# REVISITING THE SIEGEL UPPER HALF PLANE II

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**Abstract**

In this paper we study the automorphisms of Siegel upper half plane of complex dimension 3. We give the normal forms and classify the set of fixed points of such transformations. [1]

## 1 Introduction

This paper is a continuation of our paper [3] and we use its notations. Let $\mathbf{Sym}(n, \mathbb{C})$ be the space on $n \times n$ complex symmetric matrices. Let $\mathbf{SD}_n := \{Z \in \mathbf{Sym}(n, \mathbb{C}) : ||Z||_2 < 1\}$ be the Siegel $n$-disk. Then $\overline{\mathbf{SD}}_n = \{Z \in \mathbf{Sym}(n, \mathbb{C}) : \ ||Z||_2 \le 1\}$ and $\partial\mathbf{SD}_n = \{Z \in \mathbf{Sym}(n, \mathbb{C}) : \ ||Z||_2 = 1\}$. The Shilov boundary of $\mathbf{SD}_n$, denoted by $\partial_n\mathbf{SD}_n$, is $\mathbf{USym}(n) := \mathbf{U}_n \cap \mathbf{Sym}(n, \mathbb{C})$, the set of $n \times n$ unitary symmetric matrices. Let $\mathbf{SH}_n := \{Z \in \mathbf{Sym}(n, \mathbb{C}) : \ \operatorname{Im} Z > 0\}$ be the Siegel $n$-upper half plane. Then $\mathrm{Cl}(\mathbf{SH}_n)$ is the compactification of $\mathbf{SH}_n$ which is diffeomorphic to $\overline{\mathbf{SD}}_n$. Then $\partial\mathbf{SH}_n, \partial_n\mathbf{SH}_n$ are diffeomorphic to $\partial\mathbf{SD}_n, \partial_n\mathbf{SH}_n$ respectively. $\partial_n\mathbf{SH}_n$ is the Shilov boundary of $\mathbf{SH}_n$. $\partial_1\mathbf{SD}_2 := \partial\mathbf{SD}_2\backslash\partial_2\mathbf{SD}_2$ and $\partial_1\mathbf{SH}_2 := \partial\mathbf{SH}_2\backslash\partial_2\mathbf{SH}_2$ are the other strata of $\partial\mathbf{SD}_2$ and $\partial\mathbf{SH}_2$ respectively. (See [3, §3].)

Recall that the symplectic group $\mathbf{Sp}(n, \mathbb{R})$ acts as a group of generalized Möbius transformations on $\mathbf{SH}_n$, where the action of $M$ and $-M$ coincide. The action of $M \in \mathbf{Sp}(n, \mathbb{R})$ extends continuously to $\mathrm{Cl}(\mathbf{SH}_n)$. Since $\overline{\mathbf{SD}}_n$ is homeomorphic to the closed ball, the Brouwer fixed point theorem implies that $M$ has at least one fixed point in $\mathrm{Cl}(\mathbf{SH}_n)$.

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$\mathbf{Sp}(n,\mathbb{R})$ acts on $\mathbf{USym}(n)$. The action of a discrete group $\Gamma \subset \mathbf{Sp}(2,\mathbb{R})$ on $\mathbf{USym}(2)$ is a three dimensional generalization of the action of the Kleinian group on $S^2$. We believe that it is interesting and important to study the action of $\mathbf{Sp}(2,\mathbb{R})$ and its discrete subgroups on $\mathbf{SH}_2$ and $\mathrm{Cl}(\mathbf{SH}_2)$. This paper is the first step in this program.

The main purpose of this paper is to give a normal form of any element $M \in \mathbf{Sp}(2,\mathbb{R})$ under the conjugation in $\mathbf{Sp}(2,\mathbb{R})$ and to classify its fixed points. This is done by using direct computations and some geometric facts of the action of $\mathbf{Sp}(2,\mathbb{R})$. It turns out that the family of different conjugacy classes in $\mathbf{Sp}(2,\mathbb{R})$ is much richer then the family of conjugacy classes of $\mathbf{Sp}(1,\mathbb{R}) = \mathbf{SL}(2,\mathbb{R})$ or $\mathbf{SL}(2,\mathbb{C})$. Let

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \odot \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} := \left(\begin{array}{cc|cc} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ \hline c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{array}\right). \tag{1.1}$$

Clearly, $X \odot Y \in \mathbf{Sp}(2,\mathbb{R})$ if and only $X, Y \in \mathbf{SL}(2,\mathbb{R})$. (From here to the end of this section we assume that $X, Y \in \mathbf{SL}(2,\mathbb{R})$.) Then $\mathbf{SL}(2,\mathbb{R}) \times \mathbf{SL}(2,\mathbb{R}) \to \mathbf{SL}(2,\mathbb{R}) \odot \mathbf{SL}(2,\mathbb{R}) \subset \mathbf{Sp}(2,\mathbb{R})$ is a faithful representation. We show that many (but not all) $M \in \mathbf{Sp}(2,\mathbb{R})$ are conjugate (in $\mathbf{Sp}(2,\mathbb{R})$) to $X \odot Y$. We characterize $M \in \mathbf{Sp}(2,\mathbb{R})$ which are conjugate to $X \odot Y$. $X \odot Y$ act on $\mathbf{H} \times \mathbf{H} \subset \mathbf{SH}_2$ and its compactification $\mathrm{Cl}(\mathbf{H}) \times \mathrm{Cl}(\mathbf{H}) \subset \mathrm{Cl}(\mathbf{SH}_2)$. (Note that $\mathbf{H} = \mathbf{SH}_1,\ \mathbf{D} = \mathbf{SD}_1$ are the upper half plane and the unit disk respectively.) If $\xi, \eta \in \mathrm{Cl}(\mathbf{H})$ are fixed points of $X, Y$ respectively then $\xi \times \eta$ is a fixed point of $X \odot Y$. We call $\xi \times \eta$ the ordinary fixed point of $X \odot Y$. We show that the set of the fixed $X \odot Y$ is a set of ordinary fixed points if and only if $Y$ is not conjugate in $\mathbf{SL}(2,\mathbb{R})$ to $X^{-1}$. The transformation $X \odot X^{-1}$ has additional (nonordinary) fixed points, whose structure depends on the type of $X$: hyperbolic, parabolic, elliptic or $X = \pm I_2$. (See [1] for the definitions and properties of hyperbolic, parabolic and elliptic transformations for the action of $\mathbf{SL}(2,\mathbb{R})$ on $\mathbf{H}$.)

$M \in \mathbf{Sp}(2,\mathbb{R})$ is called hyperbolic if all eigenvalues of $M$ lie outside the unit circle. Hyperbolic transformations were already studied in [3]. They have two distinguished hyperbolic fixed points in the Shilov boundary of $\mathbf{SH}_2$. One is attracting and one is repelling. The domain of attraction (repulsion) includes $\mathbf{SH}_2$. There are three types of hyperbolic $M$: (Ia), (Ib) and (Ic). (Ia) $M$ has real spectrum and is diagonable. Then $M$ is conjugate to $X \odot Y$, where $X$ and $Y$ are hyperbolic. If $Y$ is not conjugate to $X^{-1}$ then $X \odot Y$ has exactly four (ordinary) fixed points in the Shilov boundary. If $Y$ is conjugate to $X^{-1}$ then $X \odot Y$ has two isolated hyperbolic fixed points (attracting and repelling) and a circle $S^1$ of fixed points all in the Shilov boundary. (Ib) $M$ has two double real eigenvalues and is not diagonable. Then $M$ has three fixed points in the Shilov boundary. (Ic) $M$ does not have real eigenvalues. Then $M$ has two fixed points in the Shilov boundary.

For a nonhyperbolic $M \in \mathbf{Sp}(2,\mathbb{R}),\ M \neq \pm I_4$ the set of fixed points is one of the following types: (a) one or two points; (b) one or two disjoint copies of $\overline{\mathbf{D}}$; (c) two copies of $\overline{\mathbf{D}}$ intersecting at one point lying on the boundary of each $\mathbf{D}$; (d) $\overline{\mathbf{D}} \times \overline{\mathbf{D}}$. We identify the location of the fixed points for any $M \in \mathbf{Sp}(2,\mathbb{R})$: either in $\partial_2\mathbf{SH}_2$, or in $\partial_1\mathbf{SH}_2$ or in $\mathbf{SH}_2$.

There is an overlap between some of our results on the fixed points of the action of elements $M \in \mathbf{Sp}(2,\mathbb{R})$ on $\mathbf{SH}_2$ and the forms of special representatives of the conjugacy class of $M$ in $\mathbf{Sp}(2,\mathbb{R})$, and the two papers of Gottschling [4] and [5]. In his papers Gottschling considered only the fixed points of $M \in \mathbf{Sp}(n,\mathbb{Z})$ and the forms of special representative of the conjugacy class $M$ in $\mathbf{Sp}(n,\mathbb{Z})$. He was not concerned with the exact location of the fixed points with respect to the stratification of $\partial\,\mathrm{Cl}(\mathbf{SH}_n)$.

We now summarize briefly the contents of our paper. §2 studies the geometry of the Shilov boundary of $\mathbf{SD}_2$. In §3 we give a number of normal forms of $M \in \mathbf{Sp}(2,\mathbb{R})$, which depend on the location of a fixed point of $M$. §4 classifies the set of the fixed points of $M$ (Theorem 4.1). §5 we discuss the projective presentation of the boundary of $\mathbf{SH}_2$. §6 devoted to the proof Theorem 4.1.

Most of the results here were obtained by the first author in his Ph.D. thesis [2] under the direction of the second author. The first author was supported by an FCT-Praxis XXI scholarship during his studies at UIC.

## 2 Geometry of the Shilov boundary of $\mathbf{SD}_2$

Let $U \in \mathbf{U}_n$. Then the spectral decomposition of $U$ is:

$$U = VDV^*,\ V \in \mathbf{U}_n,\ D = \mathrm{diag}(\lambda_1,\dots,\lambda_n),\ \lambda_i = e^{\sqrt{-1}\theta_i},\ \theta_i \in \mathbb{R}, \quad i = 1,\dots,n. \tag{2.1}$$

**Lemma 2.1** *Let $U \in \mathbf{USym}(n)$. Then in the decomposition* (2.1) *we can choose $V \in \mathbf{SO}(n,\mathbb{R})$.*

**Proof.** Assume that $\lambda$ is an eigenvalue of $U$ of multiplicity $k$. Without loss of generality we may assume that $\lambda_i = \lambda, i = 1,\dots,k$ and $\lambda_j \neq \lambda, j > k$. Assume that $U$ has the spectral decomposition (2.1). Then the first $k$ columns $v^1,\dots,v^k$ of $V$ form an orthonormal basis for the eigenspace of $U$ corresponding to $\lambda$. As $U^T = U = \overline{V}DV^T$ it follows that $\overline{v}^1,\dots,\overline{v}^k$ forms also an orthonormal basis of the eigenspace of $U$ corresponding to $\lambda$. Hence span $(v^1,\dots,v^k)$ has an orthonormal basis $\pm q^1,\dots,\pm q^k \in \mathbb{R}^n$. By picking a real orthonormal basis for each eigenspace of $U$ and choosing the signs $\pm$ accordingly we deduce the lemma. $\square$

**Lemma 2.2** *The map $F : \mathbf{Sym}(n,\mathbb{R}) \to \mathbf{USym}(n)$ given by $A \mapsto e^{\sqrt{-1}A}$ is a surjection.*

**Proof.** Clearly $F$ is into. As any $A \in \mathbf{Sym}(n,\mathbb{R})$ is of the form $A = Q^TDQ,\ Q \in \mathbf{SO}(n,\mathbb{R}),\ D \in \mathbf{D}(n,\mathbb{R})$ we obtain that $F(A) = Q^TF(D)Q$. Use Lemma 2.1 to deduce the lemma. $\square$

For $n > 1$ $F$ is not a covering map. It can be shown from the proof of Lemma 2.2 that $F$ fails to be a local homeomorphism at all points where the number of distinct eigenvalues of $F(A)$ is strictly less than the number of distinct eigenvalues of $A$. We discuss this fact for the case $n = 2$:

**Theorem 2.3** $\mathbf{USym}(2)$ *fibers over the circle, with the fibre $\mathcal{F} := \mathbf{USym}(2) \cap \mathbf{SL}(2,\mathbb{C})$, which is homeomorphic to $S^2$. The gluing homeomorphism of $\mathcal{F}$ is given by the involution map $A \mapsto -A$. The fundamental group of* $\mathbf{USym}(2)$ *is isomorphic to $\mathbb{Z}$.*

**Proof.** It is straightforward to show that $\mathbf{USym}(2)$ is diffeomorphic to the homogeneous space $\mathbf{U}_2/\mathbf{O}(2,\mathbb{R})$. Hence $\mathbf{USym}(2)$ is a compact manifold. We first consider $\mathcal{F}$. Let $\mathbf{Sym}_0(2,\mathbb{R})$ be the subspace of all real symmetric matrices of zero trace:

$$\mathbf{Sym}_0(2,\mathbb{R}) := \{A(x) : \quad A(x) = \begin{pmatrix} x_1 & x_2 \\ x_2 & -x_1 \end{pmatrix}, \quad x = (x_1, x_2)^{\mathrm{T}} \in \mathbb{R}^2\}. \tag{2.2}$$

The proof of Lemma 2.2 implies that the restriction of the map $F$ - $F_0 : \mathbf{S}_0(2,\mathbb{R}) \to \mathcal{F}$ is onto. A straightforward argument shows that $F_0$ is not a local diffeomorphism at $A$ if and only if the eigenvalues of $A$ are $\pm\pi n, \quad n = 1, \ldots$. Note that all $2 \times 2$ real symmetric matrices with eigenvalues $\pm\pi n$ are mapped by $F_0$ to $(-1)^n I_2$. In view of equation (2.2) we can identify $\mathbf{Sym}_0(2,\mathbb{R})$ with $\mathbb{R}^2$. Let $S_{0,2}$ be the space obtained from $\mathbb{R}^2$ by collapsing each of the following circles $\mathrm{tr}(A^2(x)) = 2(x_1^2 + x_2^2) = 2\pi^2 n^2$ to the points $\zeta_n$ for $n = 1, \ldots$. Then $S_{0,2}$ is a countable union of the 2-spheres $S_n^2$: $\pi^2(n-1)^2 \le x_1^2 + x_2^2 \le \pi^2 n^2$ for $n \in \mathbb{N}$, such that the spheres $S_n^2, S_{n+1}^2$ are attached at the point $\zeta_n$ for $n \in \mathbb{N}$. $S_{0,2}$ is a CW-complex. $F_0$ induces the continuous map $\tilde{F}_0 : S_{0,2} \to \mathcal{F}$. A straightforward argument shows that $\tilde{F}_0|S_n^2$ is a homeomorphism for $n \in \mathbb{N}$. Hence $\mathcal{F}$ is homeomorphic to $S^2$. Given any $B \in \mathbf{USym}(2)$, it is of the form $\lambda A = -\lambda(-A)$, where $A \in \mathcal{F}$ and $\lambda$ on the unit circle $S^1$. For $\lambda = e^{\sqrt{-1}\pi t}, 0 < t < 1$ we let the fiber $\mathcal{F}$ over $\lambda$ be identified with the set $\lambda\mathcal{F}$. Two fibers $\mathcal{F}$ over the points 1 and $-1$ of the circle $S^1$, are identified by the involution $A \mapsto -A$. Hence $\mathbf{USym}(2)$ fibers over the circle $T^1 = \mathbb{R}/\mathbb{Z}$ with the fibre $S^2$. View $T^1$ as the upper half part of $S^1$ where the points 1 and $-1$ are identified and is denoted by $S_+^1$. The map $A \mapsto \sqrt{\det\ \mathrm{A}} \in S_+^1$ gives the projection map $P : \mathbf{USym}(2) \to S_+^1$. Assume that an image of a closed path $\gamma$ starting at $I_2$ under $P$ has a winding number of zero. Then the projections of $\gamma$ on $S_+^1$ and $\mathcal{F}$ respectively are a closed path on $S_+^1$ starting at 1 and a closed path on $\mathcal{F}$ starting at $I_2$ respectively. The closed path on $\mathcal{F}$ is contractible to a constant path at $I_2$. Since the winding number of $P(\gamma)$ is zero, the closed path on $S_+^1$ is contractible to a constant path. Hence the homotopy class of $\gamma$ is determined by the winding number of $P(\gamma)$ on $S_+^1$. That is, the fundamental group of $\mathbf{USym}(2)$ is isomorphic to $\mathbf{Z}$. The projection of a generator of this group on $S_+^1$, $\mathcal{F}$ respectively, is a path of length on $\pi$ on $S_+^1$ from 1 to $-1$ and a path from $I_2$ to $-I_2$ on $\mathcal{F}$. $\square$

Similarly

**Proposition 2.4** *$\mathbf{USym}(n)$ is a compact $\binom{n+1}{2}$ dimensional manifold which fibers over the circle, with a smooth fiber $\mathcal{F}_n = \mathbf{USym}(n) \cap \mathbf{SL}(n,\mathbb{C}) \sim \mathbf{SU}_n/\mathbf{SO}(n,\mathbb{R})$.*

# 3 Normal forms in $\mathbf{Sp}(2,\mathbb{R})$

The arguments in the beginning of §1 imply:

**Lemma 3.1** *Let $A \in \mathbf{Sp}(n,\mathbb{R})$ act on $\mathrm{Cl}(\mathbf{SH}_n)$. Then $A$ has a fixed point is either in $\mathbf{SH}_n$ or its boundary.*

We now restrict our attention to $n = 2$. Since $\mathbf{Sp}(2,\mathbb{R})$ acts transitively on $\partial_2\mathbf{SH}_2$ and $\partial_1\mathbf{SH}_2$ it follows that

$$\partial_2\mathbf{SH}_2 = \mathbf{Sp}(2,\mathbb{R})(0),\ (0 = \mathrm{diag}(0,0)), \quad \partial_1\mathbf{SH}_2 = \mathbf{Sp}(2,\mathbb{R})(\sqrt{-1}\,\mathrm{diag}(1,0)). \tag{3.1}$$

The two components of the finite boundary of $\mathbf{SH}_2$ are (see [3, §3]):

$$\begin{aligned} \mathrm{fin}(\partial_2\mathbf{SH}_2) &:= \partial_2\mathbf{SH}_2 \cap \mathrm{fin}(\partial\mathbf{SH}_2) = \mathbf{Sym}(2,\mathbb{R}), \qquad (3.2)\\ \mathrm{fin}(\partial_1\mathbf{SH}_2) &:= \partial_1\mathbf{SH}_2 \cap \mathrm{fin}(\partial\mathbf{SH}_2) = \{Z \in \mathbf{Sym}(2,\mathbb{C}) : \mathrm{Im}\, Z \geq 0,\ \mathrm{rank}\ \ \mathrm{Im}\, Z = 1\}. \end{aligned}$$

In §5 we give a description of the infinite boundary of $\mathbf{SH}_2$.

For $A \in \mathbf{GL}(n,\mathbb{R})$ we denote by $\mathrm{spec}(A) \subset \mathbb{C}$ and $\rho(A)$ the spectrum and the spectral radius of $A$ respectively. $\pm I_n \neq A \in \mathbf{GL}(n,\mathbb{R})$ is called hyperbolic or elliptic if $A$ does not have eigenvalues on the unit circle or $A$ has all eigenvalues on the unit circle and $A$ is similar to a diagonal matrix respectively. $A, B \in \mathbf{GL}(n,\mathbb{R})$ are called similar if $A$ and $B$ are conjugate in $\mathbf{GL}(n,\mathbb{R})$. Let $\mathbf{G}$ be a subgroup of $\mathbf{GL}(n,\mathbb{R})$. Then $A, B \in \mathbf{G}$ are called conjugate (in $\mathbf{G}$) if $A$ and $B$ are conjugate in $\mathbf{G}$. Recall $M \in \mathbf{Sp}(2,\mathbb{R})$ if and only if and only if

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbf{M}(4,\mathbb{R}),\ A^TC, B^TD \in \mathbf{Sym}(2,\mathbb{R}),\ A^TD - C^TB = I_2. \tag{3.3}$$

Furthermore

$$M(Z) := (AZ + B)(CZ + D)^{-1}, \quad Z \in \mathbf{SH}_2. \tag{3.4}$$

Assume that $M \in \mathbf{Sp}(2,\mathbb{R})$. As $M^{-1}$ is similar to $M^T$ it follows that the 4 eigenvalues of $M$ are $\lambda_1(M),\ \lambda_2(M),\ \lambda_3(M) = \lambda_2(M)^{-1},\ \lambda_4(M) = \lambda_1(M)^{-1}$ and $\rho(M) = |\lambda_1(M)| \geq |\lambda_2(M)| \geq 1 \geq |\lambda_3(M)| \geq |\lambda_4(M)|$. Let $M_1, M_2 \in \mathbf{SL}(2,\mathbb{R})$ Then (1.1) implies that $M_1 \odot M_2 \in \mathbf{Sp}(2,\mathbb{R})$. Clearly, $M_1 \odot M_2 = P(M_1 \oplus M_2)P^T$, where $P$ is a permutation matrix which exchange the second and the third rows. Hence $M_1 \odot M_2$ is similar to $M_1 \oplus M_2$. Furthermore $M_1 \odot M_2$ is conjugate to $M_2 \odot M_1$ by the symplectic matrix $P \oplus P$, $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Let $\theta(M_1 \times M_2) := M_1 \odot M_2$. Then $\theta : \mathbf{SL}(2,\mathbb{R}) \times \mathbf{SL}(2,\mathbb{R}) \to \mathbf{Sp}(2,\mathbb{R})$ is a faithful representation. Recall the well known normal forms of conjugacy classes in $\mathbf{SL}(2,\mathbb{R}) = \mathbf{Sp}(1,\mathbb{R})$:

**Proposition 3.2** *Let $X \in \mathbf{SL}(2,\mathbb{R})$, $X \neq \pm I_2$. Then $X$ is conjugate to one and only one of the following normal forms in $\mathbf{SL}(2,\mathbb{R})$:*

$$\begin{pmatrix} 1/\alpha & 0 \\ 0 & \alpha \end{pmatrix},\ |\alpha| > 1, \tag{3.5}$$

$$\pm\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},\ \pm\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \tag{3.6}$$

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix},\ \begin{pmatrix} a & -b \\ b & a \end{pmatrix},\ a^2 + b^2 = 1,\ b > 0. \tag{3.7}$$

Note that the two canonical forms in (3.7) and each of the two corresponding canonical forms in (3.6) are similar by $\mathrm{diag}(1,-1) \in \mathbf{O}(2,\mathbb{R})$. $X \in \mathbf{SL}(2,\mathbb{R})$ is called parabolic if $X$ is conjugate to one of the forms in (3.6). In what follows we give normal forms of $M \in \mathbf{Sp}(2,\mathbb{R})$ according to the location of a fixed point of $M$, the spectrum of $M$ and the conjugacy of $M$ to $X \odot Y$.

**Theorem 3.3** *Let $M \in \mathbf{Sp}(2,\mathbb{R})$. Then*
**I**. *$M$ is conjugate to*

$$\begin{pmatrix} A & 0 \\ C & A^{-\mathrm{T}} \end{pmatrix} \tag{3.8}$$

*if and only if $M$ has a fixed point in the Shilov boundary of $\mathbf{SH}_2$. Furthermore we have the following forms for each type:*

- **Type 1**. *$M$ is hyperbolic. Then $M$ is conjugate to*

$$\begin{pmatrix} A & 0 \\ 0 & A^{-\mathrm{T}} \end{pmatrix}. \tag{3.9}$$

  *Furthermore*

  **(a)** *$M$ is conjugate to $X \odot Y$ with $X$ and $Y$ hyperbolic if an only if $M$ has real spectrum and $M$ is similar to a diagonal matrix, i.e. $A = \mathrm{diag}(\lambda_4(M), \lambda_3(M))$.*

  **(b)** $A = \begin{pmatrix} \lambda_4(M) & 0 \\ 1 & \lambda_4(M) \end{pmatrix}$ *if an only if $M$ has real spectrum and $M$ is not similar to a diagonal matrix.*

  **(c)** *$\rho(M)A$ has the first form of (3.7) if and only if $\mathrm{spec}(M) \subset \mathbb{C}\backslash\mathbb{R}$.*

- **Type 2**. *$M$ is conjugate to*

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 2a & 0 & 0 \\ 0 & \delta & 2a & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{3.10}$$

  *where $|a| < 1$ and $\delta = 0, \pm 1$.*

  **(a)** *$\delta = 0$ if and only if $M$ is conjugate to $X \odot X^{-1}$, where $X$ is elliptic. ($M$ has two double nonreal eigenvalues on the unit circle and $M$ is diagonable.)*

  **(b)** *$\delta = \pm 1$ if and only $M$ has two nonreal double eigenvalues on the unit circle and $M$ is not diagonable.*

- **Type 3**. *$M$ is conjugate to $X \odot Y$ where is $Y$ hyperbolic and $X$ is either parabolic or $\pm I_2$.* ($\mathrm{spec}(M)$ *is real and contains either* $1$ *or* $-1$ *but* $\mathrm{spec}(M) \not\subset \{1,-1\}$). *Equivalently, $M$ is conjugate to*

$$\pm \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ \delta & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/\alpha \end{pmatrix}, \ \alpha \neq \pm 1, \ \delta = 0, \pm 1. \tag{3.11}$$

- **Type 4**. *$M$ is conjugate to $X \odot Y$ where $X$ and $Y$ are parabolic or $\pm I_2$.* (spec$(M) \subset \{1, -1\}$ *and $M$ is not similar to a Jordan block of order* 4). *Equivalently $M$ or $-M$ is conjugate to*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ \delta_1 & 0 & 1 & 0 \\ 0 & \delta_2 & 0 & \alpha \end{pmatrix}, \ \alpha = \pm 1, \ \delta_1, \delta_2 = 0, \pm 1. \tag{3.12}$$

( *If $\alpha = -1$ then $M$ is conjugate to one of the above forms.*)

- **Type 5**. *$M$ is similar to a Jordan block of order* 4 *if and only if $M$ is conjugate to* (3.10) *with $a = \pm 1$ and $\delta = \pm 1$.* (spec$(M) = \{1\}$, $\{-1\}$.)

**II**. *$M$ has a fixed point in $\partial_1 \mathbf{SH}_2$ if and only if $M$ is conjugate $X \odot Y$ with the following two possibilities:*

**(a)** *$X$ elliptic or $\pm I_2$ and $Y$ hyperbolic, parabolic or $\pm I_2$,* (spec$(M)$ *has two eigenvalues on the unit circle and two real eigenvalues and either $M$ is diagonable or $M$ has exactly one Jordan block of order two with the eigenvalue* $\pm 1$). *Equivalently $M$ is conjugate to*

$$\begin{pmatrix} a_1 & 0 & -c_1 & 0 \\ 0 & a_4 & 0 & 0 \\ c_1 & 0 & a_1 & 0 \\ 0 & \delta & 0 & a_4^{-1} \end{pmatrix}, \quad a_1^2 + c_1^2 = 1, \ \delta = 0, \pm 1, \tag{3.13}$$

*where $\delta$ can only be nonzero if $a_4 = \pm 1$.*

**(b)** *with $Y = X^{-1}$ and $X$ parabolic,* (spec$(M) = \{1\}$ ,$\{-1\}$, *and $M$ has two Jordan blocks of order two*).

**III**. *$M$ is conjugate to $X \odot Y$, where $X$ and $Y$ are either elliptic or $\pm I_2$, if and only if $M$ has a fixed point inside $\mathbf{SH}_2$.* (*$M$ is either elliptic or $\pm I_4$*). *Equivalently $M$ is conjugate to*

$$\begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ -b_1 & 0 & a_1 & 0 \\ 0 & -b_2 & 0 & a_2 \end{pmatrix}, \quad a_1^2 + b_1^2 = a_2^2 + b_2^2 = 1. \tag{3.14}$$

*Furthermore, $M$ is conjugate to the form* (3.8) *if an only if either $M$ is conjugate to $X \odot X^{-1}$ with an elliptic $X$, or $M$ is conjugate to* $(\pm I_2) \odot (\pm I_2)$.

**Proof.** . According to Lemma 3.1 $M$ has a fixed point $\zeta \in \mathrm{Cl}(\mathbf{SH}_2)$. Then there exists $T \in \mathbf{Sp}(2, \mathbb{R})$ such that $T(\zeta) \in \{0, \sqrt{-1}\,\mathrm{diag}(1,0), \sqrt{-1} I_2\}$. Hence $M_1 = T^{-1} M T$ has a fixed point in the set $\{0, \sqrt{-1}\,\mathrm{diag}(1,0), \sqrt{-1} I_2\}$.

**I**. $M_1$ has a fixed point 0. Then (3.4) implies that $M_1$ is of the form (3.8).

**Type 1**. By [3, Prop. 5.4] any hyperbolic $M_1$ is conjugate to the form (3.9), where the eigenvalues of $A$ are inside the unit disk. Suppose first that spec$(M_1) \subset \mathbb{R}$. Then

$\operatorname{spec}(A) \subset \mathbb{R}$. Furthermore $M_1$ is similar to diagonal matrix if and only if $A$ is similar to a diagonal matrix. Assume that $M$ is similar to a diagonal matrix. Then $A = PA_1P^{-1}$, $A_1 = \operatorname{diag}(\lambda_4(M), \lambda_3(M))$, $P \in \mathbf{GL}(2,\mathbb{R})$ and $S^{-1}M_1S = A_1 \oplus A_1^{-T}$, $S = P \oplus P^{-T} \in \mathbf{Sp}(2,\mathbb{R})$. Hence $M$ is conjugate to the form (a). Note that the form (a) is equal to $X \odot Y$, $X = \operatorname{diag}(\lambda_4(M), 1/\lambda_4(M))$, $Y = \operatorname{diag}(\lambda_3(M), 1/\lambda_3(M))$. Suppose that $M$ is not similar to a diagonal matrix. Hence $A$ is not similar to a diagonal matrix. In particular $\lambda_3(M) = \lambda_4(M)$ and $A$ is similar to $\begin{pmatrix} \lambda_4(M) & 0 \\ 1 & \lambda_4(M) \end{pmatrix}$. Use the above arguments to show that $M$ is conjugate to the form (b). Assume now that $\operatorname{spec}(M) \subset \mathbb{C}\backslash\mathbb{R}$. Then $\rho(M)|\lambda_3(M)| = \rho(M)|\lambda_4(M)| = 1$ and $\rho(M)A$ is similar to the first matrix of (3.7). Use the above arguments to show that $M$ is conjugate to the form (c).

**Type 2**. $M$ is not hyperbolic and $M$ does not have real eigenvalues. Then $\operatorname{spec}(A) \subset S^1\backslash\{1,-1\}$ and $M$ has two nonreal double eigenvalues on $S^1$. Then $A = PA_1P^{-1}$, $A_1 = \begin{pmatrix} 0 & 1 \\ -1 & 2a \end{pmatrix}$ and $P \in \mathbf{GL}(2,\mathbb{R})$. Then $M_1 = (P \oplus P^{-T})^{-1}M(P \oplus P^{-T}) \in \mathbf{Sp}(2,\mathbb{R})$, where $M_1$ is of the form (3.8) with $A = A_1$. Let

$$M_2 = R^{-1}M_1R = \begin{pmatrix} A_1 & 0 \\ C_1 & A_1^{-T} \end{pmatrix}, \; R = \begin{pmatrix} I_2 & 0 \\ Z & I_2 \end{pmatrix} \in \mathbf{Sp}(2,\mathbb{R}) \iff$$
$$Z \in \mathbf{Sym}(2,\mathbb{R}). \tag{3.15}$$

Choose $Z \in \mathbf{Sym}(2,\mathbb{R})$ so that the first column of $C_1$ is zero. Since $M_2 \in \mathbf{Sp}(2,\mathbb{R})$ it follows that $C_1 = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$.

**(a)** $x = 0$. Then $M_2$ is in the form (3.10) with $\delta = 0$. Thus $M$ is elliptic with nonreal double eigenvalues. We claim that in this case $M_2$ is conjugate to $X \odot X^{-1}$, where $X$ is an elliptic matrix of the first form in (3.7) having two distinct eigenvalues of $M$. Clearly, $A_1 = QXQ^{-1}$ for some $Q \in \mathbf{GL}(2,\mathbb{R})$. Then $X \oplus X = (Q \oplus Q^{-T})^{-1}M_2(Q \oplus Q^{-T})$ and

$$R(X \oplus X)R^{-1} = X \odot X^{-1}, \quad R = \begin{pmatrix} \frac{1}{2} & 0 & 1 & -1 \\ \frac{1}{2} & 0 & 1 & 1 \\ 0 & \frac{1}{2} & 1 & 1 \\ 0 & -\frac{1}{2} & 1 & -1 \end{pmatrix} \in \mathbf{Sp}(2,\mathbb{R}). \tag{3.16}$$

**(b)** $x \neq 0$. Then conjugate $M_2$ by $P \oplus P^{-T}$, where $P = \sqrt{|x|}I_2$ to obtain the form (3.10) with $\delta = \pm 1$. Clearly $M$ ($M_2$) has two nonreal double eigenvalues on $S^1$ and $M$ ($M_2$) is not diagonable.

**Type 3**. $M$ is not hyperbolic, $\operatorname{spec}(M) \subset \mathbb{R}$ and $\operatorname{spec}(M) \not\subset \{1,-1\}$. Hence $A$ has real eigenvalues, and one eigenvalue is equal to $\pm 1$ and the other eigenvalue is different from $\pm 1$. By considering $-M$ instead of $M$ if needed, we may assume that $\operatorname{spec}(A) = \{1,\alpha\}$, $\alpha \neq \pm 1$. Then $A = PA_1P^{-1}$, $A_1 = \operatorname{diag}(1,\alpha)$, $P \in \mathbf{GL}(2,\mathbb{R})$. The matrix $M_1 = (P \oplus P^{-T})^{-1}M(P \oplus P^{-T})$ is of the form (3.8) with $A = A_1$. Let $M_2$ be given by (3.15). Choose $Z \in \mathbf{Sym}(2,\mathbb{R})$ such that the second column of $C_1$ is equal to zero. Since $M_2 \in \mathbf{Sp}(2,\mathbb{R})$ it follows that $C_1 = \operatorname{diag}(x,0)$. If $x = 0$ then $M$ is of the from (3.11). If $x \neq 0$ conjugate $M_2$ by $P \oplus P^{-T}$, where $P = \sqrt{|x|}I_2$ to obtain the form (3.10) with $\delta = \pm 1$.

Assume that $M$ is in the form (3.11). If $\delta = 0$ then $M = \pm(I_2 \odot \mathrm{diag}(\alpha, \alpha^{-1}))$. If $\delta = \pm 1$ then $M = \pm(\begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix} \odot \mathrm{diag}(\alpha, \alpha^{-1}))$.

Vice versa, assume that $M$ is conjugate to $X \odot Y$ where $Y$ is hyperbolic and $X$ is either $\pm I_2$ or parabolic. Conjugate $X$ and $Y$ in $\mathbf{SL}(2, \mathbb{R})$ to deduce that we may assume that $X$ and $Y$ are in their canonical forms given by Proposition 3.2. Then $X \odot Y$ is of the form (3.11).

**Type 4**. $\mathrm{spec}(M) \subset \{1, -1\}$. Then $\mathrm{spec}(A) \subset \{1, -1\}$. Assume first that $\mathrm{spec}(A) = \{1, -1\}$. Then the above arguments show that $M$ is conjugate to $M_1$ of the (3.8) with $A = A_1 = \mathrm{diag}(1, -1)$. Let $M_2$ be given by (3.15). Then one can choose $Z \in \mathbf{Sym}(2, \mathbb{R})$ such that $C_1 = \mathrm{diag}(\delta_1, \delta_2)$. Conjugate $M_2$ by $P \oplus P^{-T}$ where $P$ is a diagonal matrix, to deduce that we may assume that $\delta_1, \delta_2 \in \{0, 1, -1\}$. Clearly (3.12) is of the form $X \odot Y$, where $\mathrm{spec}(X) = \{1\}$, $\mathrm{spec}(Y) = \{-1\}$, $X$ is either $I_2$ or in one of the canonical forms in (3.6) and $Y$ is either $-I_2$ or in one of the canonical forms in (3.6).

Assume now $\mathrm{spec}(A) = \{1\}$. (If $\mathrm{spec}(A) = \{-1\}$ consider $-M$.) Assume first that $A = I_2$. Then $C$ in the form (3.8) is real symmetric. Conjugate by $M$ by $Q \oplus Q$, $Q \in \mathbf{O}(2, \mathbb{R})$ to deduce that $C$ can be chosen a diagonal matrix $C = \mathrm{diag}(\delta_1, \delta_2)$. Conjugate again by $P \oplus P^{-T}$, where $P$ is a diagonal matrix, to deduce that we may assume that $\delta_1, \delta_2 \in \{0, 1, -1\}$. Then (3.12) is of the form $X \odot Y$, where $\mathrm{spec}(X) = \mathrm{spec}(Y) = \{1\}$, $X$ or $Y$ is either $I_2$ or in one of the canonical forms in (3.6).

Vice versa, assume that $M$ is conjugate to $X \odot Y$ where $X$ and $Y$ are parabolic or $\pm I_2$. By conjugating $X$ and $Y$ to their canonical forms in $\mathbf{SL}(2, \mathbb{R})$ we deduce that $X \odot Y$ is of the form $\pm M$, where $M$ is given by (3.12).

Assume that $A \neq I_2$. Then $A = P A_1 P^{-1}$, where $A_1 = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$, $P \in \mathbf{GL}(2, \mathbb{R})$. Then $M_1 = (P \oplus P^{-T})^{-1} M (P \oplus P^{-T})$ is of the form (3.8) with $A = A_1$. The arguments in the proof of Type 2 form yield that $M$ is conjugate to $M_2$ of the form (3.10) with $a = 1$ and $\delta = 0, \pm 1$. Assume first that $\delta = 0$. Let $X = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Clearly, $A_1 = Q X Q^{-1}$ for some $Q \in \mathbf{GL}(2, \mathbb{R})$. Then $X \oplus X^{-T} = (Q \oplus Q^{-T})^{-1} M_2 (Q \oplus Q^{-T})$ and

$$R(X \oplus X^{-T}) R^{-1} = X \odot X^{-1}, \quad R = \begin{pmatrix} \frac{1}{2} & 0 & 0 & -1 \\ \frac{1}{2} & 0 & 1 & 1 \\ 0 & \frac{1}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 \end{pmatrix} \in \mathbf{Sp}(2, \mathbb{R}). \tag{3.17}$$

**Type 5**. It is straightforward to show that in the case $\delta = \pm 1$, $a = \pm 1$ the form (3.10) is similar to one Jordan block of order 4.

**II**. $M_1$ has a fixed point $L := \sqrt{-1}\,\mathrm{diag}(1, 0)$. Let Stab $L \subset \mathbf{Sp}(2, \mathbb{R})$ be the stabilizer of $L$ with respect to the action (3.4), i.e. all $M$ of the form (3.3) such that $AL + B = L(CL + D)$.

A straightforward calculation shows that

$$M=\begin{pmatrix} a_1 & a_2 & -c_1 & 0\\ 0 & a_4 & 0 & 0\\ c_1 & c_2 & a_1 & 0\\ c_3 & c_4 & d_3 & d_4 \end{pmatrix}. \tag{3.18}$$

The condition (3.3) yields

$$a_1^2+c_1^2=1,\ a_4d_4=1,\ c_3=d_4(a_1c_2-c_1a_2),\ d_3=-d_4(c_1c_2+a_1a_2). \tag{3.19}$$

Then

$$R=\begin{pmatrix} I_2 & 0\\ Z & I_2\end{pmatrix}\in \mathrm{Stab}\ L \iff Z=\begin{pmatrix} 0 & x\\ x & y\end{pmatrix}. \tag{3.20}$$

Suppose first that $c_1\neq 0$ in the matrix $M$ given in (3.18). Then it is possible to choose the entries $x,y$ of $Z$ in $R$ such that $M_1:=R^{-1}MR$ is of the form (3.18) with $a_2=d_3=0$. The equalities (3.19) yield that $c_2=0,\ c_3=0$. Hence $M=X\odot Y$, and $X=\begin{pmatrix} a_1 & -c_1\\ c_1 & a_1\end{pmatrix}$, $Y=\begin{pmatrix} a_4 & 0\\ c_4 & a_4^{-1}\end{pmatrix}$, where $X$ is elliptic and $Y$ is hyperbolic, parabolic or $\pm I_2$. Use the subgroup $\theta(\{I_2\}\times\mathbf{SL}(2,\mathbb{R}))\subset\mathbf{Sp}(2,\mathbb{R})$ to conjugate $Y$ to its normal form in $\mathbf{SL}(2,\mathbb{R})$. Then we have the form (3.13) with $\delta=0,\pm1$. In particular $\delta\neq 0$ if only $d_4=\pm1$.

Assume now that $c_1=0$. Then $a_1=\pm1$ is a double eigenvalue of $M$. The other two eigenvalues of $M$ are $a_1,\ a_1^{-1}=d_4$. Observe that the set of all $M$ of the form

$$Q=P\oplus P^{-T},\quad P=\begin{pmatrix} 1 & z\\ 0 & 1\end{pmatrix} \tag{3.21}$$

form a subgroup in Stab $L$. Suppose first that $a_1\neq a_4$. Choose $Q$ of the above form to obtain $M_1=Q^{-1}MQ$ of the form (3.18) with $c_1=a_2=d_3=0$. Choose $R$ of the above form to obtain $M_2=R^{-1}M_1R$ of the form (3.18) with $c_1=a_2=d_3=c_2=c_3=0$. Then $M_2=X\odot Y$ and $X=\pm I_2,\ Y=\begin{pmatrix} a_4 & 0\\ c_4 & a_4^{-1}\end{pmatrix}$. Bring $Y$ to its normal lower triangular form with $c_4=0,\pm1$ as in Proposition 3.2.

It is left to discuss the case where $a_1=a_4=d_4=\pm1$. By considering $-M$ we may assume that $a_1=a_4=d_4=1$. As $M$ fixes 0 and its spectrum is $\{1\}$ we deduce that $M$ is conjugate either to (I4) or to (I5). The cases (I4) and (I5) of Theorem 4.1 analyze the fixed points of $M$ of the form (I4) and (I5). (Their proof is independent of the form of $M$ of type (II).) It is shown that $M$ of type (I5) has one fixed point in the Shilov boundary. Hence our $M$ is conjugate to one of the forms of (I4). If $M$ is conjugate to $X\odot Y$, where either $X=Y=I_2$ or $X(Y)$ is parabolic (with eigenvalues 1) and $Y(X)=I_2$ then $M$ is conjugate to the form (IIa). It is left to consider the case where $M$ is conjugate to $X\odot Y$ where $X$ and $Y$ are parabolic with eigenvalues 1. Thus $M$ is either conjugate to either $X\odot X$ or $X\odot X^{-1}$. The case (I4) of Theorem claims that $X\odot X$ has one fixed point in the Shilov boundary, while $X\odot X^{-1}$ has many fixed points in $\partial_1\mathbf{SH}_2$. Hence the case (b) follows.

**III**. $M$ fixes the point $\sqrt{-1}I_2$. This is a well known case discussed also in [3]. Then $M \in \mathbf{K_2} = \mathbf{Sp}(2,\mathbb{R}) \cap \mathbf{SO}(4,\mathbb{R})$ and $M$ has the form (3.3) with $C = -B, D = A$, $A^{\mathrm{T}}B \in \mathbf{Sym}(2,\mathbb{R})$ and $A^{\mathrm{T}}A + B^{\mathrm{T}}B = I_2$. By considering the action on the Siegel disk $\mathbf{SD}_2$, The action of $M$ on $\mathbf{SH}_2$ corresponds to the action of $M' = U \oplus \overline{U}$ on $\mathbf{SD}_2$, where $U = A + iB$ is unitary and fixing the point $0 \in \mathbf{SD}_2$. The group that fixes $0 \in \mathbf{SD}_2$ is $\tilde{\mathrm{U}}_2 = \{T \oplus \overline{T} : T \in \mathbf{U}_2\}$. Since any unitary matrix is unitarily diagonalizable, take $V \in \mathbf{U}_2$ such that $VUV^* = D = \mathrm{diag}(\lambda,\mu)$, $|\lambda| = |\mu| = 1$. Then $V \oplus \overline{V} \in \tilde{\mathrm{U}}_2$ and $(V \oplus \overline{V})(U \oplus \overline{U})(V^* \oplus V^{\mathrm{T}}) = D \oplus \overline{D}$. Hence $M$ is conjugate to

$$\begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ -b_1 & 0 & a_1 & 0 \\ 0 & -b_2 & 0 & a_2 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{pmatrix} \odot \begin{pmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{pmatrix},$$

where $\lambda = a_1 + \sqrt{-1}b_1,\ \mu = a_2 + \sqrt{-1}b_2,\ |\lambda| = |\mu| = 1$. Thus every element in $\mathbf{K}_2$ is conjugate to $X \odot Y$, where $X, Y \in \mathbf{SL}(2,\mathbb{R})$ are elliptic or $\pm I_2$. The case (III) of Theorem 4.1 imply $M \in \mathbf{K}_2$ has a fixed point on the Shilov boundary of $\mathbf{SH}_2$ if and only if either $M$ is conjugate to $X \odot X^{-1}$ with an elliptic $X$, or $M$ is conjugate to $(\pm I_2) \odot (\pm I_2)$. $\square$

**Corollary 3.4** *An elliptic $M \in \mathbf{Sp}(2,\mathbb{R})$ fixes a point in $\mathbf{SH}_2$.*

# 4 Fixed points: classification

In this section we present the classification of fixed points of $M \in \mathbf{Sp}(2,\mathbb{R})$. The proofs of the results will be given in the last section of this paper.

Notice that in case $M = X \odot Y$, if $\xi$ is a fixed point for $X$ and $\eta$ a fixed point for $Y$, then $\xi \times \eta \in \mathrm{Cl}(\mathbf{H} \times \mathbf{H})$ will be a fixed point for $X \odot Y$. If $M$ is equal to $T(X \odot Y)T^{-1}$ for some $T \in \mathbf{Sp}(2,\mathbb{R})$ we shall refer to $T(\xi \times \eta)$ as an *ordinary* fixed point. Observe next that the closure of $\mathbf{SH}_2$ is a semi algebraic set in $\mathbf{Gr}(4,2,\mathbb{C})$ - the Grassmannian of two dimensional subspaces in $\mathbb{C}^4$ [3]. $M$ acts on $\mathbf{Gr}(4,2,\mathbb{C})$ as a bihomolorphism. Hence the set of fixed points of $M$ in $\mathbf{Gr}(4,2,\mathbb{C})$ is a projective variety. Therefore the set of the fixed points of $M$ in $\mathrm{Cl}(\mathbf{SH}_2)$ is compact semi algebraic set. We shall show that if $M \neq \pm I_4$ the set of fixed points of $M$ in $\mathrm{Cl}(\mathbf{SH}_2)$ is a a closed semi algebraic set which is a finite disjoint union of sets of the following types: (a) a point; (b) a smooth connected closed one dimensional manifold - $S^1$; (c) a closed disk $\overline{\mathbf{D}}$; (d) two copies of $\overline{\mathbf{D}}$ intersecting at one point lying on the boundaries of each $\mathbf{D}$; (e) $\overline{\mathbf{D}} \times \overline{\mathbf{D}}$.

**Theorem 4.1** *Let $M \in \mathbf{Sp}(2,\mathbb{R})$. Then the set of the fixed points of $M$ in $\mathrm{Cl}(\mathbf{SH}_2)$ is of the following type according to the classification given in Theorem 3.3*

- **(I1a)** *If $M$ has four distinct eigenvalues ($M$ is similar to $X \odot Y$ and $X$ and $Y^{-1}$ are not conjugate) then $M$ has four (ordinary) fixed points in the Shilov boundary of $\mathbf{SH}_2$. One point is a hyperbolic attractor and one point is a hyperbolic repeller. If $M$ has two real double eigenvalues ($M$ is conjugate to $X \odot X^{-1}$) then the set of fixed points of $M$ of consists of one hyperbolic attracting (ordinary) fixed point, one hyperbolic repelling* *(ordinary) fixed point and (ordinary and nonordinary) fixed set $S^1$, all lying in the Shilov boundary of $\mathbf{SH}_2$.*

- **(I1b)** *$M$ has three fixed points in the Shilov boundary of $\mathbf{SH}_2$. One point is a hyperbolic attractor and one point is a hyperbolic repeller.*

- **(I1c)** *$M$ has two fixed points in the Shilov boundary of $\mathbf{SH}_2$. One point is a hyperbolic attractor and one point is a hyperbolic repeller.*

- **(I2)** *If $M$ is conjugate to $X \odot X^{-1}$, where $X$ is elliptic ($\delta = 0$), then $M$ has $\overline{\mathbf{D}}$ of (ordinary and nonordinary) fixed points. The open disk $\mathbf{D}$ lies in $\mathbf{SH}_2$ and its boundary $S^1$ lies in the Shilov boundary. If $\delta = \pm 1$ then $M$ has one fixed point in the Shilov boundary.*

- **(I3)** *If $M$ is conjugate to $X \odot Y$, where $X$ is parabolic ( $M$ is not diagonable), then $M$ has two (ordinary) fixed points in the Shilov boundary. If $M$ is conjugate to $X \odot Y$, where $X = \pm I_2$, then then the set of (ordinary) fixed points consists of two disjoint closed disks $\overline{\mathbf{D}}$. The intersection of each $\overline{\mathbf{D}}$ with $\partial_2 \mathbf{SH}_2$ is $S^1$. Other points of each $\overline{\mathbf{D}}$ lie in $\partial_1 \mathbf{SH}_2$.*

- **(I4)** *$M$ is conjugate to $X \odot Y$. If $X = Y = \pm I_2$ then $M$ fixes every point of $\mathrm{Cl}(\mathbf{SH}_2)$. If $X = -Y = \pm I_2$ then $M$ fixes exactly $\overline{\mathbf{D}} \times \overline{\mathbf{D}}$, which is the ordinary set of fixed points. The torus $S^1 \times S^1$ lies in the Shilov boundary. The two open disjoint three manifolds $S^1 \times \mathbf{D}$, $\mathbf{D} \times S^1$ lie in $\partial_1 \mathbf{SH}_2$. The open four manifold $\mathbf{D} \times \mathbf{D}$ lies in $\mathbf{SH}_2$. If $X$ $(Y)$ is parabolic and $Y$ $(X) = \pm I_2$ then $\overline{\mathbf{D}}$ is the set of (ordinary) fixed points of $M$. $M$ has exactly $S^1$ fixed points in the Shilov boundary and all other fixed points are in $\partial_1 \mathbf{SH}_2$. If $X$ and $Y$ are parabolic and $Y$ is not conjugate to $X^{-1}$ then $M$ has one (ordinary) fixed point in the Shilov boundary. If $Y$ is conjugate to $X^{-1}$ then $M$ fixes exactly two copies of $\overline{\mathbf{D}}$, which intersect at one point $\xi$ lying in the boundary of each $\mathbf{D}$. $\xi$ is the ordinary fixed point of $M$ lying in the Shilov boundary. Each copy of $\mathbf{D}$ lies in $\partial_1 \mathbf{SH}_2$, while each $S^1$ lies in the Shilov boundary.*

- **(I5)** *$M$ has one fixed point in the Shilov boundary.*

- **(II)** *$M$ is conjugate to $X \odot Y$.*

- **(a)** *$X$ is elliptic. If $Y$ is hyperbolic then $M$ has two (ordinary) fixed points in $\partial_1 \mathbf{SH}_2$. If $Y$ is parabolic then $M$ has one (ordinary) fixed point in $\partial_1 \mathbf{SH}_2$. If $Y = \pm I_2$ then $\overline{\mathbf{D}}$ is the set of (ordinary) fixed points of $M$. The open disk $\mathbf{D}$ lies in $\mathbf{SH}_2$ while its boundary $S^1$ lies in $\partial_1 \mathbf{SH}_2$. (The case where $X = \pm I_2$ is covered by the cases (I3)and (I4).)*

- **(b)** *is covered in (I4).*

- **(III)** *If $X$ and $Y$ are elliptic and $Y$ is not conjugate to $X^{-1}$ then $M$ has one fixed point in $\mathbf{SH}_2$. The case where $X$ and $Y$ are elliptic and $Y$ is conjugate to $X^{-1}$ is covered in (I2). The case $X$ $(Y)$ is elliptic and $Y$ $(X) = \pm I_2$ is covered in (IIa) ( $X \odot Y$ is conjugate to $Y \odot X$ ). The case where $X, Y = \pm I_2$ is covered in (I4).*

**Corollary 4.2** *Assume that $M \in \mathbf{Sp}(2,\mathbb{R})$ has a finite number of fixed points in $\mathrm{Cl}(\mathbf{SH}_2)$. Then all of them lie either in the same stratum of the boundary or inside $\mathbf{SH}_2$.*

**Corollary 4.3** *Assume that $M \in \mathbf{Sp}(2,\mathbb{R})$ is conjugate to $X \odot Y$. Then all the fixed points of $M$ are ordinary fixed points if and only if $Y$ is not conjugate to $X^{-1}$.*

# 5 The boundary in the projective model

Recall the identification $\mathbf{Gr}(4,2,\mathbb{C})$ with $\mathbf{M}_f(4,2,\mathbb{C})/\mathbf{GL}(2,\mathbb{C})$ discussed [3, §2]. We write $A \in \mathbf{M}_f(4,2,\mathbb{C})$ as $(U^{\mathrm{T}}, V^{\mathrm{T}})^{\mathrm{T}}$, where $U, V \in \mathbf{M}(2,\mathbb{C})$. Then

$$[(U_1^{\mathrm{T}}, V_1^{\mathrm{T}})^{\mathrm{T}}] = [(U_2^{\mathrm{T}}, V_2^{\mathrm{T}})^{\mathrm{T}}] \iff (U_1^{\mathrm{T}}, V_1^{\mathrm{T}})^{\mathrm{T}} = (U_2^{\mathrm{T}}, V_2^{\mathrm{T}})^{\mathrm{T}} P, \quad P \in \mathbf{GL}(2,\mathbb{C}). \tag{5.1}$$

Recall the projective model $\mathbf{SPH}_2$ of $\mathbf{SH}_2$ in [3, §2].

**Proposition 5.1** *Every point in the boundary of $\mathbf{SPH}_2$ is uniquely presented by one of the following kind matrices:*

$$(Z, I_2)^{\mathrm{T}}, \quad (I_2, -Z)^{\mathrm{T}}, \quad (I_2 + Z, I_2 - Z)^{\mathrm{T}}. \tag{5.2}$$

*In the first kind $Z \in \mathbf{Sym}(2,\mathbb{C})$ and $\mathrm{Im}\, Z$ singular positive semi definite. The set of the first kind matrices describes the finite boundary of $\mathbf{SH}_2$, which is a semi algebraic set of dimension 5. In the second kind $Z \in \mathbf{Sym}(2,\mathbb{C})$, $\det Z = 0$ and $\mathrm{Im}\, Z$ singular positive semi definite. The set of the second kind matrices is a semi algebraic set of dimension 3. In the third kind $Z \in \mathbf{Sym}(2,\mathbb{R})$ with $\mathrm{spec}(Z) = \{1, -1\}$. The set of the third kind matrices is $S^1$ and it lies in Shilov boundary of $\mathbf{SPH}_2$.*

**Proof.** In what follows we let $Z = X + \sqrt{-1}Y,\ X, Y \in \mathbf{Sym}(2,\mathbb{R})$. Then $Z$ lying in the finite boundary of $\mathbf{SH}_2$ if and only if $Y \geq 0,\ \det \mathrm{Y} = 0$. Clearly this boundary is 5 dimensional semi algebraic set. Every point $Z$ in this boundary is uniquely presented by $(Z, I_2)^{\mathrm{T}}$.

Recall next that $\mathbf{Sp}(2,\mathbb{R})$ acts on $\mathbf{Gr}(4,2,\mathbb{C})$ by multiplication from the left. L et $M = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. Then $M(Z) = -Z^{-1}$ for any $Z \in \mathbf{SH}_2$. Recall also that $M$ acts on each of the boundary stratum of $\mathbf{SH}_2$. Assume now that $Y \geq 0$ and $\det \mathrm{Z} = 0$. Hence $\det \mathrm{Y} = 0$. Assume first that $Y = \mathrm{diag}(y, 0),\ y > 0$. Then $X = \mathrm{diag}(x, 0)$. Hence all such $Z$ are of the form $O^{\mathrm{T}}(\mathrm{diag}(x,0) + \sqrt{-1}\,\mathrm{diag}(y,0))O,\ y \geq 0,\ O \in \mathbf{SO}(2,\mathbb{R})$. Thus all singular $Z$ lying in the finite boundary of $\mathbf{SH}_2$ is a semi algebraic variety of dimension 3. Hence $Q = (Z, I_2)^{\mathrm{T}}$ lies on the boundary of $\mathbf{SPH}_2$. Then $MQ = (I_2, -Z)^{\mathrm{T}}$ lies in the boundary of $\mathbf{SPH}_2$. Use (5.1) to see that $[MQ]$ can not be presented by any first matrix given in (5.2).

Let $Q = (U^{\mathrm{T}}, V^{\mathrm{T}})^{\mathrm{T}}$ and assume that $[Q]$ is lying on the boundary of $\mathbf{SPH}_2$. Suppose furthermore that $[Q]$ is not presented by either first or second kind of (5.2). Hence $\det \mathrm{U} = \det \mathrm{V} = 0$. Then $[MQ]$ is not presented by either first or second matrix of (5.2). Recall the complex symplectic maps $\Phi_2, \Phi_2^{-1}$ connecting the models $\mathbf{SH}_2$ and $\mathbf{SD}_2$ [3, §2]. The infinite part of the boundary $\mathrm{Cl}(\mathbf{SH}_2)$ is the image of $\{T \in \partial\mathbf{SD}_2 : \det\,(\mathrm{T} - \mathrm{I}) = 0\}$ by

$\Phi_1^{-1}$. It is straightforward to check that the map $Z \mapsto -Z^{-1}$ in $\mathbf{SH}_2$ is conjugate by $\Phi_1$ to the map $W \mapsto -W$ in $\mathbf{SD}_2$. Hence $[Q]$ corresponds to the point in the infinite part of the boundary $\mathrm{Cl}(\mathbf{SH}_2)$, which is the image of $\{T \in \partial\mathbf{SD_2} : \det\,(\mathrm{T}-\mathrm{I}) = \det\,(\mathrm{T}+\mathrm{I}) = 0\}$ by $\Phi_2^{-1}$. That is $T \in \mathbf{USym}(2,\mathbb{C}),\ \mathrm{spec}(T) = \{1,-1\}$. Consider the map $W \to \sqrt{-1}W$ in $\mathbf{SD}_2$. Then $\Phi_1^{-1}(-\sqrt{-1}T) = Y \in \mathbf{Sym}(2,\mathbb{R}),\ \mathrm{spec}(Y) = \{1,-1\}$. It is straightforward to show that $W \to \sqrt{-1}W$ in $\mathbf{SD}_2$ is conjugate by $\Phi_2^{-1}$ to $Z \to N(Z)$ in $\mathbf{SH}_2$, where $N = \begin{pmatrix} \frac{\sqrt{2}I_2}{2} & \frac{-\sqrt{2}I_2}{2} \\ \frac{\sqrt{2}I_2}{2} & \frac{\sqrt{2}I_2}{2} \end{pmatrix}$. Hence $[N^{-1}(Y,I_2)^{\mathrm{T}}] = [\frac{\sqrt{2}}{2}(I+Y,I-Y)^{\mathrm{T}}] = [(I+Y,I-Y)^{\mathrm{T}}]$. The set of all $2\times 2$ unitary symmetric matrices with the spectrum $\{1,-1\}$ is $S^1$. Hence the third kind matrices in (5.2) with $Y \in \mathbf{Sym}(2,\mathbb{R}),\ \mathrm{spec}(Y) = \{1,-1\}$ present faithfully all boundary points in $\mathrm{Cl}(\mathbf{SH}_2)$ which correspond to $T \in \mathbf{USym}(2),\ \mathrm{spec}(T) = \{1,-1\}$. All these points correspond to the points in Shilov boundary of $\mathbf{SD}_2$. $\square$

Sometimes we apply the symplectic transformation $M = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$ to $\mathrm{Cl}(\mathbf{SPH_2})$ to get an equivalent presentation of $\mathbf{SPH_2}$ and its boundaries. Note that $M$ fixes the boundary points of the third kind and it maps the boundary points of the second kind to the boundary points of the first kind (the finite boundary points of $\mathbf{SH}_2$) with $\det \mathrm{Z} = 0$.

# 6 Fixed points: proofs

We need the following well known facts whose proof are left to the reader.

**Lemma 6.1**

1. *If* $Z \in \mathbf{Sym}(n,\mathbb{C})$ *and* $\mathrm{Im}\,Z$ *is positive definite then* $Z$ *is invertible.*
2. *If* $A \in \mathbf{M}(n,\mathbb{R})$ *has a nonreal eigenvalue* $\lambda$ *then any eigenvector* $v$ *associated with* $\lambda$ *can not be real.*
3. *Let* $\mathbb{F} = \mathbb{R},\mathbb{C}$. *If* $Z \in \mathbf{M}(2,\mathbb{F}),\ Z \neq 0,\ \det \mathrm{Z} = 0$ *then there exist two vectors* $u,v \in \mathbb{F}^2$ *such that* $Z = uv^{\mathrm{T}}$. $u,v$ *are uniquely defined up to scalar multiplication. Moreover, if* $Z \in \mathbf{Sym}(2,\mathbb{C})$ *then* $Z = uu^{\mathrm{T}}$ *where* $u$ *is determined up to sign. If in addition* $\mathrm{Im}\,Z \geq 0$ *then* $Z = zvv^{\mathrm{T}}$, $v$ *is real,* $v^{\mathrm{T}}v = 1$ *and* $\mathrm{Im}\,z \geq 0$.
4. *If* $Z \in \mathbf{Sym}(2,\mathbb{R}),\ \mathrm{spec}(Z) = \{1,-1\}$ *then*

$$I + Z = 2uu^{\mathrm{T}},\ I - Z = 2vv^{\mathrm{T}}, \quad u,v \in \mathbb{R}^2,\ u^{\mathrm{T}}u = v^{\mathrm{T}}v = 1,\ u^{\mathrm{T}}v = 0.$$

   *In particular any boundary point of* $\mathbf{SPH}_2$ *of the the third kind has a unique representative of the form* $(uu^{\mathrm{T}}, vv^{\mathrm{T}})^{\mathrm{T}}$.

**Proof of Theorem 4.1.**

**(I1)** We assume that $M$ is of the form (3.9), where $\rho(A) < 1$. Then the action of $M$ on $\mathbf{Sym}(2,\mathbb{C})$ is given by

$$Z \mapsto AZA^{\mathrm{T}}. \tag{6.3}$$

As $\lim_{m\to\infty} A^m = 0$ it follows that $Z = 0$ is unique hyperbolic attracting point in $\mathbf{Sym}(2,\mathbb{C})$. Hence $[(0, I_2)^{\mathrm{T}}]$ is the unique fixed point of the first kind. $M$ acts as follows on the boundary points of the second kind: $[(I_2, -Z)^{\mathrm{T}}] \mapsto [(I_2, -A^{-\mathrm{T}}ZA^{-1})^{\mathrm{T}}]$. Hence $[(I_2, 0)^{\mathrm{T}}]$ is the unique hyperbolic repelling point of the second kind. See also [3]. We now consider the fixed points of $M$ of the third kind. According to the last part of Lemma 6.1 the third kind boundary points are of the form $[Q]$ where $Q = (uu^{\mathrm{T}}, vv^{\mathrm{T}})^{\mathrm{T}}$. Then $M[Q] = [Q]$ if and only if there exists $P \in \mathbf{GL}(2,\mathbb{C})$, $a, b \in \mathbb{C}^*$ such that

$$Au = au,\ P^{\mathrm{T}}u = \frac{1}{a}u, \quad A^{-\mathrm{T}}v = bv,\ P^{\mathrm{T}}v = \frac{1}{b}v. \tag{6.4}$$

**(a)** Let $A = \mathrm{diag}(\lambda_4(M), \lambda_3(M))$ where $\lambda_4(M), \lambda_3(M) \in \mathbb{R}$ and $|\lambda_4(M)| \le |\lambda_3(M)| < 1$. Suppose first that $\lambda_3(M) \ne \lambda_4(M)$, i.e. $X$ is not conjugate to $Y^{-1}$ $(Y)$. Since $A \in \mathbf{Sym}(2,\mathbb{R})$ it follows that $u, v$ are two linearly independent eigenvectors of $A$. Clearly the linearly independent eigenvectors of $A$ are the standard basis vectors $e_1 = (1,0)^{\mathrm{T}}$, $e_2 = (0,1)^{\mathrm{T}}$. Then we have the following two choices corresponding to two points of the third kind:

$$\begin{aligned} &u = e_1,\ a = \lambda_4(M), \quad v = e_2,\ b = \frac{1}{\lambda_3(M)}, \quad P = \mathrm{diag}(\frac{1}{\lambda_4(M)}, \lambda_3(M)),\\ &u = e_2,\ a = \lambda_3(M), \quad v = e_1,\ b = \frac{1}{\lambda_4(M)}, \quad P = \mathrm{diag}(\lambda_4(M), \frac{1}{\lambda_3(M)}). \end{aligned}$$

Assume now that $\lambda_3(M) = \lambda_4(M) = \alpha$. Then $A = \alpha I_2$. Let $u, v$ be an orthonormal basis of $\mathbb{R}^2$. Then $a = \alpha$, $b = \frac{1}{\alpha}$ in (6.4). There exists a unique $P \in \mathbf{Sym}(2,\mathbb{R})$ which satisfies the two conditions of (6.4). Hence any point of the third kind is a fixed point of $M$.

**(b)** $u$ is an eigenvector of $A$ hence $u = e_2$ and $a = \lambda_4(M)$. $v$ is an eigenvector of $A^{\mathrm{T}}$ hence $v = e_1$ and $b = \frac{1}{\lambda_4}$. Then $P = \mathrm{diag}(\lambda_4(M), \frac{1}{\lambda_4(M)})$ satisfies (6.4). Thus $M$ one fixed point of the third kind.

**(c)** Since $A$ has nonreal eigenvalues $A$ can not have real eigenvectors. Hence the first equality of (6.4) can not hold. Thus $A$ does not have fixed points of the third kind.

**(I2)** Assume first that $M$ is conjugate to $X \odot X^{-1}$ where $X$ is elliptic. That is $M = \mathrm{diag}(A, A^{-\mathrm{T}})$ and $A = \begin{pmatrix} 0 & 1 \\ -1 & 2a \end{pmatrix}$, where $|a| < 1$. Then $M$ acts on $\mathbf{Sym}(2,\mathbb{C})$ as in (6.3). A straightforward calculation shows $M$ fixes the following matrices in $\mathbf{Sym}(2,\mathbb{C})$:

$$Z = \begin{pmatrix} z & az \\ az & z \end{pmatrix}, \quad z \in \mathbb{C}. \tag{6.5}$$

Let $z = x + \sqrt{-1}y$, $x, y \in \mathbb{R}$. Then $\mathrm{Im}\, Z = \begin{pmatrix} y & ay \\ ay & y \end{pmatrix}$. Hence $\mathrm{Im}\, Z > 0 \iff \mathrm{Im}\, z > 0$, and $\mathrm{Im}\, Z$ is in the finite boundary of $\mathbf{SH}_2$ if and only if $y = 0 \iff \mathrm{Im}\, Z = 0$. Then set

$\operatorname{Im} z > 0$ is exactly $\mathbf{H} \sim \mathbf{D}$. The set $y = 0$ corresponds to $\mathbb{R}$ which is $S^1$ minus a point at infinity.

Consider now the fixed points of $M$ of the second kind. A straightforward calculation show that $Z$ must be of the form (6.5). As $\det \mathrm{Z} = (1 - \mathrm{a}^2)\mathrm{z}^2 = 0$ we obtain a unique point $Z = 0$ which lie in the Shilov boundary. The arguments in (I1c) yield that $M$ does not have fixed points of the third kind. Thus $\overline{\mathbf{D}}$ is the fixed point set of $M$.

Let $A$ be as above and assume that $M = \begin{pmatrix} A & 0 \\ C & A^{-\mathrm{T}} \end{pmatrix}$, $C = \pm E$ and $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. For the fixed point of the first kind we have the condition

$$AZ(CZ + A^{-\mathrm{T}})^{-1} = Z \quad \Longleftrightarrow \quad AZ = Z(CZ + A^{-\mathrm{T}}). \tag{6.6}$$

Consider first the case $C = E$. Suppose first that $Z$ is invertible. Then $EZ + A^{-\mathrm{T}}$ is similar to $A$. Note that $A^{-\mathrm{T}}$ is similar to $A$. Hence $A^{-\mathrm{T}}$ has the same trace and determinant as $A$. Clearly

$$Z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}, \quad CZ = \begin{pmatrix} z_2 & z_3 \\ 0 & 0 \end{pmatrix} \tag{6.7}$$

Hence the trace condition yields $z_2 = 0$. The determinant condition yields that $z_3 = 0$. Hence $Z$ is singular contrary to our assumption.

Assume now that $\det \mathrm{Z} = 0$, $Z \neq 0$ and $\operatorname{Im} Z \geq 0$. The third part of Lemma 6.1 implies that $Z = zvv^{\mathrm{T}}$ for some $v \in \mathbb{R}^2$ and $z \in \mathbb{C}^*$. The condition (6.6) implies that $Av = \alpha v$. This is impossible as $A$ has no real eigenvalues. Hence $[(0, I_2)^{\mathrm{T}}]$ is the only fixed point of $M$ of the first kind. Same results hold if $C = -E$.

Assume now that $[(I_2, -Z)^{\mathrm{T}}]$ is a fixed point of the second kind. Then $A^{-\mathrm{T}}Z - ZA = C$, which is equal to (6.7), yields $\begin{pmatrix} 2az_1 + 2z_2 & z_3 - z_1 \\ z_3 - z_1 & -2z_2 - 2az_3 \end{pmatrix} = \begin{pmatrix} 0 & \pm 1 \\ 0 & 0 \end{pmatrix}$. This system of equations is unsolvable as $0 = z_3 - z_1 = \pm 1$.

Assume now that $[(uu^{\mathrm{T}}, vv^{\mathrm{T}})^{\mathrm{T}}]$ is a fixed point of the third kind. Then $Auu^{\mathrm{T}} = uu^{\mathrm{T}}P$ and $Cuu^{\mathrm{T}} + A^{-\mathrm{T}}vv^{\mathrm{T}} = vv^{\mathrm{T}}P$, for some $P \in \mathbf{GL}(2, \mathbb{C})$. The first equation yields that $u \in \mathbb{R}^2$ is an eigenvector of $A$ which is impossible.

**(I3)** Let $M$ be of the form (3.8) where $A = \operatorname{diag}(1, \alpha)$, $C = \operatorname{diag}(\delta, 0)$ and $\alpha \neq \pm 1$, $\delta = 0, \pm 1$. For fixed points of the first kind we obtain again the equation $AZ = Z(CZ + A^{-\mathrm{T}})$. With the notation as in (6.7) we get $\begin{pmatrix} z_1 & z_2 \\ \alpha z_2 & \alpha z_3 \end{pmatrix} = \begin{pmatrix} \delta z_1^2 + z_1 & \delta z_1 z_2 + \frac{z_2}{\alpha} \\ \delta z_1 z_2 + z_2 & \delta z_2^2 + \frac{z_3}{\alpha} \end{pmatrix}$. If $z_2 \neq 0$, we simplify the equations from entries (1,2) and (2,1), canceling $z_2$, obtaining $1 - \frac{1}{\alpha} = \delta z_1 = \alpha - 1$. Hence $\alpha^2 - 2\alpha + 1 = 0 \Rightarrow \alpha = 1$ contrary to our assumption. So $z_2 = 0$ and we obtain the conditions: $\delta z_1^2 = 0$ and $\frac{\alpha^2 - 1}{\alpha} z_3 = 0$. If $\delta \neq 0$ then $M$ has the unique fixed point with $Z = 0$. If $\delta = 0$ then $z_3 = 0$ and $z_1$ is a free variable. As $\operatorname{Im} Z \geq 0$ we get that $z_1 \in \overline{\mathbf{H}}$. For $z_1 \in \mathbf{H} \sim \mathbf{D}$ we obtain that $Z \in \partial_1 \mathbf{SH}_2$. If $z \in \mathbb{R}$ then $Z$ is in the Shilov boundary. Note that $\overline{\mathbf{H}}$ is the closed disk $\overline{\mathbf{D}}$ minus a point.

For fixed points of the second kind we get the equation is $AZ - ZA = C$. In terms of the entries of $Z$ we get the conditions $0z_1 = \delta$ and $z_2 = z_3 = 0$. So we will only have fixed points if $\delta = 0$. Assume that $\delta = 0$. Again we obtain a set of fixed points of the second kind in the boundary of $\mathbf{SPH}_2$ equal to a copy of $\overline{\mathbf{H}}$. Points corresponding to $\mathbf{H} \sim \mathbf{D}$ lie in $\partial_1 \mathbf{SH}_2$ while the points corresponding to $\mathbb{R}$ lie in the Shilov boundary.

For the fixed points of the third kind $[(uu^{\mathrm{T}}, vv^{\mathrm{T}})^{\mathrm{T}}]$ we obtain the equation

$$Auu^{\mathrm{T}} = uu^{\mathrm{T}} P,\ Cuu^{\mathrm{T}} + A^{-\mathrm{T}} vv^{\mathrm{T}} = vv^{\mathrm{T}} P, \quad P \in \mathbf{GL}(2, \mathbb{C}).$$

Hence either $u = e_1 = (1,0)^{\mathrm{T}},\ v = e_2 = (0,1)^{\mathrm{T}}$ or $u = e_2,\ v = e_1$. The second matrix equation for the first choice is $\operatorname{diag}(\delta, \frac{1}{\alpha}) = \operatorname{diag}(0,1)P$. It is only solvable if $\delta = 0$. Assume $\delta = 0$. Then $P = \operatorname{diag}(1, \frac{1}{\alpha})$ satisfies the two matrix equation. This choice for $u$, $v$ and $P$ gives one fixed point of the third kind, which is in the Shilov boundary. This fixed point completes one set of fixed points of the form $\overline{\mathbf{H}}$ to $\overline{\mathbf{D}}$ for the case $\delta = 0$. Assume the second choice second choice $u = e_2,\ v = e_1$. Then the second matrix equation is $\operatorname{diag}(0,1) = \operatorname{diag}(0,1)$ P. Both matrix equations are solvable for $P = \operatorname{diag}(\alpha, 1)$. Hence $M$ has a fixed point of the third kind in this case, which is in the Shilov boundary. If $\delta = 0$ this fixed point completes another set $\overline{\mathbf{H}}$ of fixed points to the set $\overline{\mathbf{D}}$.

**(I4)** Let $M$ be of the form (3.8) where $A = \operatorname{diag}(1, \alpha),\ C = \operatorname{diag}(\delta_1, \delta_2)$ and $\alpha = \pm 1,\ \delta_1, \delta_2 = 0, \pm 1$. Note that $A^{-\mathrm{T}} = A,\ A^2 = I_2$. We first consider the fixed points of the second kind $[(I_2, -Z)^{\mathrm{T}}]$, $\operatorname{Im} Z \geq 0$ allowing $Z$ to be invertible. Then we consider the fixed points of the first kind $[(Z, I_2)^{\mathrm{T}}]$, $\operatorname{Im} Z \geq 0$ and det Z $= 0$. For fixed points of the second kind we obtain the equation $(C - AZ) = -ZA$. Hence $\delta_1 = \delta_2 = 0$. If $\alpha = 1$ then $M = I_4 = I_2 \odot I_2$. Clearly $M$ fixes $\mathrm{Cl}(\mathbf{SH}_2)$. Note that in this case the ordinary fixed points of $I_2 \odot I_2$ are $\mathrm{Cl}(\mathbf{H}) \times \mathrm{Cl}(\mathbf{H})$. If $\alpha = -1$ then the fixed points of $M$ are of the form $Z = \operatorname{diag}(z_1, z_2)$. The condition $\operatorname{Im} Z \geq 0$ implies that this set is $\overline{\mathbf{H}} \times \overline{\mathbf{H}}$. The set $\mathbf{H} \times \mathbf{H} \sim \mathbf{D} \times \mathbf{D}$ lies in $\mathbf{SH}_2$. The sets $\mathbb{R} \times \mathbf{H} \sim \mathbb{R} \times \mathbf{D}$, $\mathbf{H} \times \mathbb{R} \sim \mathbf{D} \times \mathbb{R}$ lie in $\partial_1 \mathbf{SH}_2$. The set $\mathbb{R} \times \mathbb{R}$ lies in the Shilov boundary.

We now consider the fixed point of the first kind $Z \in \overline{\mathbf{SH}}_2$, det Z $= 0$. Recall the form of $Z = zuu^{\mathrm{T}}$ from the the third part of Lemma 6.1. $Z$ satisfies the matrix equation $AZ = Z(CZ + A)$. Clearly $Z = 0$ is a solution. Under our assumptions $Z = 0$ is an ordinary fixed point of $X \odot Y$. We are looking for other solutions. So we assume that $z \neq 0$. Then $u$ is an eigenvector of $A$.

Suppose first that $\alpha = -1$. Then either $u = e_1$ or $u = e_2$. Suppose first that $u = e_1$. Then $e_1^{\mathrm{T}} C e_1 = 0 \Rightarrow \delta_1 = 0$. If $\delta_1 = 0$ then we get $\overline{\mathbf{H}}$ as the set of fixed points of the form $Z = z e_1 e_1^{\mathrm{T}},\ z \in \overline{\mathbf{H}}$. Assume now that $u = e_2$. Then $e_2^{\mathrm{T}} C e_2 = 0 \Rightarrow \delta_2 = 0$. If $\delta_2 = 0$ then we get $\overline{\mathbf{H}}$ as the set of fixed points of the form $Z = z e_2 e_2^{\mathrm{T}},\ z \in \overline{\mathbf{H}}$. If $C = 0$ then the set of the fixed points are two copies of $\overline{\mathbf{H}}$ which intersect at one boundary point $z = 0$. This set should be viewed as $\infty \times (\overline{\mathbf{H}} \backslash \{0\}) \cup (\overline{\mathbf{H}} \backslash \{0\}) \times \infty \cup \infty \times \infty$ as will be explained later. (We view the Riemann sphere $\mathbb{P}$ as $\mathbb{C} \cup \infty$.)

Assume now that $\alpha = 1$. Then $A = I_2$ and $u$ can be any real vector of length 1. Then the matrix equation yields $u^t C u = 0$. If $\delta_1 \delta_2 = 1$ then $u^{\mathrm{T}} C u = \pm 1$ and we do not have fixed point for $z \neq 0$. If $\delta_1 \delta_2 = -1$ we have four vectors $u = \pm w,\ u = \pm Cw$, where $w = \frac{1}{\sqrt{2}}(1,1)^{\mathrm{T}}$, which give rise to two distinct matrices $ww^{\mathrm{T}}$ and $Cww^{\mathrm{T}}C$. Then the set of all fixed points for the corresponding $M$, which is of the form $X \odot X^{-1}$ where $X$ is parabolic, is two copies of $\overline{\mathbf{H}}$ which intersect at $z = 0$. If $\delta_1 \delta_2 = 0$ and $\delta_1^2 + \delta_2^2 = 1$ then $uu^{\mathrm{T}}$ is either $e_1 e_1^{\mathrm{T}}$ or $e_2 e_2^{\mathrm{T}}$. In all these cases the set of fixed point is $\overline{\mathbf{H}}$. The set $\mathbf{H} \sim \mathbf{D}$ lies in $\partial_1 \mathbf{SH}_2$ while the its boundary $\mathbb{R}$ lies in the Shilov boundary. If $C = 0$ then all $Z = zuu^{\mathrm{T}}$ are fixed.

We now consider the fixed point of the third kind $[(uu^{\mathrm{T}}, vv^{\mathrm{T}})^{\mathrm{T}}]$. Then we have two matrix equations $Auu^{\mathrm{T}} = uu^{\mathrm{T}}P$ and $Cuu^{\mathrm{T}} + Avv^{\mathrm{T}} = vv^{\mathrm{T}}P$, for some $P \in \mathbf{GL}(2, \mathbb{C})$. The first matrix equation yields that $u$ is an eigenvector of $A$. Since $A \in \mathbf{Sym}(2, \mathbb{R})$ and $u^{\mathrm{T}}v = 0$ it follows that $v$ is also an eigenvector of $A$. Then the second matrix equation yields that $Cu = cv$ for some $c \in \mathbb{R}$.

Suppose first that $\alpha = -1$. Hence either $u = e_1,\ v = e_2$ or $u = e_2,\ v = e_1$. Then $Cu = cv \Rightarrow Cu = 0$. In this case $P = A$ will satisfy the above two matrix equations. Hence if $\delta_1\delta_2 = \pm 1$ $M$ will not have fixed points of the third kind. In this case $M = X \odot Y$, where $X$ and $Y$ are parabolic and $X$ is not conjugate to $Y^{-1}$, $M$ has one ordinary fixed point in the Shilov boundary.

If $\delta_1\delta_2 = 0$ and $\delta_1^2 + \delta_2^2 = 1$ then $M$ will have exactly one fixed point of the third kind. In that case the fixed points of $M$ of the first and the third kind form $\overline{\mathbf{D}}$, with $\mathbf{D}$ in $\partial_1\mathbf{SH}_2$ and $S^1$ in the Shilov boundary. These fixed points are ordinary fixed points of $M$.

If $\delta_1 = \delta_2 = 0$ then $M$ has two fixed points of the third kind. In that case the set of all fixed points of $M$ form $\overline{\mathbf{D}} \times \overline{\mathbf{D}}$, where $\mathbf{D} \times \mathbf{D}$ are in $\mathbf{SH}_2$, $\mathbf{D} \times S^1$, $S^1 \times \mathbf{D}$ are in $\partial_1\mathbf{SH}_2$ and $S^1 \times S^1$ are in the Shilov boundary. Indeed the fixed set of $M$ of the second kind is $\begin{pmatrix} 1 & 0 & -z_1 & 0 \\ 0 & 1 & 0 & -z_3 \end{pmatrix}^{\mathrm{T}}$, where $\operatorname{Im} z_1 \geq 0, \operatorname{Im} z_3 \geq 0$. The fixed singular matrices of the first kind are $\begin{pmatrix} -\frac{1}{z_1} & 0 & 1 & 0 \\ 0 & -\frac{1}{z_3} & 0 & 1 \end{pmatrix}^{\mathrm{T}}$, where $z_1, z_3 \neq 0$, $\operatorname{Im} z_1, \operatorname{Im} z_3 \geq 0$ and either $z_1$ or $z_3$ is $\infty$. The two fixed points of the third kind are $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}^{\mathrm{T}}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{\mathrm{T}}$ corresponding to the points $z_1 = \infty, z_3 = 0,\ z_1 = 0, z_3 = \infty$. All the fixed points of $M$ are ordinary fixed points.

Assume now that $\alpha = 1$. It is enough to consider the case $C \neq 0$. Then $u$ can be any real vector of length 1. If $\delta_1\delta_2 = 1$ then $u^{\mathrm{T}}Cu = \pm 1$ while $v^{\mathrm{T}}u = 0$. The second matrix equation yields $Cu = cv$. Hence $M$ does not have any fixed points of the third kind. Thus if $C = \pm I_2$, i.e. $M$ is conjugate to $X \odot Y$, where $X$ and $Y$ are parabolic and $X$ and $Y^{-1}$ are not conjugate, then $M$ has one ordinary fixed point in the Shilov boundary.

If $\delta_1\delta_2 = -1$ then there are two possible solutions for $u$ and $v$ up to a sign: either $u = \pm w,\ v = \pm Cw$ or $u = \pm Cw,\ v = \pm w$ ($w = \frac{1}{\sqrt{2}}(1,1)^{\mathrm{T}}$). It is straightforward to check that in each case there exist $P \in \mathbf{GL}(2, \mathbb{R})$ which satisfies both matrix equations. Hence $M$ has two fixed points of the third kind. Thus if $\delta_1 = -\delta_2 = \pm 1$, i.e. $M = X \odot X^{-1}$ and $X$ is parabolic, then the set of fixed points of $M$ is union of two closed disks $\overline{\mathbf{D}}$ which intersect at the ordinary fixed point of $X \odot X^{-1}$ lying on the boundary of each $\overline{\mathbf{D}}$. Each $\mathbf{D}$ lies in $\partial_1\mathbf{SH}_2$ and each $S^1$ lies in the Shilov boundary.

If $\delta_1\delta_2 = 0,\ \delta_1^2 + \delta_2^2 = 1$ then $Cu = cv \Rightarrow u^{\mathrm{T}}Cu = 0$ for $u$ equal either $\pm e_1$ or $\pm e_2$ but not both. In each case $Cu = 0$. It easily follows that $M$ has exactly one fixed point of the third kind. In that case, i.e. $M = X \odot I_2$ or $M = I_2 \odot X$ and $X$ is parabolic with $\operatorname{spec}(X) = \{1\}$, the set of fixed points of $M$ is consists of ordinary fixed points $\overline{\mathbf{D}}$. $\mathbf{D}$ lies in $\partial_1\mathbf{SH}_2$ and $S^1$ lies in the Shilov boundary.

**(I5)** $M$ is of the form (3.8) where $A = \begin{pmatrix} 0 & 1 \\ -1 & \pm 2 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & \pm 1 \\ 0 & 0 \end{pmatrix}$. As in the proof of (I4) we first consider all fixed point of the second kind $[(I_2, -Z)^{\mathrm{T}}]$, where $\mathrm{Im}\, Z \geq 0$. For these points we get the matrix equation $C = A^{-\mathrm{T}} Z - Z A^{-1}$. Note that $A^{-\mathrm{T}} Z - Z A^{-1}$ is skew symmetric while $C$ is not skew symmetric. Hence this matrix equation is not solvable. We now consider the fixed point of the first kind with $Z = zuu^{\mathrm{T}}$. Then $Z$ satisfies the matrix equation $AZ = Z(CZ + A^{-\mathrm{T}})$. $z = 0$ is trivial solution. We now look for the solutions $z \neq 0$, $\mathrm{Im}\, z \geq 0$. Then $u = \frac{1}{\sqrt{2}}(1, \pm 1)^{\mathrm{T}}$ is the unique eigenvector of $A$ corresponding to the eigenvalue $\lambda = \pm 1$. Observe next that $u^{\mathrm{T}} A^{-\mathrm{T}} = \lambda u^{\mathrm{T}}$. Hence $AZ - ZA^{-\mathrm{T}} = 0 \Rightarrow ZCZ = 0$. As $u^{\mathrm{T}} C u \neq 0$ we do not have any nontrivial solution $Z = zuu^{\mathrm{T}}$. We now consider the two matrix equations for the fixed points of the third kind: $Auu^{\mathrm{T}} = uu^{\mathrm{T}} P$ and $Cuu^{\mathrm{T}} + A^{-\mathrm{T}} vv^{\mathrm{T}} = vv^{\mathrm{T}} P$, for some $P \in \mathbf{GL}(2, \mathbb{C})$. Again $u$ must be the unique eigenvalue of $A$ given above. As $v^{\mathrm{T}} u = 0$ the absolute values of each coordinate of $u$ and $v$ are $\frac{1}{\sqrt{2}}$. A straightforward calculations shows that $Cuu^{\mathrm{T}} + A^{-\mathrm{T}} vv^{\mathrm{T}}$ has always rank two. Hence the second matrix equation is not solvable. Thus $M$ has one fixed point in the Shilov boundary.

**(IIa)** $M$be of the form (3.3) and (3.13). As $M = X \odot Y$, where $X$ is elliptic, then $c_1 \neq 0$. Recall that if $\delta = \pm 1$ then $a_4 = \pm 1$. The matrix equation for the fixed point of the first kind is $AZ + B = Z(CZ + D)$:

$$\begin{pmatrix} a_1 z_1 - c_1 & a_1 z_2 \\ a_4 z_2 & a_4 z_3 \end{pmatrix} = \begin{pmatrix} c_1 z_1^2 + \delta z_2^2 + a_1 z_1 & c_1 z_1 z_2 + \delta z_2 z_3 + z_2/a_4 \\ c_1 z_1 z_2 + \delta z_2 z_3 + a_1 z_2 & c_1 z_2^2 + \delta z_3^2 + z_3/a_4 \end{pmatrix}.$$

Subtract the equation coming from entry (2,1) from the one coming from entry (1,2) to get $z_2(a_1 - a_4) = z_2(\frac{1}{a_4} - a_1)$. If $z_2 \neq 0$ then $2a_1 = a_4 + 1/a_4 \Rightarrow |a_1| \geq 1$ contrary to our assumption that $c_1 \neq 0$. Hence $z_2 = 0$. The equation from entry (1,1) gives us $z_1 = \pm\sqrt{-1}$. Since $\mathrm{Im}\, Z \geq 0$ we deduce that $z_1 = \sqrt{-1}$. Equation from entry (2,2) is $z_3(\delta z_3 + a_4^{-1} - a_4) = 0$, so either $z_3 = 0$ or $\delta z_3 = a_4 - a_4^{-1}$. Assume that $z_3 \neq 0$. If $\delta \neq 0$ then $a_4 = \pm 1$ and we obtain that $z_3 = 0$ contrary to our assumption. If $\delta = 0$ then $z_3 \neq 0$ is a solution if and only if $a_4 = \pm 1$, i.e. $M = X \odot (\pm I_2)$. In this case $z_3 \in \overline{\mathbf{H}}$. For $z_3 \in \mathbf{H}$ $Z \in \mathbf{SH}_2$ and for $z \in \mathbb{R}$ $Z$ is in $\partial_1 \mathbf{SH}_2$. If $a_4 \neq \pm 1$ then $M$ has one fixed point of the first kind $Z = \mathrm{diag}(\sqrt{-1}, 0) \in \partial_1 \mathbf{SH}_2$.

For the fixed points of the second kind of the boundary the equation is $C - DZ = -Z(A - BZ)$: $\begin{pmatrix} c_1 - a_1 z_1 & -a_1 z_2 \\ -z_2/a_4 & \delta - z_3/a_4 \end{pmatrix} = \begin{pmatrix} -c_1 z_1^2 - a_1 z_1 & -c_1 z_1 z_2 - a_4 z_2 \\ -c_1 z_1 z_2 - a_1 z_2 & -c_1 z_2^2 - a_4 z_3 \end{pmatrix}$.

The equation from the (1,1) entry implies that $z_1 = \sqrt{-1}$. Then we can easily conclude that $z_2 = 0$ from equation (1,2). As the boundary point of the second kind is of the form $Z = zuu^{\mathrm{T}}$ we deduce that $z = \sqrt{-1}$ and $u = e_1$. That is $z_3$ must be equal to 0. The equation from the (2,2) entry implies that $\delta = 0$, i.e. either $Y = \pm I_2$ or $Y$ is hyperbolic.

For the fixed points of the third kind the equations are

$$Auu^{\mathrm{T}} + Bvv^{\mathrm{T}} = uu^{\mathrm{T}} P, \quad Cuu^{\mathrm{T}} + Dvv^{\mathrm{T}} = vv^{\mathrm{T}} P, \quad \text{for some } P \in \mathbf{GL}(2, \mathbb{C}). \tag{6.8}$$

Hence the two matrices appearing on the left-hand side of each equation have to be rank one matrices. As $u, v$ is an orthonormal basis for $\mathbb{R}^2$ we deduce that the two vectors in each pair $Au,\ Bv$ are linear combinations of $u$ and the vectors $Cu,\ Dv$ are linear combinations of $v$. (By changing the coordinates one can assume that $u = e_1,\ v = e_2$. In that case this claim is straightforward.) Recall that $u = (s,t)^{\mathrm{T}},\ v = (t,-s)^{\mathrm{T}},\ s^2+t^2 = 1$. Clearly $Au = (a_1 s, a_4 t)^{\mathrm{T}},\ Bv = (-c_1 t, 0)^{\mathrm{T}}$. Hence $t = 0$ and we may assume that $s = 1$. $(c_1, a_4 \neq 0.)$ As $Cu = (c_1, 0)^{\mathrm{T}},\ Dv = (0, -a_4^{-1})$ we deduce that $Cu$ and $Dv$ are linearly independent. Hence $M$ does not have a fixed point of the third kind.

We now summarize our results for $M = X \odot Y$ where $X$ is elliptic. If $Y = \pm I_2$ then the fixed points of $M$ of the first and the second kind form $\overline{\mathbf{D}}$, where $\mathbf{D}$ lies in $\mathbf{SH}_2$ and $S^1$ in $\partial_1 \mathbf{SH}_2$. If $Y$ is hyperbolic then $M$ have two fixed points of the first and the second kind in $\partial_1 \mathbf{SH}_2$. If $Y$ is parabolic then $M$ have one fixed point of the first kind in $\partial_1 \mathbf{SH}_2$. All fixed points are ordinary.

**(III)** We assume that $M$ is of the form (3.3) and (3.14). $M = X \odot Y$ where $X,\ Y$ are elliptic and $X$ is not conjugate to $Y^{-1}$. Hence $b_1, b_2 \neq 0$ and $(a_1, b_1) \neq (a_2, -b_2)$. The matrix equation for the fixed point of the first kind is $AZ + B = Z(CZ + D)$:

$$\begin{pmatrix} a_1 z_1 + b_1 & a_1 z_2 \\ a_2 z_2 & a_2 z_3 + b_2 \end{pmatrix} = \begin{pmatrix} -b_1 z_1^2 - b_2 z_2^2 + a_1 z_1 & -b_1 z_1 z_2 - b_2 z_2 z_3 + a_2 z_2 \\ -b_1 z_1 z_2 - b_2 z_2 z_3 + a_1 z_2 & -b_1 z_2^2 - b_2 z_3^2 + a_2 z_3 \end{pmatrix}.$$

Subtract the equation coming from entry (2,1) from the one coming from entry (1,2) to get

$$z_2(a_1 - a_2) = z_2(a_2 - a_1). \tag{6.9}$$

Suppose first that $a_1 \neq a_2$. Then $z_2 = 0$, i.e. $Z = \mathrm{diag}(z_1, z_3)$. The equation for entries (1,1) and (2,2) yield that $z_1, z_2 = \pm\sqrt{-1}$. As $\mathrm{Im}\, Z \geq 0$ we deduce that $Z = \sqrt{-1} I_2$. Assume now that $a_1 = a_2$. Then $b_1 = b_2$. So $A = a_1 I_2,\ B = b_1 I_2,\ C = -b_1 I_2,\ D = a_1 I_2$. The matrix equation $AZ+B = Z(CZ+D)$ yield that $Z^2 = -I_2$. Then $Z$ is a diagonable matrix with eigenvalues $\pm\sqrt{-1}$. As $Z \in \mathbf{Sym}(2,\mathbb{C}),\ \mathrm{Im}\, Z \geq 0$ we deduce that both eigenvalues of $Z$ must equal to $\sqrt{-1}$. Hence $Z = \sqrt{-1} I_2$.

For the fixed points of the second kind of the boundary the equation is $-C + DZ = Z(A - BZ)$:

$$\begin{pmatrix} b_1 + a_1 z_1 & a_1 z_2 \\ a_2 z_2 & b_2 + a_2 z_3 \end{pmatrix} = \begin{pmatrix} -b_1 z_1^2 - b_2 z_2^2 + a_1 z_1 & -b_1 z_1 z_2 - b_2 z_2 z_3 + a_2 z_2 \\ -b_1 z_1 z_2 - b_2 z_2 z_3 + a_1 z_2 & -b_1 z_2^2 - b_2 z_3^2 + a_2 z_3 \end{pmatrix}.$$

Subtract the equation coming from entry (2,1) from the one coming from entry (1,2) to get (6.9). As before we conclude that the only solution $Z \in \mathbf{Sym}(2,\mathbb{C}),\ \mathrm{Im}\, Z \geq 0$ is $Z = \sqrt{-1} I_2$. This solution is not a *boundary point* of the second kind.

For the fixed points of the third kind we get the equations (6.8). Note that $C = -D$ are diagonal invertible while $A = D$ are diagonal. The conditions that $Au,\ Bv$ are linear combinations of $u$ and the vectors $Cu,\ Dv$ are linear combinations of $v$ imply the existence of $c, d \in \mathbb{R}$ such that $Au = cBv$ and $Av = dBu$. Therefore the diagonal matrix $B^{-1}A$ represented in the basis $u, v$ as a matrix with zero diagonal. Hence the trace of $B^{-1}A$ is zero, i.e. $\frac{a_1}{b_1} = -\frac{a_2}{b_2}$. As $(a_1, b_1) \neq (a_2, -b_2)$ we get that $a_2 = -a_1$ and $b_2 = b_1$. Then the condition that $Bv = b_1 v$ is collinear with $u$ contradicts the fact that $u, v$ are linearly independent. Thus $M$ does not have fixed point of the third kind. Hence $M$ has a unique ordinary fixed point in $\mathbf{SH}_2$ $\Box$

## References


[1] A. Beardon, *The Geometry of Discrete Groups*, Springer Verlag · New York, 1983.

[2] P. J. Freitas, *On the action of the symplectic group on the Siegel upper half plane*, PhD thesis, Univ. of Illinois at Chicago, 1999.

[3] S. Friedland and P. J. Freitas, Revisiting the Siegel Upper Half Plane I, *to appear.*

[4] E. Gottschling, Über die fixpunkte der Siegelschen modulgruppe, *Math. Annalen* 143 (1961), 111-149.

[5] E. Gottschling, Über die fixpunktegruppen der Siegelschen modulgruppe, *Math. Annalen* 143 (1961), 399-430.