Problem 1. Recall that a tournament $D = (V, A)$ is a directed graph obtained from the complete graph by $K_n$, $n = |V|$, by directing each edge of $K_n$.

(a) (5 pts) How many tournaments are there if $|V| = n$?
(b) (15 pts) Show by induction that any tournament has a directed Hamiltonian path.
(c) (5 pts) Show that either $D$ is strongly connected or you can reverse a direction of one edge in $D$ to obtain $D'$, such that $D'$ is strongly connected.

Problem 2. Let $G = (V, E)$ be a connected graph which has exactly one cycle.

a. (10pts) Show that there exists an edge $e$ in $G$ such that $G - e$ is a spanning tree of $G$.

b. (5pts) Characterize those $e$ which satisfy part (a) of the problem.

c. (10 pts) Show that $G$ has a vertex of degree one if and only if at least one of its vertices has degree 3 at least.

Problem 3.

a. (5 pts.) Let $T = (V, E)$ be a tree on $n$ vertices, with the degree sequence $d(v_1), \ldots, d(v_n)$.

Show that these $n$ numbers are positive integers that satisfy $\sum_{i=1}^{n} d(v_i) = 2(n - 1)$.

b. (10 pts) show that if $d_1, \ldots, d_n$ are $n$ positive integers with sum $2(n - 1)$ then there exists a tree so that $d_1, \ldots, d_n$ is its degree sequence.

c. (10 pts.) Let $G_w = (V, E)$ be a connected weighted graph, where each edge $e \in E$ has a positive weight $w(e)$. Assume that any two different edges in $G$ have different weights. Show that a spanning tree of $G$ with the minimum weight is unique.

Problem 4. Let $G = (V_1 \cup V_2, E)$ be a bipartite graph.

(a) (3 pts) Let $H = (V_1 \cup V_2, E')$ be a spanning subgraph of $G$, i.e. $E' \subseteq E$. Show

$$\sum_{u \in V_1} d_H(u) = \sum_{v \in V_2} d_H(v).$$

(0.1)

(b) (11 pts) Let $p : V_1 \to \mathbb{Z}_+$, $q : V_2 \to \mathbb{Z}_+$ be two functions on $V_1, V_2$ which take nonnegative integer values. Assume that $p(u) \leq d_G(u), q(v) \leq d_G(v)$ for $u \in V_1, v \in V_2$ respectively. Suppose furthermore that $\sum_{u \in V_1} p(u) = \sum_{v \in V_2} q(v)$. Consider the problem of existence of a spanning graph $H = (V_1 \cup V_2, E')$ of $G$ such that $d_H(u) = p(u), d_H(v) = q(v)$ for each $u \in V_1, v \in V_2$. Show that this problem can be stated in terms of flows.

c. (11 pts) Show that such subgraph exists if and only if for any two subsets $S \subseteq V_1, T \subseteq V_2$ on has the following inequality

$$\sum_{u \in S} p(u) + a(V_1 \setminus S, T) \geq \sum_{v \in T} q(v).$$

(0.2)

Here $a(S, T)$ is the number of edges in $E$ that connect a vertex in $S$ to a vertex in $T$. 
