf{X}$ is orthogonal to any $\mathbf{y} \in \mathbf{Y}$.
- The Parallelogram Law $||\mathbf{u}+\mathbf{v}||^2 = \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}\rangle = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 + 2\langle \mathbf{u}, \mathbf{v}\rangle$.
- The Pythagorean Law: $\langle \mathbf{u}, \mathbf{v}\rangle = 0 \Rightarrow ||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$.
- Scalar projection of $\mathbf{u}$ on $\mathbf{v} \neq 0$: $\frac{\langle \mathbf{u}, \mathbf{v}\rangle}{||\mathbf{v}||}$. Vector projection of $\mathbf{u}$ on $\mathbf{v} \neq 0$: $\frac{\langle \mathbf{u}, \mathbf{v}\rangle \mathbf{v}}{\langle \mathbf{v}, \mathbf{v}\rangle}$.
- The distance between $\mathbf{u}$ and $\mathbf{v}$ is defined by $||\mathbf{u} - \mathbf{v}||$.

4.21 Orthonormal sets in IPS

Let $\mathbf{V}$ be an inner product space (*IPS*). $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbf{V}$ is called an *orthogonal set* (OS) if $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ if $\mathbf{i} \neq j$, i.e. any two vectors in this set is an orthogonal pair. As in the case of dot product in $\mathbb{R}^n$ we have.
Theorem. An orthogonal set of nonzero vectors is linearly independent.

$\mathbf{v}_1, ..., \mathbf{v}_n \in \mathbf{V}$ is called an *orthonormal set* (ONS) if $\mathbf{v}_1, ..., \mathbf{v}_n$ is an orthogonal set and each $\mathbf{v}_i$ has length 1, i.e. $\mathbf{v}_1, \ldots, \mathbf{v}_n$ ONS $\iff$ $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \delta_{ij}$ for $i, j = 1, ..., n$.
Example: In $\mathrm{C}[-\pi, \pi]$ with $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$ the set

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx$$

is a nonzero ONS.

An orthonormal basis in $\mathrm{C}[-\pi, \pi]$ is

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \ldots, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin nx}{\sqrt{\pi}}, \ldots.$$

4.22 Fourier series

Every $f(x) \in \mathrm{C}[-\pi, \pi]$ can be expanded in Fourier series

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx),$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx.$$

a_n, b_n are called the even and the odd Fourier coefficients of f respectively.
Parseval equality is

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right) = \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2\, dx.$$

Dirichlet's theorem: If $f \in \mathrm{C}^1((-\infty, \infty))$ and $f(x + 2\pi) = f(x)$, i.e. f is differentiable and periodic, then the Fourier series converge absolutely for each $x \in \mathbb{R}$ to $f(x)$.

This is an infinite version of the identity Theorem 4.12, part *1*, $\mathbf{u} = \sum_{i=1}^{\infty} \langle \mathbf{u}, \mathbf{v}_i \rangle \mathbf{v}_i$ where $\mathbf{v}_1, \ldots, \mathbf{v}_n, \ldots$ is an orthonormal basis in a complete IPS. Such a complete infinite dimensional IPS is called a Hilbert space.

4.23 Short biographies of related mathematcians

4.23.1 Johann Carl Friedrich Gauss

Born: 30 April 1777 in Brunswick, Duchy of Brunswick (now Germany). Died: 23 Feb 1855 in Göttingen, Hanover (now Germany).

The method of least squares, established independently by two great mathematicians, Adrien Marie Legendre (1752 – 1833) of Paris and Carl Friedrich Gauss.

In June 1801, Zach, an astronomer whom Gauss had come to know two or three years previously, published the orbital positions of Ceres, a new "small planet" which was discovered by G Piazzi, an Italian astronomer on 1 January, 1801. Unfortunately, Piazzi had only been able to observe 9 degrees of its orbit before it disappeared behind the Sun. Zach published several predictions of its position, including one by Gauss which differed greatly from the others. When Ceres was rediscovered by Zach on 7 December 1801 it was almost exactly where Gauss had predicted. Although he did not disclose his methods at the time, Gauss had used his least squares approximation method.

http://www-history.mcs.st-and.ac.uk/Biographies/Gauss.html

4.23.2 Augustin Louis Cauchy

Born: 21 Aug 1789 in Paris, France. Died: 23 May 1857 in Sceaux (near Paris), France. His achievement is summed as follows:- "... Cauchy's creative genius found broad expression not only in his work on the foundations of real and complex analysis, areas to which his name is inextricably linked, but also in many other fields. Specifically, in this connection, we should mention his major contributions to the development of mathematical physics and to theoretical mechanics... we mention ... his two theories of elasticity and his investigations on the theory of light, research which required that he develop whole new mathematical techniques such as Fourier transforms, diagonalisation of matrices, and the calculus of residues."

Cauchy was first to state the Cauchy-Schwarz inequality, and stated it for sums.

http://www-history.mcs.st-and.ac.uk/Biographies/Cauchy.html

4.23.3 Hermann Amandus Schwarz

Born: 25 January 1843 in Hermsdorf, Silesia (now Poland). Died: 30 November 1921 in Berlin, Germany

His most important work is a Festschrift for Weierstrass's 70th birthday. @articleSchwarz1885, author = "H. A. Schwarz", title = "Ueber ein die Flächen kleinsten Flächeninhalts betreffendes Problem der Variationsrechnung", journal = "Acta societatis scientiarum Fennicae", volume = "XV", year = 1885, pages = "315–362" Schwarz answered the question of whether a given minimal surface really yields a minimal area. An idea from this work, in which he constructed a function using successive approximations, led Emile Picard to his existence proof for solutions of differential equations. It also contains the inequality for integrals now known as the "Schwarz inequality". Schwarz was the third person to state the Cauchy-Schwarz inequality, stated it for integrals over surfaces.

4.23.4 Viktor Yakovlevich Bunyakovsky

Born: 16 December 1804 in Bar, Podolskaya gubernia, (now Vinnitsa oblast), Ukraine. Died: 12 December 1889 in St. Petersburg, Russia. Bunyakovskii was first educated at home and then went abroad, obtaining a doctorate from Paris in 1825 after working under Cauchy.

Bunyakovskii published over 150 works on mathematics and mechanics. He is best known in Russia for his discovery of the Cauchy-Schwarz inequality, published in a monograph in 1859 on inequalities between integrals. This is twenty-five years before Schwarz's work. In the monograph Bunyakovskii gave some results on the functional form of the inequality.

@articleBunyakovskii1859, author = " V. Bunyakovski**ui**", title = "Sur quelques inégalités concernant les intégrales ordinaires et les intégrales aux différences finies", journal = "Mém. Acad. St. Petersbourg", year = 1859, volume = 1

4.23.5 Gram and Schmidt

Jorgen Pedersen Gram. Born: 27 June 1850 in Nustrup (18 km W of Haderslev), Denmark. Died: 29 April 1916 in Copenhagen, Denmark. Gram is best remembered for the Gram-Schmidt orthogonalization process which constructs an orthogonal set of from an independent one. The process seems to be a result of Laplace and it was essentially used by Cauchy in 1836.

http://www-history.mcs.st-and.ac.uk/Biographies/Gram.html

Erhard Schmidt. Born: 13 Jan 1876 in Dorpat, Germany, (now Tartu, Estonia). Died: 6 Dec 1959 in Berlin, Germany. Schmidt published a two part paper on integral equations in 1907 in which he reproved Hilbert's results in a simpler fashion, and also with less restrictions. In this paper he gave what is now called the Gram-Schmidt orthonormalisation process for constructing an orthonormal set of functions from a linearly independent set.

http://www-history.mcs.st-and.ac.uk/Biographies/Schmidt.html

4.23.6 Jean Baptiste Joseph Fourier

Born: 21 March 1768 in Auxerre, Bourgogne, France. Died: 16 May 1830 in Paris, France. It was during his time in Grenoble that Fourier did his important mathematical work on the theory of heat. His work on the topic began around 1804 and by 1807 he had completed his important memoir "On the Propagation of Heat in Solid Bodies". The memoir was read to the Paris Institute on 21 December 1807 and a committee consisting of Lagrange, Laplace, Monge and Lacroix was set up to report on the work. Now this memoir is very highly regarded but at the time it caused controversy.

There were two reasons for the committee to feel unhappy with the work. The first objection, made by Lagrange and Laplace in 1808, was to Fourier's expansions of functions as trigonometrical series, what we now call Fourier series. Further clarification by Fourier still failed to convince them.

http://www-history.mcs.st-and.ac.uk/Biographies/Fourier.html

4.23.7 J. Peter Gustav Lejeune Dirichlet

Born: 13 Feb 1805 in Düren, French Empire, (now Germany). Died: 5 May 1859 in Göttingen, Hanover (now Germany). Dirichlet is also well known for his papers on conditions for the convergence of trigonometric series and the use of the series to represent arbitrary functions. These series had been used previously by Fourier in solving differential equations. Dirichlet's work is published in Crelle's Journal in 1828. Earlier work by Poisson on the convergence of Fourier series was shown to be non-rigorous by Cauchy. Cauchy's work itself was shown to be in error by Dirichlet who wrote of Cauchy's paper:- "The author of this work himself admits that his proof is defective for certain functions for which the convergence is, however, incontestable". Because of this work Dirichlet is considered the founder of the theory of Fourier series.

http://www-history.mcs.st-and.ac.uk/Biographies/Dirichlet.htm

4.23.8 David Hilbert

Born: 23 Jan 1862 in Königsberg, Prussia, (now Kaliningrad, Russia). Died: 14 Feb 1943 in Göttingen, Germany.

Today Hilbert's name is often best remembered through the concept of Hilbert space. Irving Kaplansky, writing in [2], explains Hilbert's work which led to this concept: "Hilbert's work in integral equations in about 1909 led directly to 20th-century research in functional analysis (the branch of mathematics in which functions are studied collectively). This work also established the basis for his work on infinite-dimensional space, later called Hilbert space, a concept that is useful in mathematical analysis and quantum mechanics. Making use of his results on integral equations, Hilbert contributed to the development of mathematical physics by his important memoirs on kinetic gas theory and the theory of radiations."

http://www-history.mcs.st-and.ac.uk/Biographies/Hilbert.html

5 DETERMINANTS

5.1 Introduction to determinant

For a square matrix $A \in \mathbb{R}^{n\times n}$ the determinant of A, denoted by det A, (in Hefferon book $|A| :=$ det A), is a real number such that det A $\neq 0 \iff A$ is invertible.

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \mathrm{ad} - \mathrm{bc}, \tag{5.34}$$

$$\det \begin{bmatrix} 2 & -7 \\ 3 & -1 \end{bmatrix} - 19$$

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \mathrm{aei} + \mathrm{bfg} + \mathrm{cdh} - \mathrm{ceg} - \mathrm{afh} - \mathrm{bdi}. \tag{5.35}$$

A way to remember the formula for the determinant of a matrix of order 3 is

$$\begin{bmatrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{bmatrix}.$$

The product of diagonals starting from a, b, c, going south west have positive signs, the products of diagonals starting from c, a, b and going south east have negative signs.

The next rules are

I. The determinant of diagonal matrix, upper triangular matrix and lower triangular is equal to the product of the diagonal entries.

II. $\det \mathrm{A} \neq 0 \iff$ The Row Echelon Form of A has the maximal number of possible pivots $\iff$ The Reduced Row Echelon Form of A is the identity matrix.

A is called *singular* if $\det \mathrm{A} = 0$.

III. The determinant of a matrix having at least one zero row or column is 0.

IV. $\det \mathrm{A} = \det \mathrm{A}^\top$: The determinant of A is equal to the determinant of $A^\top$.

V. $\det \mathrm{AB} = \det \mathrm{A} \det \mathrm{B}$: The determinant of the product of matrices is equal to the product of determinants.

VI. If A is invertible then $\det \mathrm{A}^{-1} = \frac{1}{\det \mathrm{A}}$.

$$I = A^{-1}A \Rightarrow 1 = \det \mathrm{I} = \det(\mathrm{A}^{-1}\mathrm{A}) = \det \mathrm{A}^{-1} \det \mathrm{A}$$

(We will demonstrate some of these properties later.)

5.2 Determinant as a multilinear function

Claim 1: View $A \in \mathbb{R}^{n \times n}$ as composed of n-columns $A = [\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n]$. Then det A is a multilinear function in each column separately. Thar is, fix all columns except the column $\mathbf{c}_i$. Let $\mathbf{c}_i = a\mathbf{x} + b\mathbf{y}$, where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$. Then

$$\begin{aligned} &\det [\mathbf{c}_1, \ldots, \mathbf{c}_{\mathrm{i}-1}, \mathrm{a}\mathbf{x} + \mathrm{b}\mathbf{y}, \mathbf{c}_{\mathrm{i}+1}, \ldots, \mathbf{c}_\mathrm{n}] = \\ &a \det [\mathbf{c}_1, \ldots, \mathbf{c}_{\mathrm{i}-1}, \mathbf{x}, \mathbf{c}_{\mathrm{i}+1}, \ldots, \mathbf{c}_\mathrm{n}] + \mathrm{b} \det [\mathbf{c}_1, \ldots, \mathbf{c}_{\mathrm{i}-1}, \mathbf{y}, \mathbf{c}_{\mathrm{i}+1}, \ldots, \mathbf{c}_\mathrm{n}] \end{aligned}$$

for each $i = 1, \ldots, n$.

Claim 2: det A is a skew-symmetric, (anti-symmetric): The exchange of any two columns of A changes the sign of determinant. For example: $\det [\mathbf{c}_2, \mathbf{c}_1, \ldots, \mathbf{c}_n] = -\det [\mathbf{c}_1, \mathrm{c}_2, \ldots, \mathbf{c}_n]$. (The skew symmetricity yields that the determinant of A is zero if A has two identical columns.)

Claim 3: $\det \mathrm{I_n} = 1$.

Claim 4: These three properties determine uniquely the determinant function.

Remark: The above claims hold for rows as in Hefferon.

Clearly, these properties hold for the determinant of matrices of order 2: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \mathrm{ad} - \mathrm{bc}$. It is linear in the columns $\mathbf{c}_1 = (a, c)^\top, \mathbf{c}_2 = (b, d)^\top$ and in the rows $(a, b), (c, d)$. Also $\det \begin{bmatrix} b & a \\ d & c \end{bmatrix} = \det \begin{bmatrix} c & d \\ a & b \end{bmatrix} = \mathrm{bc} - \mathrm{ad} = -\det \mathrm{A}$

Proposition 5.1 *Let $A \in \mathbb{R}^{n\times n}$ then* $\det \mathrm{A} = 0$ *if one of the following conditions hold.*

1. *A has a zero column.*
2. *A has two identical columns.*
3. *A has two linearly dependent columns.*
4. *A has a zero row.*
5. *A has two identical rows.*
6. *A has two linearly dependent rows.*

Proof.

1. Fix all columns except the zero one, which is column i. Since the determinant is a linear function in the column i its value for the zero vector must be zero.

2. Assume that $\mathbf{c}_i = \mathbf{c}_j$ for some $i \neq j$. If we interchange this two columns we still have the same matrix A. One the other hand the skew-symmetricity property of det A implies that $\det \mathrm{A} = -\det \mathrm{A}$. Hence $2\det \mathrm{A} = 0 \Rightarrow \det \mathrm{A} = 0$.

3. Suppose that $\mathbf{c}_i, \mathbf{c}_j$ linearly dependent. So without loss of generality we ay assume that $\mathbf{c}_j = a\mathbf{c}_i$. Let B be the matrix where we replace the column $\mathbf{c}_j$ in A by $\mathbf{c}_i$. Fix all columns of A except the column j. The multilinearity of det A implies that $\det \mathrm{A} = \mathrm{a}\det \mathrm{B}$. By the previous result $\det \mathrm{B} = 0$ since B has two identical columns. Hence $\det \mathrm{A} = 0$.

The other claims of the proposition follows from the above vanishing properties of the determinant by considering $A^\top$ and recalling that $\det A^\top = \det A$.
□

5.3 Computing determinants using elementary row or columns operations

Proposition 5.2 *Let $A \in \mathbb{R}^{n\times n}$. Let B and C be the matrix obtained from A using an elementary row and column operations respectively denoted by E. So $B = EA, C = AE$. Then.*

1. $\det B = \det C = -\det A$ *if B is obtained from A by interchanging two different rows, and C is obtained from A by interchanging two different columns. (Elementary row or column operation of type I.)*

2. $\det B = \det C = a \det A$ *if B is obtained from A by multiplying one row of A by a, and C is obtained from A by multiplying one column of A by a. (Elementary row or column operation of type II.)*

3. $\det B = \det C = \det A$ *if B is obtained from A by adding to one row a multiple of another row, and C is obtained from A by adding one column a multiple of another column. (Elementary row or column operation of type III.)*

Proof. We prove the statements of the Proposition for the columns. The statements for the rows follows from the statement for the columns and the equality $\det A^\top = \det A$.

1. Follows from the skew-symmetricity of the determinant function as a function on columns.

2. Follows from the multilinearity of the determinant as a function in columns. Let F be obtained from A by adding $a\mathbf{c}_i$ to the column $\mathbf{c}_j$. Denote by B the obtained by replacing the column $\mathbf{c}_j$ in A by the column $a\mathbf{c}_i$. The linearity of the determinant in the j column implies $\det F = \det A + \det B$. Part *3* of Proposition 5.1 yields that $\det B = 0$. Hence $\det F = \det A$.

□

Recall that any elementary matrix E can be obtained form the identity matrix using the elementary row operation represented by E, i.e. $E = EI$. Hence we deduce from the above Proposition.

Corollary 5.3 *1.* $\det E_I = -1$ *where E_I corresponds to interchanging two rows: $R_i \leftrightarrow R_j$. (Example: $R_1 \leftrightarrow R_1$ corresponds to* $\det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 0 \cdot 0 - 1 \cdot 1 = -1$.*)*

2. $\det \mathrm{E_{II}} = \mathrm{a}$ *where E_{II} corresponds to multiplying a row by a: $R_i \to aR_i$. (Example: $R_2 \to aR_2$ corresponds to* $\det \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} = \mathrm{a}.$*)*

3. $\det \mathrm{E_{III}} = 1$ *where E_{III} corresponds to adding to one row a multiple of another row: $R_i + aR_j \to R_i$. (Example: $R_2 + aR_1 \to R_2$ corresponds to* $\det \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} = 1.$

Theorem 5.4 *Let $A \in \mathbb{R}^{n\times n}$ and perform k ERO:*

$$A \overset{ERO_1}{\to} A_1 \overset{ERO_2}{\to} A_2 \overset{ERO_3}{\to} \ldots A_{k-1} \overset{ERO_k}{\to} A_k \tag{5.36}$$

Assume that the elementary row operation number i, denoted as ERO_i is given by the elementary matrix E_i. Then

$$\det \mathrm{A_k} = (\det \mathrm{E_k})(\det \mathrm{E_{k-1}}) \ldots (\det \mathrm{E_1}) \det \mathrm{A}. \tag{5.37}$$

Proof. Clearly $A_i = E_i A_{i-1}$ where $A_0 = A$. Combine Proposition 5.2 with Corollary 5.3 to deduce that $\det \mathrm{A_i} = (\det \mathrm{E_i}) \det \mathrm{A_{i-1}}$ for $i = 1, \ldots, k$. This implies (5.37). □

Theorem 5.5 *Let $A \in \mathbb{R}^{n\times n}$. Then the following conditions hold.*

1. *Assume that A is invertible, i.e. there exists as sequence of elementary row operations such that* (5.36) *holds, where $A_k = I_n$ is the RREF of A. Then*

$$A^{-1} = E_k E_{k-1} \ldots E_1, \quad A = E_1^{-1} E_2^{-1} \ldots E_k^{-1} \tag{5.38}$$
$$\det \mathrm{A} = (\det \mathrm{E_1})^{-1} \ldots (\det \mathrm{E_k})^{-1} \neq 0. \tag{5.39}$$

2. $\det \mathrm{A} = 0$ *if and only if* $\operatorname{rank} A < n$*, i.e. A is singular.*

3. *The determinant of an upper or lower triangular matrix is a product of its diagonal entries.*

Proof. Use elementary row operations as (5.36) to bring A to its RREF, which is A_k. If $\operatorname{rank} A = n$ then $A_k = I_n$ and we have equality (5.38). Use Theorem 5.4 and the assumption that $\det \mathrm{I_n} = 1$ to deduce *1.* (Observe that Corollary 5.3 yields that the determnant of an elementary matrix is nonzero.)

Assume now that $\operatorname{rank} A < n$. Then the last row of A_k is zero. Use Proposition 5.1 to deduce that $\det \mathrm{A_k} = 0$. Combine (5.37) with the fact that the determinant of an elementary matrix is nonzero to deduce *2* that $\det \mathrm{A} = 0$.

Assume that A is a lower triangular with the diagonal entries $a_{11}, \ldots, a_{nn}$ if $a_{ii} = 0$ for some i, then A has at most $n-1$ pivots in its REF. Hence

$\operatorname{rank} A < n$ and $\det A = 0 = \prod_{i=1}^{n} a_{ii}$. Suppose that all diagonal entries of A are different from zero. First divide row i by a_{ii} to obtain the lower trianguar matrix B with 1 on the diagonal. Use Theorem 5.4 to deduce that $\det A = (\prod_{i=1}^{n} a_{ii}) \det B$. Now bring B to its RREF I_n by ding Gauss elimination, which consists of adding to row j a multiple of row i, where $i < j$. According to Proposition (5.2) such elementary row operations do not change the value of the determinant. Hence $\det B = \det I_n = 1$. Similar arguments apply if A is an upper triangular matri. This proves *3*. $\square$

Proposition 5.6 *Let $A \in \mathbb{R}^{n \times n}$ be an invertible matrix such that $A = F_1 F_2 \ldots F_k$, where $F_1, \ldots, F_k$ are elementary matrices. Then*

$$\det A = (\det F_1)(\det F_2) \ldots (\det F_k). \tag{5.40}$$

Proof. Let $E_i = F_i^{-1}$. Note that E_i is also an elementary matrix. Furthermore , Corollary 5.3 yields that $\det E_i = (\det F_i)^{-1}$. Clearly (5.38) holds. Apply Part *1* of Theorem 5.5 to deduce (5.40). $\square$

Theorem 5.7 *Let $A, B \in \mathbb{R}^{n \times n}$. Then* $\det (AB) = (\det A)(\det B)$.

Proof. Assume first that B is singular, i.e. $\det B = 0$. So $\operatorname{rank} B < n$ and there exists a nonzero $\mathbf{x} \in \mathbb{R}^n$ such that $B\mathbf{x} = \mathbf{0}$. Then $(AB)\mathbf{x} = A(B\mathbf{x}) = A\mathbf{0} = \mathbf{0}$. So AB is singular, i.e. $\operatorname{rank}(AB) < n$. Hence $= 0 \det (AB) = (\det A) 0 = (\det A)(\det B)$.

Assume next that B is invertible and A is singular. So $A\mathbf{y} = \mathbf{0}$ for some nonzero $\mathbf{y}$. Then $(AB)(B^{-1}\mathbf{y}) = A\mathbf{y} = 0$. As $B^{-1}\mathbf{y} \neq \mathbf{0}$ it follows that AB singular. So $= \det (AB) = 0(\det B) = (\det A)(\det B)$.

It is left to prove the theorem when A and B are invertible. Then

$$A = F_1 F_2 \ldots F_k, \quad G_1 G_2 \ldots G_l,$$

where each F_i and G_j is elementary. So $AB = F_1 \ldots F_k G_1 \ldots G_l$. By Proposition 5.6

$$\det (AB) = (\prod_{i=1}^{k} \det F_i)(\prod_{j=1}^{k} l \det G_j) = (\det A)(\det B).$$

$\square$

5.4 Permutations

Definition: A bijection, i.e. $1-1$ and onto map, $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$, is called a permutation of the set $\{1, 2, \ldots, n\}$. The set of all permutations of $\{1, 2, \ldots, n\}$ is called the symmetric group on n-elements, and is denoted by S_n.

$\sigma(i)$ is the image of the number i for $i = 1, \ldots, n$. (Note that $1 \leq \sigma(i) \leq n$ for $i = 1, \ldots, n$. $\iota \in \mathrm{S}_n$ is called the identity element, (or map), if $\iota(i) = i$ for $i = 1, \ldots, n$.

Proposition 5.8 *The number of elements in* S_n *is* $n! = 1 \cdot 2 \cdots n$.

Proof. $\sigma(1)$ can have n choices: $1, \ldots, n$. $\sigma(2)$ can have all choices: $1, \ldots, n$ except $\sigma(1)$, i.e. $n-1$ choices. $\sigma(3)$ can have all choices except $\sigma(1), \sigma(2)$, i.e. $\sigma(3)$ has $n-3$ choices. Hence total number of σ-s is $n(n-1)\ldots 1 = n!$. $\square$

Definition. $\tau \in \mathrm{S}_n$ is a transposition, if there exists $1 \leq i < j \leq n$ so that $\tau(i) = j, \tau(j) = i$, and $\tau(k) = k$ for all $k \neq i, j$.

Since $\sigma, \omega \in \mathrm{S}_n$ are bijections, we can compose them $\sigma \circ \omega$, which is an element in S_n, $((\sigma \circ \omega)(i) = \sigma(\omega(i)))$. We denote this composition by $\sigma\omega$ and view this composition as a product in S_n.

Claim. Any $\sigma \in \mathrm{S}_n$ is a product of transpositions. There are many different products of transpositions to obtain σ. All these products of transpositions have the same parity of elements. (Either all products have even number of elements only, or have odd numbers of elements only.)

Definiiton. For $\sigma \in \mathrm{S}_n$, $\mathrm{sgn}(\sigma) = 1$ if σ is a product of even number of transpositions. $\mathrm{sgn}(\sigma) = -1$ if σ is a product of odd number of transpositions.

Proposition 5.9 $\mathrm{sgn}(\sigma\omega) = \mathrm{sgn}(\sigma)\mathrm{sgn}(\omega)$.

Proof. Express σ and ω is a product of transpositions. Then $\sigma\omega$ is also a product of transpositions. Now count the parity. $\square$

5.5 S_2

S_2 consists of two element: the identity ι: $\iota(1) = 1, \iota(2) = 2$ and the transposition τ: $\tau(1) = 2, \tau(2) = 1$. Note $\tau^2 = \tau\tau = \iota$ since $\tau(\tau(1)) = \tau(2) = 1, \tau(\tau(2)) = \tau(1) = 2$. So ι is a product of any any even number of τ, i.e. $\iota = \tau^{2m}$, while $\tau = \tau^{2m+1}$ for $m = 0, 1, \ldots$.

Note that this is true for any transposition $\tau \in \mathrm{S}_n, n \geq 2$. Thus $\mathrm{sgn}(\iota) = 1, \mathrm{sgn}(\tau) = -1$ for any $n \geq 2$.

5.6 S_3

S_3 consists of 6 elements. Identity: ι. There are three transpositions in S_3:

$$\tau_1(1) = 1, \quad \tau_1(2) = 3, \quad \tau_1(3) = 2,$$
$$\tau_2(1) = 3, \quad \tau_2(2) = 2, \quad \tau_2(3) = 1,$$
$$\tau_3(1) = 2, \quad \tau_3(2) = 1, \quad \tau_3(3) = 3.$$

(τ_j fixes j.)

There are two cyclic permutations

$$\sigma(1) = 2, \quad \sigma(2) = 3, \quad \sigma(3) = 1,$$
$$\omega(1) = 3, \quad \omega(2) = 1, \quad \omega(3) = 2.$$

Note $\omega\sigma = \sigma\omega = \iota$, i.e. $\sigma^{-1} = \omega$. It is straightforward to show

$$\sigma = \tau_1\tau_2 = \tau_2\tau_3, \quad \omega = \tau_2\tau_1 = \tau_3\tau_2.$$

So

$$\mathrm{sgn}(\iota) = \mathrm{sgn}(\sigma) = \mathrm{sgn}(\omega) = 1, \quad \mathrm{sgn}(\tau_1) = \mathrm{sgn}(\tau_2) = \mathrm{sgn}(\tau_3) = -1.$$

5.7 Rigorous definition of determinant

For

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{m2} & \dots & a_{nn} \end{bmatrix} \in \mathbb{R}^{n\times n}$$

define

$$\mathrm{det\ A} = \sum_{\sigma\in \mathrm{S_n}} \mathrm{sgn}(\sigma)\mathrm{a}_{1\sigma(1)}\mathrm{a}_{2\sigma(2)}\dots\mathrm{a}_{\mathrm{n}\sigma(\mathrm{n})}. \tag{5.41}$$

Note that det A has $n!$ summands in the above sum.

5.8 Cases $n = 2, 3$

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \mathrm{a}_{1\iota(1)}\mathrm{a}_{2\iota(2)} - \mathrm{a}_{1\tau(1)}\mathrm{a}_{2\tau(2)} = \mathrm{a}_{11}\mathrm{a}_{22} - \mathrm{a}_{12}\mathrm{a}_{21},$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} =$$
$$a_{1\iota(1)}a_{2\iota(2)}a_{3\iota(3)} + a_{1\sigma(1)}a_{2\sigma(2)}a_{3\sigma(3)} + a_{1\omega(1)}a_{2\omega(2)}a_{3\omega(3)}$$
$$-a_{1\tau_1(1)}a_{2\tau_1(2)}a_{3\tau_1(3)} - a_{1\tau_2(1)}a_{2\tau_2(2)}a_{3\tau_2(3)} - a_{1\tau_3(1)}a_{2\tau_3(2)}a_{3\tau_3(3)} =$$
$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$-a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33}.$$

5.9 Minors and Cofactors

For $A \in \mathbb{R}^{n\times n}$ the matrix $M_{ij} \in \mathbb{R}^{(n-1)\times(n-1)}$ denotes the submatrix of A obtained from A by deleting row i and column j. The determinant of M_{ij} is called (i,j)-minor of A. The cofactor A_{ij} is defined to be $(-1)^{i+j}\mathrm{det\ M_{ij}}$.

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, M_{32} = \begin{bmatrix} a & c \\ d & f \end{bmatrix},\ A_{32} = -af + cd.$$

Expansion of the determinant by row i, (*Laplace expansion*)

$$\mathrm{det\ A = a_{i1}A_{i1} + a_{i2}A_{i2} + ... + a_{in}A_{in}} = \sum_{\mathrm{j=1}}^{\mathrm{n}} \mathrm{a_{ij}A_{ij}}.$$

Expansion of the determinant by column p:

$$\mathrm{det\ A = a_{1p}A_{1p} + a_{2p}A_{2p} + ... + a_{np}A_{np}} = \sum_{\mathrm{j=1}}^{\mathrm{n}} \mathrm{a_{jp}A_{jp}}.$$

One can compute also the determinant of A using repeatedly the row or column expansions

Warning: Computationally the method of using row/column expansion is very inefficient. Expansion of determinant by row/column is used primarily for theoretical computations.

5.10 Examples of Laplace expansions

Expand the determinant of $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ by the second row:

$$\mathrm{det\ A = dA_{21} + eA_{22} + fA_{23}} =$$
$$d(-1)\mathrm{det} \begin{bmatrix} b & c \\ h & i \end{bmatrix} + \mathrm{e\ det} \begin{bmatrix} a & c \\ g & i \end{bmatrix} + \mathrm{f}(-1)\mathrm{det} \begin{bmatrix} a & b \\ g & h \end{bmatrix} =$$
$$(-d)(bi - hc) + e(ai - cg) + (-f)(ah - bg) = aei + bfg + cdh - ceg - afh - bdi.$$

Find det $\begin{bmatrix} -1 & 1 & -1 & 3 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ -1 & -1 & -1 & 2 \end{bmatrix}$. Expand by the row or column which has the maximal number of zeros. We expand by the first column: $\mathrm{det\ A} = \mathrm{a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} + a_{41}A_{41}} = \mathrm{a_{11}A_{11} + a_{41}A_{41}}$ since $\mathbf{a}_{21} = \mathbf{a}_{31} = 0$. Observe that $(-1)^{1+1} = 1, (-1)^{1+4} = -1$. Hence

$$\mathrm{det\ A} = (-1)\mathrm{det} \begin{bmatrix} 3 & 1 & 1 \\ 0 & 2 & 2 \\ -1 & -1 & 2 \end{bmatrix} + (-1)(-1)\mathrm{det} \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}.$$

Expand the first determinant by the second row and the second determinant by the third row.

$$\det A = (-1)\big(3 \det \begin{bmatrix} 2 & 2 \\ -1 & 2 \end{bmatrix} + (-1)\det \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\big) +$$
$$\big((-2)\det \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} + 2 \det \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}\big) = -18 + 16 + 8 = 6.$$

Another way to find det A, $A = \begin{bmatrix} -1 & 1 & -1 & 3 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ -1 & -1 & -1 & 2 \end{bmatrix}$.

Perform ERO: $R_4 - R_1 \to R_4$ to obtain $B = \begin{bmatrix} -1 & 1 & -1 & 3 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & -2 & 0 & -1 \end{bmatrix}$. So det A = det B. Expand det B by the first column to obtain det B = −det C, $C = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 2 & 2 \\ -2 & 0 & -1 \end{bmatrix}$. Perform the ERO $R_1 - 0.5R_2 \to R_1$ to obtain $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 2 \\ -2 & 0 & -1 \end{bmatrix}$. Expand det D by the first row to get det D $= (3)(2 \cdot (-1) - 2 \cdot 0) = -6$. Hence det A = 6.

5.11 Adjoint Matrix

For $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$ the adjoint matrix is defined as

$$\text{adj A} = \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{bmatrix},$$

where A_{ij} is the (i, j) cofactor of A. Note that the i-th row of adj A is $(A_{1i}\ A_{2i} \dots A_{ni})$.

Examples:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \text{adj A} = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix},$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \text{adj A} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix},$$

$$A_{33} = \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \mathrm{a}_{11}\mathrm{a}_{22} - \mathrm{a}_{12}\mathrm{a}_{21},$$

$$A_{12} = -\det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} = -\mathrm{a}_{21}\mathrm{a}_{33} + \mathrm{a}_{23}\mathrm{a}_{31}.$$

A way to remember to get the adjoint matrix correctly:

$$\text{adj A} = \begin{bmatrix} A_{11} & A_{12} & ... & A_{1n} \\ A_{21} & A_{22} & ... & A_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{n1} & A_{n2} & ... & A_{nn} \end{bmatrix}^{\top} = \begin{bmatrix} A_{11} & A_{21} & ... & A_{n1} \\ A_{12} & A_{22} & ... & A_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ A_{1n} & A_{2n} & ... & A_{nn} \end{bmatrix}$$

5.12 The properties of the adjoint matrix

Proposition 5.10 *Let* $A \in \mathbb{R}^{n\times n}$*. Then* A adj A $=$ (adjA)A $=$ (det A)$\mathrm{I_n}$*,*

Proof. Consider the (i,j) element of the product A adj A: $a_{i1}A_{j1}+a_{i2}A_{j2}+ ... + a_{in}A_{jn}$. Assume first that $i = j$. Then this sum is the expansion of the determinant of A by $i-th$ row. Hence it is equal to det A, which is the (i,i) entry of the diagonal matrix (det A)I.

Assume now that $i \neq j$. Then the above sum is the expansion of the determinant of a matrix C obtained from A by replacing the row j in A by the row i of A. Since C has two identical row, it follows that det C = 0. This shows A adj A = (det A)I. Similarly (adj A)A = (det A)I. □

Corollary 5.11 det A $\neq 0 \Rightarrow \mathrm{A}^{-1} = \frac{1}{\det \mathrm{A}}$adj A.

Example: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$. Find adj A and A^{-1}.

$$A_{11} = 24, A_{12} = -0, A_{13} = 0, A_{21} = -12, A_{22} = 6,$$
$$A_{23} = -0, A_{31} = 10 - 12 = -2, A_{32} = -5, A_{33} = 4,$$
$$\text{adj A} = \begin{bmatrix} 24 & 0 & 0 \\ -12 & 6 & 0 \\ -2 & -5 & 4 \end{bmatrix}^\top = \begin{bmatrix} 24 & -12 & -2 \\ 0 & 6 & -5 \\ 0 & 0 & 4 \end{bmatrix}.$$

Since A is upper triangular det A $= 1 \cdot 4 \cdot 6 = 24$ and

$$A^{-1} = \frac{1}{24} \begin{bmatrix} 24 & -12 & -2 \\ 0 & 6 & -5 \\ 0 & 0 & 4 \end{bmatrix}.$$

5.13 Cramer's Rule

Theorem 5.12 *Consider the linear system of n equations with n unknowns:*

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n = b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n = b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \ldots & + & a_{nn}x_n = b_n \end{array}$$

Let $A \in \mathbb{R}^{n \times n}, \mathbf{b} = (b_1, ..., b_n)^\top$ be the coefficient matrix and the column vector corresponding to the right-hand side of these system. That is the above system is $A\mathbf{x} = b$, $\mathbf{x} = (x_1, ..., x_n)^\top$. Denote by $B_j \in \mathbb{R}^{n \times n}$ the matrix obtained from A by replacing the $j-th$ column in A by: $B_j =$

$$\begin{bmatrix} a_{11} & \ldots & a_{1(j-1)} & b_1 & a_{1(j+1)} & \ldots & a_{1n} \\ a_{21} & \ldots & a_{2(j-1)} & b_2 & a_{2(j+1)} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \ldots & a_{n(j-1)} & b_n & a_{n(j+1)} & \ldots & a_{nn} \end{bmatrix}.$$

Then $x_j = \frac{\text{det B}_\text{j}}{\text{det A}}$ for $j = 1, ..., n$.

Proof. Since det A $\neq 0$, $A^{-1} = \frac{1}{\text{det A}}$adj A. Hence the solution to the system $A\mathbf{x} = \mathbf{b}$ is given by: $A^{-1}\mathbf{x} = \frac{1}{\text{det A}}$adj A $\mathbf{b}$. Writing down the formula for the matrix adj A we get: $x_j = \frac{A_{1j}b_1 + A_{2j}b_2 + \ldots + A_{nj}b_n}{\text{det A}}$. The numerator of this quotient is the expansion of det B_j by the column j. □

Example: Find the value of x_2 in the system

$$\begin{array}{ccccc} x_1 & + & 2x_2 & - & x_3 = 0 \\ -2x_1 & - & 5x_2 & + & 5x_3 = 3 \\ 3x_1 & + & 7x_2 & - & 5x_3 = 0 \end{array}.$$

$$x_2 = \frac{\det \begin{bmatrix} 1 & 0 & -1 \\ -2 & 3 & 5 \\ 3 & 0 & -5 \end{bmatrix}}{\det \begin{bmatrix} 1 & 2 & -1 \\ -2 & -5 & 5 \\ 3 & 7 & -5 \end{bmatrix}}.$$

Expand the determinant of the denominator by the second column to obtain $\det \begin{bmatrix} 1 & 0 & -1 \\ -2 & 3 & 5 \\ 3 & 0 & -5 \end{bmatrix} = 3 \det \begin{bmatrix} 1 & -1 \\ 3 & -5 \end{bmatrix} = 3(-5+3) = -6$. On the coefficient matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ -2 & -5 & 5 \\ 3 & 7 & -5 \end{bmatrix}$ perform the ERO $R_1 + 3R_2 \to R_2, R_2 - 3R_1 \to R_3$ to obtain $A_2 = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 0 & 1 & -2 \end{bmatrix}$. Expand det A_2 by the first column to obtain det A $=$ det $A_2 = 1(2-3) = -1$. So $x_2 = 6$. (Note that A^{-1} was computed before in the notes. Check the answer by comparing it to $A^{-1}(0,3,0)^\top = (-9,6,3)^\top$.)

5.14 History of determinants

Historically, determinants were considered before matrices. Originally, a determinant was defined as a property of a system of linear equations. The determinant "determines" whether the system has a unique solution (which occurs precisely if the determinant is non-zero). In this sense, two-by-two determinants were considered by Cardano at the end of the 16th century and larger ones by Leibniz about 100 years later. Following him Cramer (1750) added to the theory, treating the subject in relation to sets of equations.

It was Vandermonde (1771) who first recognized determinants as independent functions. Laplace (1772) gave the general method of expanding a determinant in terms of its complementary minors: Vandermonde had already given a special case. Immediately following, Lagrange (1773) treated determinants of the second and third order. Lagrange was the first to apply determinants to questions outside elimination theory; he proved many special cases of general identities.

Gauss (1801) made the next advance. Like Lagrange, he made much use of determinants in the theory of numbers. He introduced the word determinants (Laplace had used resultant), though not in the present signification, but rather as applied to the discriminant of a quantic. Gauss also arrived at the notion of reciprocal (inverse) determinants, and came very near the multiplication theorem.

The next contributor of importance is Binet (1811, 1812), who formally stated the theorem relating to the product of two matrices of m columns and n rows, which for the special case of m = n reduces to the multiplication theorem. On the same day (Nov. 30, 1812) that Binet presented his paper to the Academy, Cauchy also presented one on the subject. (See Cauchy-Binet formula.) In this he used the word determinant in its present sense, summarized and simplified what was then known on the subject, improved the notation, and gave the multiplication theorem with a proof more satisfactory than Binet's. With him begins the theory in its generality.

Source: http://en.wikipedia.org/wiki/Determinant
(See section History)

6 Eigenvalues and Eigenvectors

6.1 Definition of eigenvalues and eigenvectors

Let $\mathbb{F}$ be the field, and $\mathbb{C}$ be the filed of complex numbers. Let $A \in \mathbb{F}^{n\times n}$. $\mathbf{x} \in \mathbb{F}^n$ is called an eigenvector (*characteristic vector*) if $\mathbf{x} \neq 0$ and there exists $\lambda \in \mathbb{C}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. λ is called an eigenvalue (*characteristic value* of A). In this Chapter we will deal mostly with the two fields $\mathbb{F} = \mathbb{R}, \mathbb{C}$. For the sake of generality we will state some results fr general fields $\mathbb{F}$.

Proposition 6.1 *λ is an eigenvalue of A if and only if* $\det(\mathrm{A} - \lambda\mathrm{I}) = 0$.

Proof. Let $B(\lambda) := A - \lambda I$. Then $\mathbf{x}$ is an eigenvector of A iff and only iff $\mathbf{x} \in N(B(\lambda))$, i.e. $\mathbf{x}$ is in the null space of $B(\lambda)$. Suppose first that $B(\lambda)\mathbf{x} = \mathbf{0}$. Since $\mathbf{x} \neq 0$, $B(\lambda)$ is singular, hence $\det \mathrm{B}(\lambda) = 0$. Vice versa suppose that $\det \mathrm{B}(\lambda) = 0$ for some λ. Then there exists a nonzero $\mathbf{x}$ such that $B(\lambda)\mathbf{x} = \mathbf{0}$ i.e. $\mathbf{x}$ is an eigenvector of A. $\square$

The polynomial $p(\lambda) := \det(\mathrm{A} - \lambda\mathrm{I})$ is called a characteristic polynomial of A.

$$p(\lambda) = (-1)^n(\lambda^n - \sigma_1\lambda^{n-1} + \sigma_2\lambda^{n-2} + \ldots + (-1)^n\sigma_n)$$

is a polynomial of degree n. The fundamental theorem of algebra states that $p(\lambda)$ has n roots (*eigenvalues*) $\lambda_1, \lambda_2, \ldots, \lambda_n$ and

$$p(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda).$$

Given an eigenvalue λ then a basis to the null space $\mathrm{N}(A - \lambda I)$ is a basis for the eigenspace of eigenvectors of A corresponding to λ.

6.2 Examples of eigenvalues and eigenvectors

Example 1. Consider the Markov chain given by $A = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}$. (%70 of healthy remain healthy and %20 of sick recover.)

$$A - \lambda I = \begin{bmatrix} 0.7-\lambda & 0.2 \\ 0.3 & 0.8-\lambda \end{bmatrix},$$
$$\det(\mathrm{A} - \lambda \mathrm{I}) = (0.7-\lambda)(0.8-\lambda) - 0.2 \cdot 0.3 = \lambda^2 - 1.5\lambda + 0.5$$

is the characteristic polynomial of A. So $\det(\mathrm{A} - \lambda\mathrm{I}) = (\lambda - 1)(\lambda - 0.5)$. Eigenvalues of A are the zeros of the characteristic polynomial, i.e. solutions of $\det(\mathrm{A} - \lambda\mathrm{I}) = 0$: $\lambda_1 = 1,\ \lambda_2 = 0.5$.

To find a basis for the null space of $A - \lambda_1 I = A - I$, denoted by $N(A - \lambda_1 I)$, we need to bring the matrix $A - I$ to RREF:
$A - I = \begin{bmatrix} -0.3 & 0.2 \\ 0.3 & -0.2 \end{bmatrix}$. So $B = \begin{bmatrix} 1 & -\frac{2}{3} \\ 0 & 0 \end{bmatrix}$ is RREF of $A - I$.

$N(B)$ corresponds to the system $x_1 - \frac{2}{3}x_2 = 0$. Since x_1 is a lead variable and x_2 is free $x_1 = \frac{2x_2}{3}$. By choosing $x_2 = 1$ we get the eigenvector $\mathbf{x}_1 = (\frac{2}{3}, 1)^\top$ which corresponds to the eigenvalue 1.

Note that the steady state of the Markov chain corresponds o the coordinates of $\mathbf{x}_1$. More precisely the ratio of heathy to sick is $\frac{x_1}{x_2} = \frac{2}{3}$.

To find a basis for the null space of $A - \lambda_2 I = A - 0.5I$, denoted by $N(A - \lambda_2 I)$ we need to bring the matrix $A - 0.5I$ to RREF: $A - 0.5I = \begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.3 \end{bmatrix}$. So $C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ is RREF of $A - 0.5I$.

$N(C)$ corresponds to the system $x_1 + x_2 = 0$. Since x_1 is a lead variable and x_2 is free $x_1 = -x_2$. By choosing $x_2 = 1$ we get the eigenvector $\mathbf{x}_2 = (-1, 1)^\top$ which corresponds to the eigenvalue 0.5.

Example 2: Let $A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$. So $A - \lambda I = \begin{bmatrix} 2-\lambda & -3 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & -3 & 2-\lambda \end{bmatrix}$.
Expand $\det(\mathrm{A} - \lambda\mathrm{I})$ by the first row:

$$(2-\lambda)\big((-2-\lambda)(2-\lambda)+3\big) + (-1)(-3)\big(1(2-\lambda)-1\big) + 1\big(-3+(2+\lambda)\big) =$$
$$(2-\lambda)(\lambda^2-1) + 3(1-\lambda) + (\lambda-1) =$$
$$(\lambda-1)\big((2-\lambda)(\lambda+1) - 3 + 1\big) = (\lambda-1)(-\lambda^2+\lambda) = -\lambda(\lambda-1)^2.$$

So $\lambda_1 = 0$ is a simple root and $\lambda_2 = 1$ is a double root.

$$A - \lambda_1 I = A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}.$$

RREF of A is

$$B = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The null space $N(B)$ is given by $x_1 = x_3, x_2 = x_3$, where x_3 is the free variable. Set $x_3 = 1$ to obtain that $\mathbf{x}_1 = (1,1,1)^\top$ is an eigenvector corresponding to $\lambda_1 = 0$.

$$A - \lambda_2 I = \begin{bmatrix} 1 & -3 & 1 \\ 1 & -3 & 1 \\ 1 & -3 & 1 \end{bmatrix}.$$

RREF of $A - \lambda_2 I$ is

$$B = \begin{bmatrix} 1 & -3 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The null space $N(B)$ given by $x_1 = 3x_2 - x_3$, where x_2, x_3 are the free variable. Set $x_2 = 1, x_3 = 0$ to obtain that $\mathbf{x}_2 = (3,1,0)^\top$. Set $x_2 = 0, x_3 = 1$ to obtain that $\mathbf{x}_3 = (-1,0,1)^\top$. So $\mathbf{x}_2, \mathbf{x}_3$ are two linearly independent eigenvectors corresponding to the double zero $\lambda_2 = 1$.

6.3 Similarity

Definition. Let $\mathbf{V}$ be a vector space with a basis $[\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_n]$ over a field $\mathbb{F}$. Let $T : \mathbf{V} \to \mathbf{V}$ be a linear transformation. Then the representation matrix $A = [\mathbf{a}_1\ \mathbf{a}_2 \ldots \mathbf{a}_n] \in \mathbb{F}^{n\times n}$ of T in the basis $[\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_n]$ is given as follows. The column j of A, denoted by $\mathbf{a}_j \in \mathbb{R}^n$, is the coordinate vector of $T(\mathbf{v}_j)$. That is, $T(\mathbf{v}_j) = [\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_n]\mathbf{a}_j$ for $j = 1, \ldots, n$.

Change a basis in $\mathbf{V}$: $[\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_n] \xrightarrow{Q} [\mathbf{u}_1\ \mathbf{u}_2 \ldots \mathbf{u}_n]$. Then the representation matrix of T in the bases $[\mathbf{u}_1\ \mathbf{u}_2 \ldots \mathbf{u}_n]$ is given by the matrix QAQ^{-1}.

Definition. $A, B \in \mathbb{F}^{n\times n}$ are called similar if $B = QAQ^{-1}$ for some invertible matrix $Q \in \mathbb{F}^{n\times n}$.

Definition. For $A \in \mathbb{F}^{n\times n}$ the trace of A is the sum of the diagonal elements of A.

Proposition 6.2 *Two similar matrices A and B have the same characteristic polynomial. In particular A and B have the same trace and the same determinant.*

Proof. Fix $\lambda \in \mathbb{F}$. Clearly

$$\begin{aligned} &\det\,(\mathrm{B} - \lambda\mathrm{I}) = \det\,(\mathrm{QAQ^{-1}} - \lambda\mathrm{I}) = \det\,(\mathrm{Q}(\mathrm{A} - \lambda\mathrm{I})\mathrm{Q}^{-1}) = \\ &\det \mathrm{Q}(\det\,(\mathrm{A} - \lambda\mathrm{I}))\det \mathrm{Q}^{-1} = \det \mathrm{Q}(\det\,(\mathrm{A} - \lambda\mathrm{I}))(\det \mathrm{Q})^{-1} = \det\,(\mathrm{A} - \lambda\mathrm{I}). \end{aligned} \tag{6.42}$$

Express $\det\,(\mathrm{A} - \lambda\mathrm{I})$ as a sum of $n!$ product of elements of $A - \lambda I$ (§6.1) we get $\det\,(\mathrm{A} - \lambda\mathrm{I}) = (-1)^{\mathrm{n}}\lambda^{\mathrm{n}} + (-1)^{\mathrm{n}-1}\,\mathrm{tr}\,\mathrm{A}\ \lambda^{\mathrm{n}-1} + \ldots + \det\ \mathrm{A}$. Hence $\mathrm{tr}\,A := a_{11} + a_{22} + \ldots + a_{nn}$. In view of (6.42) we deduce that A and B have the same trace and the same determinant. □

Theorem 6.3 *Suppose that $A, B \in \mathbb{F}^{n\times n}$ have the same characteristic polynomial $p(\lambda)$. If $p(\lambda)$ has n distinct roots then A and B are similar.*

We will prove this result later. However if $p(\lambda)$ has multiple roots than it is possible that A and B have the same characteristic polynomial but A and B are not similar.

6.4 Characteristic polynomials of upper triangular matrices

Suppose that A is upper triangular. Hence $A - \lambda I$ is also upper triangular. Thus $\det(\mathrm{A} - \lambda\mathrm{I}) = (\mathrm{a}_{11} - \lambda)(\mathrm{a}_{22} - \lambda)\dots(\mathrm{a}_{\mathrm{nn}} - \lambda)$ So the eigenvalues of upper or lower triangular matrix are given by its diagonal entries, (counted with multiplicities!)

Example 1:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}, \ A - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ 0 & a_{22} - \lambda & a_{23} \\ 0 & 0 & a_{33} - \lambda \end{bmatrix}$$
$$\det(\mathrm{A} - \lambda\mathrm{I}) = (\mathrm{a}_{11} - \lambda)(\mathrm{a}_{22} - \lambda)(\mathrm{a}_{33} - \lambda).$$

Let $A = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}$. (See §**??**.) Recall $\det(\mathrm{A}-\lambda\mathrm{I}) = (1-\lambda)(0.5-\lambda)$. Let $D = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$. So $\det(\mathrm{A} - \lambda\mathrm{I}) = \det(\mathrm{D} - \lambda\mathrm{I})$. We show that A and D are similar. Recall that $A\mathbf{x}_1 = \mathbf{x}_1, A\mathbf{x}_2 = 0.5\mathbf{x}_2$. Let $X = (\mathbf{x}_1 \ \mathbf{x}_2) = \begin{bmatrix} \frac{2}{3} & -1 \\ 1 & 1 \end{bmatrix}$. So $AX = XD$ (Check it!). This is equivalent to the fact that $\mathbf{x}_1, \mathbf{x}_2$ are the corresponding eigenvectors). As $\det \mathrm{X} = \frac{5}{3} \neq 0$ X is invertible and $A = XDX^{-1}$. So A and D are similar.

Example 2:: *Matrices nonsimilar to diagonal mattrices.*

Let
$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Both matrices are upper triangular so $\det(\mathrm{A} - \lambda\mathrm{I}) = \det(\mathrm{B} - \lambda\mathrm{I}) = \lambda^2$. Since $TAT^{-1} = 0 = A \neq B$, A and B are not similar.

Proposition 6.4 *: B is not similar to a diagonal matrix.*

Proof. Suppose that B is similar to $D = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. As $\det(\mathrm{B} - \lambda\mathrm{I}) = \lambda^2 = \det(\mathrm{D} - \lambda\mathrm{I}) = (\mathrm{a} - \lambda)(\mathrm{b} - \lambda)$ we must have $a = b = 0$, i.e. $D = A$. We showed above that A and B are not similar.

6.5 Defective matrices

Defininiton λ_0 is called a defective eigenvalue of $A \in \mathbb{F}^{n\times n}$ if the multiplicity of λ_0 in $\det(\mathrm{B} - \lambda\mathrm{I})\ (= 0)$ is strictly greater than $\dim N(B - \lambda_0 I)$. $B \in \mathbb{F}^{n\times n}$ is called defective if and only if one of the following conditions hold.

1. $\det(\mathrm{B} - \lambda\mathrm{I})$ is not of the form $\prod_{i=1}^{n}(\lambda_i - \lambda)$, for some $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$.
2. $\det(\mathrm{B} - \lambda\mathrm{I}) = \prod_{\mathrm{i}=1}^{\mathrm{n}}(\lambda_\mathrm{i} - \lambda)$, for $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$, and B has at least one defective eigenvalue.

Note that $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is defective since the only eigenvalue $\lambda_0 = 0$ is defective: $\mathrm{rank}(B - 0I) = \mathrm{rank}\, B = 1, \dim N(B) = 2 - \mathrm{rank}\, B = 1$, and the multiplicity of $\lambda_0 = 0$ in $\det(\mathrm{B} - \lambda\mathrm{I}) = \lambda^2$ is 2.

Definition: $A \in \mathbb{F}^{n\times n}$ is called diagonable if A is similar to a diagonal matrix $D \in \mathbb{F}^{n\times n}$. (The diagonal entries of D are the eigenvalues of A counted with multiplicities.)

Theorem 6.5 *: $B \in \mathbb{F}^{n\times n}$ is a diagonable matrix if and only if B is not defective.*

(Note that $A \in \mathbb{R}^{3\times 3}$ given in Example 2 of §6.2 is not defective, hence according to the above Theorem A is diagonable.)

To prove Theorems 6.5 and 6.3 we need the following lemma.

Lemma 6.6 *: Let $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p$ be p eigenvectors of $A\mathbb{F}^{n\times n}$ corresponding to p distinct eigenvalues. Then $\mathbf{y}_1, ..., \mathbf{y}_p$ are linearly independent.*

Proof. The proof is by induction on p. Let $p = 1$. By the definition an eigenvector $\mathbf{y}_1 \neq 0$. Hence $\mathbf{y}_1$ is lin.ind. . Assume that the lemma holds for $p = k$. Let $p = k + 1$. Assume that $A\mathbf{y}_i = \lambda_i\mathbf{y}_i, \mathbf{y}_i \neq 0, i = 1, \ldots, k+1$ and $\lambda_i \neq \lambda_j$ for $i \neq j$. Suppose that

$$a_1\mathbf{y}_1 + \ldots + a_k\mathbf{y}_k + a_{k+1}\mathbf{y}_{k+1} = 0. \tag{6.43}$$

So

$$\mathbf{0} = A\mathbf{0} = A(a_1\mathbf{y}_1 + \ldots + a_k\mathbf{y}_k + a_{k+1}\mathbf{y}_{k+1}) =$$
$$a_1A\mathbf{y}_1 + \ldots + a_kA\mathbf{y}_k + a_{k+1}A\mathbf{y}_{k+1} = a_1\lambda_1\mathbf{y}_1 + \ldots + a_k\lambda_k\mathbf{y}_k + a_{k+1}\lambda_{k+1}\mathbf{y}_{k+1}.$$

Multiply (6.43) by λ_{k+1} and subtract it from the last equality above to get

$$a_1(\lambda_1 - \lambda_{k+1})\mathbf{y}_1 + \ldots + a_k(\lambda_k - \lambda_{k+1})\mathbf{y}_k = 0.$$

The induction hypothesis implies that $a_i(\lambda_i - \lambda_{k+1}) = 0$ for $i = 1, \ldots, k$. Since $\lambda_i - \lambda_{k+1} \neq 0$ for $i < k + 1$ we get $a_i = 0, i = 1, \ldots, k$. Use these equalities in (6.43) to obtain $a_{k+1}\mathbf{y}_{k+1} = 0 \Rightarrow a_{k+1} = 0$. So $\mathbf{y}_1, \ldots, \mathbf{y}_{k+1}$ are linearly independent. $\square$

Proof of Theorem 6.3. Suppose that the characteristic polynomial of $A \in \mathbb{F}^{n\times n}$ has n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. To each eigenvalue λ_i we have an eigenvector $\mathbf{y}_i$ for $i = 1, \ldots, n$. Since $\lambda_i \neq \lambda_j$ for $i \neq j$ Lemma 6.6 yields that $\mathbf{y}_1, \ldots, \mathbf{y}_n$ are linearly independent. Let $Y \in \mathbb{F}^{n\times n}$ such that the columns of Y are $\mathbf{y}_1, \ldots, \mathbf{y}_n$. Since $\mathbf{y}_1, \ldots, \mathbf{y}_n$ are linearly independent then $\operatorname{rank} Y = n$ and Y is invertible. Let $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ be the diagonal matrix whose diagonal entries are $\lambda_1, \ldots, \lambda_n$. A straightforward calculation shows that $AY = Y\Lambda$. So $A = Y\Lambda Y^{-1}$, i.e. A is similar to Λ. Similarly B is similar to Λ. Hence A is similar to B. $\square$

Proof of Theorem 6.5. Consider first a diagonal matrix $D = \operatorname{diag}(d_1, \ldots, d_n) \in \mathbb{F}^{n\times n}$. Clearly, $\mathrm{det\ (D - \lambda I) = \prod_{i=1}^n (d_i - \lambda)}$. Let $\mathbf{e}_i = (\delta_{1i}, \ldots, \delta_{ni})^\top$. Clearly $D\mathbf{e}_i = d_i\mathbf{e}_i$ for $i = 1, \ldots, n$. So if a diagonal element λ in D appears k times, the k corresponding columns of the identity matrix I_n are the eigenvectors corresponding to λ. Clearly these vectors are linearly independent. Also λ is a root of the characteristic polynomial of multiplicity k. Suppose that A is similar to D, i.e. $A = YDY^{-1}$. So $\mathrm{det\ (A - \lambda I) = det\ (D - \lambda I) = \prod_{i=1}^n (d_i - \lambda)}$. Then a straightforward calculation shows that i-th column of Y is an eigenvector of A corresponding to d_i. That is, if A is similar to a diagonal matrix it is not defective.

Assume now that $A \in \mathbb{F}^{n\times n}$ is not defective. First $\mathrm{det\ (A - \lambda I) = prod_{i=1}^n (\lambda_i - \lambda)}$. Let $\mu_1, \ldots, \mu_k$ be the distinct roots of the characteristic polynomial of A. Assume that the multiplicity of μ_i is $n_i \geq 1$. So A has n_i linearly independent eigenvectors $\mathbf{y}_{i,1}, \ldots, \mathbf{y}_{i,n_i}$ such that $A\mathbf{y}_{i,j} = \mu_i\mathbf{y}_{i,j}$ for $j = 1, \ldots, n_i$. We claim that the n eigenvectors $\mathbf{y}_{1,1}, \ldots, \mathbf{y}_{i,n_1}, \ldots, \mathbf{y}_{k,1}, \ldots, \mathbf{y}_{k,n_k}$ are linearly independent. Indeed suppose that

$$\sum_{i=1}^{k}\sum_{j=1}^{n_i} a_{i,j}\mathbf{y}_{i,j} = \mathbf{0}.$$

We claim that all $a_{i,j} = 0$. Suppose not. If $\sum_{j=1}^{n_i} a_{i,j}\mathbf{y}_{i,j} = \mathbf{0}$, since $\mathbf{y}_{i,1}, \ldots, \mathbf{y}_{i,n_i}$ are linearly independent, it follows that $a_{i,j} = 0$ for $j = 1, \ldots, n_i$. Let I be the set of all indices i such that $\sum_{j=1}^{n_i} a_{i,j}\mathbf{y}_{i,j} \neq \mathbf{0}$. Then $\mathbf{x}_i := \sum_{j=1}^{n_i} a_{i,j}\mathbf{y}_{i,j} = \mathbf{0}$ is an eigenvector of A corresponding to μ_i. Our assumption is that $\sum_{i\in I}\mathbf{x}_i = \mathbf{0}$. But each eigenvector $\mathbf{x}_i, i \in I$ correspond to a different eigenvalue μ_i of A. Lemma 6.6 yields that the set of eigenvectors $\mathbf{x}_i, i \in I$ are linear independent. So we can not have that $\sum_{i\in I}\mathbf{x}_i = \mathbf{0}$. Hence $\mathbf{y}_{1,1}, \ldots, \mathbf{y}_{i,n_1}, \ldots, \mathbf{y}_{k,1}, \ldots, \mathbf{y}_{k,n_k}$ are linearly independent. Let Y be the matrix with the columns $\mathbf{y}_{1,1}, \ldots, \mathbf{y}_{i,n_1}, \ldots, \mathbf{y}_{k,1}, \ldots, \mathbf{y}_{k,n_k}$. So Y is invertible. Since each column of Y is an eigenvector a straight computation implies that $AY = Y\Lambda$, where Λ is a diagonal matrix, whose entries are μ_i repeating n_i times for $i = 1, \ldots, k$. So $A = Y\Lambda Y^{-1}$. $\square$

6.6 An examples of a diagonable matrix

Example: See Example 2 in §6.2. Let $A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$. det $(A - \lambda I) = -\lambda(\lambda-1)^2$. $X = [\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3] = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A = XDX^{-1} = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$

6.7 Powers of diagonable matrices

$A = XDX^{-1} \Rightarrow A^m = XD^mX^{-1}$, $D^m = \mathrm{diag}(\lambda_1^m \dots \lambda_n^m),\ m = 1, \dots$.

Iteration process:

$$\mathbf{x}_m = A\mathbf{x}_{m-1},\ m = 1, \dots \Rightarrow \mathbf{x}_m = A^m\mathbf{x}_0. \tag{6.44}$$

Problem: Under what conditions $\mathbf{x}_m$ converges to $\mathbf{x} := \mathbf{x}(\mathbf{x}_0)$?

Proposition 6.7 *Assume that $A \in \mathbb{C}^{n\times n}$ is diagonable. Then $\mathbf{x}_m$ converges to $\mathbf{x}$ for all $\mathbf{x}_0$ if and only if each eigenvalue of A satisfies either $|\lambda| < 1$ or $\lambda = 1$.*

Proof. Let $A = YDY^{-1}$, where $D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ is a diagonal matrix. In (6.44) replace $\mathbf{x}_m = Y\mathbf{y}_m$. Then the system (6.44) becomes $\mathbf{y}_m = D\mathbf{y}_{m-1}$. So $\mathbf{y}_m = D^m\mathbf{y}_0 = \mathrm{diag}(\lambda_1^m, \dots, \lambda_n^m)\mathbf{y}_0$. Assume that $\mathbf{y}_0 = (a_1, \dots, a_n)^\top$. Then the i coordinate of $\mathbf{y}_m$ is $\lambda_i^m a_i$. If $a_i \neq 0$ then the sequence $\lambda_i^m a_i, i = 0, 1, \dots,$ converges if and only if either $|\lambda_i| < 1$ or $\lambda_i = 1$. □

Markov Chains: $A \in \mathbb{R}^{n\times n}$ is called column (row) stochastic if all entries of A are nonnegative and the sum of each column (row) is 1. That is $A^\top\mathbf{e} = \mathbf{e}$, $(A\mathbf{e} = \mathbf{e})$, where $\mathbf{e} = (1, 1, \dots, 1)^\top$. Under mild assumptions, e.g. all entries of A are positive $\lim_{m\to\infty} A^m\mathbf{x}_0 = \mathbf{x}$. If A is column stochastic and $\mathbf{e}^\top\mathbf{x}_0 = 1$ then the limit vector is a unique probability eigenvector of A:

$$A\mathbf{x} = \mathbf{x}, \quad \mathbf{x} = (x_1, \dots, x_n)^\top, \quad 0 < x_1, \dots, x_n, \quad x_1 + x_2 + \dots x_n = 1.$$

Examples:

1. See Example 1 in §6.2:

$$A = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{3}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{bmatrix}. \tag{6.45}$$

$$A^k = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}^k = \begin{bmatrix} \frac{2}{3} & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & (0.5)^k \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{3}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{bmatrix},$$

$$\lim_{k\to\infty} A^k = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{3}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{2}{5} \\ \frac{3}{5} & \frac{3}{5} \end{bmatrix}.$$

(The columns give proportions of healthy and sick.)

2. See Example 2 in §6.2. $A^k = A$ since $\mathrm{diag}(0,1,1)^k = \mathrm{diag}(0^k, 1^k, 1^k) = \mathrm{diag}(0,1,1)$. (This follows also from the straightforward computation $A^2 = A$.)

A is called projection, or involution if $A^2 = A$. For projection $\lim_{k\to\infty} A^k = A$.

6.8 Systems of linear ordinary differential equations

A system of linear ordinary differential equations with constant coefficients, abbreviated as **SOLODEWCC**, is given as.

$$\begin{array}{ccccccccc} y_1' & = & a_{11}y_1 & + & a_{12}y_2 & +\ldots+ & a_{1n}y_n \\ y_2' & = & a_{21}y_1 & + & a_{22}y_2 & +\ldots+ & a_{2n}y_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\vdots & \\ y_n' & = & a_{n1}y_1 & + & a_{n2}y_2 & +\ldots+ & a_{1n}y_n \end{array}. \tag{6.46}$$

In matrix terms we write: $\mathbf{y}' = A\mathbf{y}$, where $\mathbf{y} = \mathbf{y}(t) = (y_1(t), y_2(t), ..., y_n(t))^{\mathrm{T}}$ and $A \in \mathbb{C}^{n\times n}$ is a constant matrix.

We guess a solution of the form $\mathbf{y}(t) = e^{\lambda t}\mathbf{x}$, where $\mathbf{x} = (x_1, \ldots, x_n)^\top \in \mathbb{C}^n$ is a constant vector. We assume that $\mathbf{x} \neq \mathbf{0}$, otherwise we have a constant noninteresting solution $\mathbf{x} = \mathbf{0}$. Then $\mathbf{y}' = (e^{\lambda t})'\mathbf{x} = \lambda e^{\lambda t}\mathbf{x}$. The system $\mathbf{y}' = A\mathbf{y}$ is equivalent to $\lambda e^{\lambda t}\mathbf{x} = A(e^{\lambda t}\mathbf{x})$. Since $e^{\lambda t} =\neq \mathbf{0}$, divide by $e^{\lambda t}$ to get $A\mathbf{x} = \lambda\mathbf{x}$.

Corollary 6.8 *If $\mathbf{x}(\neq \mathbf{0})$ is an eigenvector of A corresponding to the eigenvalue λ then $\mathbf{y}(t) = e^{\lambda t}\mathbf{x}$ is a nontrivial solution of the given SOLODEWCC.*

Theorem 6.9 *Assume that $A \in \mathbb{C}^{n\times n}$ is diagonable. Let $\det\,(\mathrm{A} - \lambda\mathrm{I}) = (\lambda_1 - \lambda)^{\mathrm{m}_1}(\lambda_2 - \lambda)^{\mathrm{m}_2}\ldots(\lambda_\mathrm{k} - \lambda)^{\mathrm{m}_\mathrm{k}}$, where $\lambda_i \neq \lambda_j$ for $i \neq j$, $1 \leq m_i$. (The multiplicity of λ_i), and $\dim \mathrm{N}(A - \lambda_i I) = m_i$, $\mathrm{N}(A - \lambda_i) = \mathrm{span}(\mathbf{x}_{i1}, \ldots, \mathbf{x}_{im_i})$ for $i = 1, \ldots, k$. Then the general solution of SOLODEWCC is:*

$$\mathbf{y}(t) = \sum_{i=1,j=1}^{k,m_i} c_{ij} e^{\lambda_i(t-t_0)}\mathbf{x}_{ij}. \tag{6.47}$$

The constants $c_{ij}, i = 1, \ldots, k, j = 1, \ldots, m_i$ are determined uniquely by the initial condition $\mathbf{y}(t_0) = \mathbf{y}_0$.

Proof. Since each $\mathbf{x}_{ij}$ is an eigenvector of A corresponding the eigenvalue λ_i it follows from Corollary 6.8 that any $\mathbf{y}$ given by (6.47) is a solution of (6.46). From the proof of Theorem 6.5 it follows that the eigenvectors

$\mathbf{x}_{ij}, i = 1, \ldots, k, j = 1, \ldots, m_i$ form a basis in $\mathbb{C}^n$. Hence there exists a unique linear combination of the eigenvectors of A satisfying $\sum_{i=1,j=1}^{k,m_i} c_{ij}\mathbf{x}_{ij} = \mathbf{c}$. □

Example 1:

$$\begin{array}{rcrcr} y_1' & = & 0.7y_1 & + & 0.2y_2 \\ y_2' & = & 0.3y_1 & + & 0.8y_2 \end{array} . \tag{6.48}$$

The right-hand side is given by $A = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}$ which was studied in §6.2. So $\det(\mathrm{A} - \lambda \mathrm{I}) = (\lambda - 1)(\lambda - 0.5)$.

$$A\mathbf{x}_1 = \mathbf{x}_1,\ A\mathbf{x}_2 = 0.5\mathbf{x}_2,\ \mathbf{x}_1 = (\frac{2}{3}, 1)^\top,\ \mathbf{x}_2 = (-1, 1)^\top.$$

The general solution of the system is $\mathbf{y}(t) = c_1 e^t \mathbf{x}_1 + c_2 e^{0.5t}\mathbf{x}_2$:

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1 e^t \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} + c_2 e^{0.5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

$$y_1(t) = \frac{2c_1 e^t}{3} - c_2 e^{0.5t},\ y_2(t) = c_1 e^t + c_2 e^{0.5t}.$$

Example 2:

$$\begin{array}{rcrrr} y_1' & = & 2y_1 & -3y_2 & +y_3 \\ y_2' & = & y_1 & -2y_2 & +y_3 \\ y_3' & = & y_1 & -3y_2 & +2y_3 \end{array} .$$

So $A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$ as in Example 2 in §6.2. Hence

$$\det(\mathrm{A} - \lambda \mathrm{I}) = -\lambda(\lambda - 1)^2,\ \lambda_1 = 0,\ \lambda_2 = \lambda_3 = 1,$$

$$X = [\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3] = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

The general solution is $\mathbf{y}(t) = c_1 e^0 \mathbf{x}_1 + c_2 e^t \mathbf{x}_2 + c_3 e^t \mathbf{x}_3$, where

$$y_1(t) = c_1 + 3c_2 e^t - c_3 e^t,\ y_2(t) = c_1 + c_2 e^t,\ y_3(t) = c_1 + c_3 e^t.$$

6.9 Initial conditions

Assume that the initial conditions of the system (6.46) are given at the time $t_0 = 0$: $\mathbf{y}(0) = \mathbf{y}_0$. On the assumption that A is diagonable, i.e. $X\Lambda X^{-1}$ it follows that the unknown vector $\mathbf{c}$ appearing in (6.47) satisfies $X\mathbf{c} = \mathbf{y}_0$. We solve this system either by Gauss elimination or by inverting X: $\mathbf{c} = X^{-1}\mathbf{y}_0$.

Example 1: In the system of ODE given in (6.48) find the solution satisfying *IC* $\mathbf{y}(0) = (1, 2)^\top$.

Solution. This condition is equivalent to $\begin{bmatrix} \frac{2}{3} & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{3}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{9}{5} \\ \frac{1}{5} \end{bmatrix}$$

(The inverse is taken from (6.45).) Now substitute these values of c_1, c_2 in (6.47).

6.10 Complex eigenvalues of real matrices

Proposition 6.10 *Let $A \in \mathbb{R}^{n\times n}$ and assume that $\lambda := \alpha + \mathbf{i}\beta,\ \alpha, \beta \in \mathbb{R}$ is non-real eigenvalue ($\beta \neq 0$). Then the corresponding eigenvector $\mathbf{x} = \mathbf{u} + \mathbf{i}\mathbf{v},\ \mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ ($A\mathbf{u} = \lambda\mathbf{u}$) is non-real ($\mathbf{v} \neq 0$). Furthermore $\overline{\lambda} = \alpha - \mathbf{i}\beta \neq \lambda$ is another eigenvalue of A with the corresponding eigenvector $\overline{\mathbf{x}} = \mathbf{u} - \mathbf{i}\mathbf{v}$. The corresponding contributions of the above two complex eigenvectors to the solution of $\mathbf{y}' = A\mathbf{y}$ is*

$$e^{\alpha t}C_1(\cos(\beta t)\mathbf{u} - \sin(\beta t)\mathbf{v}) + e^{\alpha t}C_2(\sin(\beta t)\mathbf{u} + \cos(\beta t)\mathbf{v}). \tag{6.49}$$

These two solutions can be obtained by considering the real linear combination of the real and the imaginary part of the complex solution $e^{\lambda t}\mathbf{x}$.

Proof. Recall the Euler's formula for e^z where $z = a + \mathbf{i}b,\ a, b \in \mathbb{R}$:

$$e^z = e^{a+\mathbf{i}b} = e^a e^{\mathbf{i}b} = e^a(\cos b + \mathbf{i}\sin b)$$

Now find the real part of the complex solution $(C_1 + \mathbf{i}C_2)e^{(\alpha+\mathbf{i}\beta)t}(\mathbf{u} + \mathbf{i}\mathbf{v})$ to deduce (6.49). $\square$

6.11 Second Order Linear Differential Systems

Let $A_1, A_2 \in \mathbb{C}^{n\times n}$ and $\mathbf{y} = (y_1(t), \ldots, y_n(t))^\top$. Then the second order differential system of linear equations is given as

$$\mathbf{y}'' = A_1\mathbf{y} + A_2\mathbf{y}'. \tag{6.50}$$

It is possible to translate this system to a system of the first order involving matrices and vectors of the double size. Let $\mathbf{z} = (y_1, \ldots, y_n, y_1', \ldots, y_n')^\top$. Then

$$\mathbf{z}' = A\mathbf{z}, \quad A = \begin{bmatrix} 0_n & I_n \\ A_1 & A_2 \end{bmatrix} \in \mathbb{C}^{2n\times 2n}. \tag{6.51}$$

Here 0_n is $n \times n$ zero matrix and I_n is $n \times n$ identity matrix. The initial conditions are $\mathbf{y}(t_0) = \mathbf{a} \in \mathbb{C}^n,\ \mathbf{y}'(t_0) = \mathbf{b} \in \mathbb{C}^n$ which are equivalent to the initial conditions $\mathbf{z}(t_0) = \mathbf{c} \in \mathbb{C}^{2n}$.

Thus the solution of the second order differential system with n unknown functions can be solved by converting this system to the first order system with $2n$ unknown functions.

6.12 Exponential of a Matrix

For $A \in \mathbb{C}^{n\times n}$ let

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!}A^k = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots. \tag{6.52}$$

If $D = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ it is straightforward to show that

$$e^D = \mathrm{diag}(e^{\lambda_1}, e^{\lambda_2}, \dots, e^{\lambda_n}). \tag{6.53}$$

Hence for a diagonable matrix A we get the indentity

$$e^A = Xe^D X^{-1}, \quad A = XDX^{-1}. \tag{6.54}$$

Similarly, we define

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!}A^k = I + tA + \frac{1}{2!}t^2A^2 + \frac{1}{3!}t^3A^3 + \dots. \tag{6.55}$$

So taking the derivative with respect to t we obtain

$$(e^{tA})' = \sum_{k=1}^{\infty} \frac{kt^{k-1}}{k!}A^k = 0 + A + \frac{1}{2!}2tA^2 + \frac{1}{3!}3t^2A^3 + \dots = Ae^{At}.$$

If A is diagonable $A = XDX^{-1}$ then $tA = X(tD)X^{-1} \Rightarrow$. So $e^{At} = X\,\mathrm{diag}(e^{\lambda_1 t}, \dots, e^{\lambda_n t})X^{-1}$.

The matrix $Y(t) := e^{(t-t_0)A}$ satisfies the matrix differential equation

$$Y'(t) = AY(t) = Y(t)A \text{with the initial condition} Y(t_0) = I. \tag{6.56}$$

(*As in the scalar case*, i.e. A is 1×1 matrix.)

The solution of $\mathbf{y}' = A\mathbf{y}$ with the initial condition $\mathbf{y}(t_0) = \mathbf{a}$ is given by $\mathbf{y}(t) = e^{(t-t_0)A}\mathbf{a}$.

6.13 Examples of exponential of matrices

Example 1

$$A = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{3}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{bmatrix},$$

$$e^A = \begin{bmatrix} \frac{2}{3} & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^1 & 0 \\ 0 & e^{0.5} \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{3}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{bmatrix} = \begin{bmatrix} \frac{2e-3e^{0.5}}{5} & \frac{2e-2e^{0.5}}{5} \\ \frac{3e-3e^{0.5}}{5} & \frac{3e+2e^{0.5}}{5} \end{bmatrix},$$

$$e^{tA} = \begin{bmatrix} \frac{2}{3} & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{0.5t} \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{3}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{bmatrix} = \begin{bmatrix} \frac{2e^t-3e^{0.5t}}{5} & \frac{2e^t-2e^{0.5t}}{5} \\ \frac{3e^t-3e^{0.5t}}{5} & \frac{3e^t+2e^{0.5t}}{5} \end{bmatrix}.$$

In the system of ODE (6.48) the solution satisfying IC $\mathbf{y}(0) = (1, 2)^\top$ is given as

$$\mathbf{y}(t) = e^{At}\mathbf{y}(0) = \begin{bmatrix} \frac{2e^t-3e^{0.5t}}{5} & \frac{2e^t-2e^{0.5t}}{5} \\ \frac{3e^t-3e^{0.5t}}{5} & \frac{3e^t+2e^{0.5t}}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{6e^t-7e^{0.5t}}{5} \\ \frac{9e^t+e^{0.5t}}{5} \end{bmatrix}.$$

Compare this solution with the solution given in §6.9 in Example 1.

Example 2. $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is defective. We compute e^B, e^{tB} using power series (6.55). Note $B^2 = 0$. Hence $B^k = 0$ for $k \geq 2$. So

$$e^B = I + B + \frac{1}{2!}B^2 + \frac{1}{3!}B^3 + \ldots = I + B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$
$$e^{tB} = I + tB + \frac{1}{2!}t^2B^2 + \frac{1}{3!}t^3B^3 + \ldots = I + tB = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

Hence the system of ODLE

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= 0 \end{aligned}$$

Has the general solution

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = e^{tB} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 t \\ c_2 \end{bmatrix}.$$

References

[1] J. Hefferon, *Linear Algebra*, http://joshua.smcvt.edu/linearalgebra/Linear Algebra