p-metrics on $\text{GL}(n, \mathbb{C})/U_n$ and their Busemann compactifications

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Abstract
Let $\text{GL}(n, \mathbb{C}) \supset U_n$ be the group of $n \times n$ complex valued invertible matrices and the subgroup of unitary matrices respectively. In this paper we study Finsler $p$-metrics on the homogeneous space $X_n = \text{GL}(n, \mathbb{C})/U_n$ for $p \in [1, \infty]$, which are induced by Schatten $p$-norms on the tangent bundle of $X_n$ and are invariant under the action of $\text{GL}(n, \mathbb{C})$. We show that for $p \in (1, \infty)$ the Busemann $p$-compactification is the visual compactification. For $p = 1, \infty$ the Busemann $p$-compactification is not the visual compactification. We give a complete description of Busemann 1-compactification.

1 Introduction
Let $\text{GL}(n, \mathbb{C})$ be the group of $n \times n$ complex valued invertible matrices. Let $\mathbf{M}(n, \mathbb{C})$ be its Lie algebra of $n \times n$ complex valued matrices. For $A \in \mathbf{M}(n, \mathbb{C})$ denote by $\sigma_1(A) \geq ... \geq \sigma_n(A) \geq 0$ the singular values of $A$. Recall that the Schatten $p$-norm on $\mathbf{M}(n, \mathbb{C})$ is given by $||A||_p = (\sum_{i=1}^{n} \sigma_i(A)^p)^{\frac{1}{p}}$. Denote by $H_n \supset H_n^+$, $U_n \subset \text{GL}(n, \mathbb{C})$ be the space of $n \times n$ hermitian matrices, the closed cone of nonnegative definite hermitian matrices and the subgroup of unitary matrices respectively. Consider the homogeneous space $X_n = \text{GL}(n, \mathbb{C})/U_n$. Recall that $X_n$ can be identified with $e^{H_n} \cap \text{GL}(n, \mathbb{C})$, which is equal to $e^{H_n} := \{ e^A : A \in H_n \}$. Then there exists a unique Finsler $p$-metric on the tangent bundle

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of $X_n$, invariant under the action of $\text{GL}(n, \mathbb{C})$, which is given by the Schatten $p$-norm on $T_1X_n = H_n$. We show that the Finsler $p$-metric on the tangent bundle of $X_n$ induces the following metric on $X_n$:

$$d_p(A, B) := \left( \sum_{i=1}^{n} |\log \sigma_i(A^{-1}B)|^p \right)^{\frac{1}{p}}, \quad A, B \in X_n, \ p \in [1, \infty]. \quad (1.1)$$

d_2$ is the classical Riemannian metric on the homogeneous space $X_n$. All $p$-metrics are uniformly Lipschitz equivalent for a fixed value of $n$. $\text{GL}(n, \mathbb{C})$ acts (from the left) as a subgroup of isometries for each $p \in [1, \infty]$.

The main object of this paper is to consider the Busemann functions and the Busemann compactifications for $d_p$, $p \in [1, \infty]$ as in [Bal]. One can view the Busemann $p$-compactification as a geometric way to add the $p$-boundary $\partial_p X_n$ to $X_n$ using the metric $d_p$, such that the space $\partial_p X_n \cup X_n$ is compact with respect to a suitable topology. For $p \in (1, \infty)$ the Busemann $p$-compactification is equivalent to the visual compactification, i.e. $\partial_p X_n$ can be identified as the ends of geodesic rays from a fixed point $o \in X_n$. For $p = 1, \infty$ the Busemann $p$-compactification is different from the visual compactification. We give the complete description of $\partial_1 X_n$:

View $\mathbb{C}^n := \{x := (x_1, \ldots, x_n)^T : \ x_i \in \mathbb{C}, \ i = 1, \ldots, n\}$ as inner product space with standard inner product $\langle x, y \rangle = y^*x$. Let $\mathbb{C}^n = U_+ + U_0 + U_-$ be a nontrivial orthonormal decomposition of $\mathbb{C}^n$, i.e. $\dim U_0 < n$. Note that each decomposition of $\mathbb{C}^n$ corresponds to the flag $\mathbb{C}^n = U_+ + U_0 + U_- \supset U_0 + U_- \supset U_-$. Let $H(U_0)$ be the real space of self-adjoint operators $T : U_0 \to U_0$. If $\dim U_0 = 0$ then $H(U_0)$ has only one element: the complex number $0$. Let $(U_+, H(U_0), U_-) := \{(U_+, T, U_-) : \ T \in H(U_0)\}$. If $\dim U_0 = 0$ we identify $(U_+, H(U_0), U_-)$ with $(U_+, U_-)$. For $A \in H_n$ and a set $S \subset \mathbb{R}$ let $P_S(A)$ be the orthogonal projection on the invariant subspace of $A$ spanned by the eigenvectors of $A$ corresponding to the eigenvalues in $S$. If $S$ does not contain an eigenvalue of $A$ then $P_S(0) = 0$. Note that if $A \in H_n^{+}$ then the eigenvalues of $A$ coincide with the singular eigenvalues of $A$.

We show that a sequence $\{e^{A_m}\}_{m=1}^{\infty} \subset e^{H_n}$ converges to a point in $\partial_1 X_n$ if and only if there exist three nonnegative integers $k_0 < n, k_+, k_- \in \mathbb{N}$ with the following properties:

$$\lim_{m \to \infty} \sigma_i(e^{A_m}) = \infty, \ i = 1, \ldots, k_+,$$

$$\lim_{m \to \infty} P_{[\sigma_{k_+}(e^{A_m}), \sigma_1(e^{A_m})]}(e^{A_m}) \mathbb{C}^n = U_+,$$

$$\lim_{m \to \infty} \sigma_i(e^{A_m}) = 0, \ i = n - k_+ + 1, \ldots, n,$$

$$\lim_{m \to \infty} P_{[\sigma_n(e^{A_m}), \sigma_{n-k_-+1}(e^{A_m})]}(e^{A_m}) \mathbb{C}^n = U_-,$$

$$\lim_{m \to \infty} \sigma_i(e^{A_m}) = \sigma_i \in (0, \infty), \ i = k_+ + 1, \ldots, k_+ + k_0,$$

$$\lim_{m \to \infty} P_{[\sigma_{k_++k_0}(e^{A_m}), \sigma_{k_+}(e^{A_m})]}(e^{A_m}) \mathbb{C}^n = U_0,$$

$$\lim_{m \to \infty} e^{A_m} P_{[\sigma_{k_++k_0}(e^{A_m}), \sigma_{k_+}(e^{A_m})]}(e^{A_m}) \mathbb{C}^n = e^T \text{ for some } T \in H(U_0).$$
Then \( \partial_1 X_n \) is a union of \((U_+ , H(U_0), U_-)\) for all possible nontrivial decompositions of \( \mathbb{C}^n \). Let \( \text{Gr}(n,k, \mathbb{C}) \) be the Grassmannian manifold of all \( k \)-dimensional subspaces of \( \mathbb{C}^n \). Then compact part of \( \partial_1 X_n \) is \( \bigcup_{k=0}^n \text{Gr}(n,k, \mathbb{C}) \), which corresponds to all the orthonormal decomposition \( \mathbb{C}^n = U_+ \oplus U_- \) and \( \dim U_+ = k \) for \( k = 0, 1, \ldots, n \).

In [FF1] the authors show that the Busemann 1-compactification of the Siegel upper half plane given as \( \text{Sp}(n, \mathbb{R})/K_n \subset X_{2n} \), where \( \text{Sp}(n, \mathbb{R}) \) is the symplectic subgroup of \( \text{GL}(2n, \mathbb{R}) \) and \( K_n = \text{Sp}(n, \mathbb{R}) \cap U_{2n} \), is the classical Satake compactification [Sat] as a bounded domain [Hel]. It is of interest to extend these results to a larger class of symmetric spaces.

We now survey briefly the contents of this paper. In \( \S 2 \) we discuss briefly the general setting of the Busemann compactification. As an example we consider the Busemann compactification of \( \mathbb{R}^n \) with respect to Hölder \( p \)-metric. In \( \S 3 \) we show that \( d_p(A,B) \) defined in (1.1) is a metric on \( X_n \). In \( \S 4 \) we give describe the Busemann functions corresponding to the boundary points in \( \partial_p X_n \) which are induced by geodesic rays from \( I \). In \( \S 5 \) we show that \( \partial_p X_n \) is the visual boundary of \( X_n \). In \( \S 6 \) we describe the Busemann 1-compactification of \( X_n \).

2 Busemann functions and compactifications

A Finsler manifold is a smooth manifold \( X \) equipped with a continuous function which assigns to each point \( x \in X \) a norm on the tangent space \( T_x X \). The integral of the norm of the tangent vector to a smooth curve \( \gamma \) in \( X \) is called the length of \( \gamma \). The distance \( d(x,y) \) between two points \( x,y \in X \) is the infimum of the lengths of the curves connecting \( x \) and \( y \). A geodesic is a curve \( \gamma \) which minimizes the distance between any two sufficiently near points on the curve. A complete geodesic space is a Finsler manifold such that it is a complete metric space with respect to \( d \) and any two points can be connected by a geodesic. A complete geodesic space is called a Hadamard manifold if any two points are connected by a unique geodesic.

Let \( X \) be a complete geodesic space and locally compact with respect to the metric \( d \). Then \( X \) admits a Busemann compactification defined as \( \text{Cl} \mathcal{Y} \) in \( C(X) \) of the set of functions \( \mathcal{Y} := \{b_y(x) : y \in X\} \) where

\[
\text{b}_y : X \rightarrow \mathbb{R}, \quad \text{b}_y(x) = d(y,x) - d(y,o).
\]  

(2.1)

Here \( o \) is a base point in \( X \) (the compactification is independent of the choice of \( o \)) and the topology on \( C(X) \) is given by uniform convergence on compact subsets of \( X \). The map \( y \mapsto \text{b}_y \) is a homeomorphism of \( X \) and \( \mathcal{Y} \). Then \( \partial \text{Cl} \mathcal{Y} := \text{Cl} \mathcal{Y} \setminus \mathcal{Y} \) is identified with the Busemann boundary \( \partial X \) [Bal, \( \S II.1 \)]. An unbounded sequence \( y_k, k = 1, \ldots \) is said to converge to \( \xi \in \partial X \) if the sequence of functions \( \text{b}_{y_k} \) converges to a function \( \text{b}_\xi \in C(X) \) (uniformly on bounded subsets of \( X \)).

As an example we consider the Busemann compactification of \( X = \mathbb{R}^n \). Fix \( p \in [0, \infty] \) and assume that the norm on \( T_x \mathbb{R}^n \) is given by \( ||(z_1, \ldots, z_n)^T||_p := (\sum_{i=1}^n |z_i|^p)^{1/p} \). Denote by \( \mathbb{R}_p^n \) the corresponding Finsler manifold. Then the distance between \( x,y \in \mathbb{R}_p^n \) is given the Hölder metric \( \delta_p(x,y) = ||x-y||_p \). \( \mathbb{R}_p^n \) is a complete geodesic space which is Hadamard if and
only if \( p \in (1, \infty) \). Let \( \partial \mathbb{R}_p^n \) be the Busemann boundary. For \( y \in \mathbb{R}_p^n, \xi \in \partial \mathbb{R}_p^n \) let \( b_{y,p}, b_{\xi,p} \) be the corresponding Busemann functions for the distance \( \delta_p \) with \( o = o = (o_1, ..., o_n)^T \). Set

\[
S_p^{n-1} := \{ x \in \mathbb{R}^n : \| \xi \|_p = 1 \}.
\]

Assume first that \( p \in (1, \infty) \). Then (2.3) holds for \( p \in (1, \infty) \).

Let \( \langle p \rangle = (\xi, x) \), \( x = (x_1, ..., x_n)^T \in \mathbb{R}^n \), \( \xi = (\xi_1, ..., \xi_n)^T \in \mathbb{R}^n \setminus \{0\} \).

Then

\[
Q_p(\xi, x) := -\sum_{i=1}^n \xi_i|x_i|^{p-2}x_i, \quad x = (x_1, ..., x_n)^T \in \mathbb{R}^n, \quad \xi = (\xi_1, ..., \xi_n)^T \in \mathbb{R}^n \setminus \{0\}.
\]  

(2.2)

Hence \( \partial \mathbb{R}_p^n \) can be identified with \( S_p^{n-1} \), which is diffeomorphic to the Euclidean sphere \( S_2^{n-1} \).

Let \( p = 1 \) and \( < n > := \{1, 2, ..., n\} \). Denote by \( 2^{<n>} \) all nonempty subsets of \( < n > \). Fix \( \alpha \in 2^{<n>} \). Then \( \{1, -1\}^\alpha \) denotes the set of all possible maps of \( \alpha \) to \( \{1, -1\} \). This set has cardinality \( 2^{|\alpha|} \), where \( |\alpha| \) is the cardinality of the set \( \alpha \). Thus an element \( \epsilon \in \{1, -1\}^\alpha \) is a set \( \{\epsilon_j\}_{j\in\alpha} \) where \( \epsilon_j = \pm 1, j \in \alpha \). Let \( \mathbb{R}^0 \) be a set consisting of one element and \( \{0\} = 0 \).

**Lemma 2.1** The Busemann boundary \( \mathbb{R}^n \) with respect to \( \delta_1 \) has the stratification

\[
\partial \mathbb{R}^n_1 = \cup_{\alpha \in 2^{<n>} \setminus \{1, -1\}^\alpha} \mathbb{R}^{<n>\setminus \alpha} \times [1, -1]^\alpha \times \mathbb{R} \setminus \{0\}
\]

(2.4)

That is, a sequence \( y_k = (y_{1,k}, ..., y_{n,k})^T, k = 1, ... \), converges to \( \xi = (\epsilon_j)_{j\in\alpha} \times (u_1, ..., u_m)^T \) if the following conditions hold:

\[
\alpha = \{\alpha_1, ..., \alpha_l\}, \quad 1 \leq \alpha_1 < \ldots < \alpha_l \leq n,
\]

\[
< n \setminus \alpha = \{\beta_1, ..., \beta_m\}, \quad 1 \leq \beta_1 < \beta_2 < \ldots < \beta_m \leq n, \quad m = n - l,
\]

\[
\lim_{k \to \infty} \epsilon_{\alpha, y_{\alpha, k}} = +\infty, \quad i = 1, ..., l,
\]

\[
\lim_{k \to \infty} y_{\beta, k} = u_j, \quad j = 1, ..., m.
\]

(2.5)

For \( x = (x_1, ..., x_n)^T \in \mathbb{R}^n \) and \( \xi \) as above let

\[
Q_1(\xi, x) := -\sum_{i=1}^l x_{\alpha_i} \epsilon_{\alpha_i} + \sum_{j=1}^m |u_j - x_{\beta_j}|.
\]

(2.6)

Then (2.3) holds for \( p = 1 \).
The proof of the lemma is straightforward. Note that (2.5) implies that the component \( \{ \rho_i \}_{i \in n} \times \mathbb{R}^{<n>\setminus \gamma} \) of the strata \( \{ 1, -1 \}^\gamma \times \mathbb{R}^{<n>\setminus \gamma} \) is a boundary of \( \{ \epsilon_j \}_{j \in \alpha} \times \mathbb{R}^{<n>\setminus \alpha} \) if and only if \( \alpha \) is a strict subset of \( \gamma \) and \( \epsilon_i = \rho_i \) for \( i \in \alpha \).

The stratification of \( \partial R^n_\infty \) is similar to the stratification of \( \partial R^n_p \). One can also define the function \( Q_\infty(\xi, x) \) on each strata of \( \partial R^n_p \) so that (2.3) holds for \( p = \infty \). Note that for \( p = 1, \infty \) \( \partial R^n_p \) does not correspond to the visual compactification \( S^{n-1}_p \).

## 3 p-metrics on \( X_n \)

Let \( M(m, n, \mathbb{F}) \) be the vector space of \( m \times n \) matrices over the field \( \mathbb{F} \), \( M(n, \mathbb{F}) := M(n, n, \mathbb{F}) \) be the algebra of \( n \times n \) matrices and \( GL(n, \mathbb{F}) \) be the group of invertible matrices. In this paper \( \mathbb{F} = \mathbb{R}, \mathbb{C} \) is either the field of real or complex numbers. Denote by \( U_n, SU_n, O_n \) and \( SO_n \) the groups of \( n \times n \) unitary, special unitary, real orthogonal and special real orthogonal matrices respectively. Let \( A = (a_{pq}) \in M(n, \mathbb{C}) \). Then \( A^T \) is transpose of \( A \) and \( A^* = \overline{A}^T \). By the spectrum of \( A \) we mean the eigenvalues \( \lambda_1(A), \ldots, \lambda_n(A) \) counted with their multiplicities and arranged in the following order: \( \text{Re } \lambda_1(A) \geq \cdots \geq \text{Re } \lambda_n(A) \). The singular values of \( A \) are the eigenvalues of \( (AA^*)^{1/2} \). Set \( \sigma(A) := (\sigma_1(A), \ldots, \sigma_n(A))^T \). For \( x = (x_1, \ldots, x_n)^T \in \mathbb{F}^n \) let \( D(x) = \text{diag}(x_1, \ldots, x_n) \) be the diagonal matrix with the diagonal entries \( x_1, \ldots, x_n \). Denote by \( D(n, \mathbb{F}) \subset M(n, \mathbb{F}) \) the space of all diagonal matrices and let \( D^+(n, \mathbb{R}) := D(n, \mathbb{R}) \cap H^+(n, \mathbb{C}) \).

\[
A = U \Sigma(A)V, \quad U, V \in U_n, \quad \Sigma(A) = D(\sigma(A))
\]

is called the singular value decomposition (SVD). (It is also called the Cartan decomposition.) If \( A \in M(n, \mathbb{R}) \) then the unitary matrices \( U, V \) in can be chosen to be orthogonal matrices. Note that \( ||A||_2 = \sigma_1(A) \) is the \( l_2 \) norm of \( A \) viewed as a linear operator \( A : \mathbb{C}^n \rightarrow \mathbb{C}^n \). Furthermore \( (AA^*)^{1/2} \) is the unique representative of the coset \( AU_n \). Use the singular value decomposition of \( A \in GL(n, \mathbb{C}) \) to deduce \( \sigma_{n-i+1}(A^{-1}) = \sigma_i(A)^{-1}, i = 1, \ldots, n \). Observe next that \( \sigma_i(A) = 1, i = 1, \ldots, n \leftrightarrow A \) is a unitary matrix.

For \( A \in M(m, n, \mathbb{F}) \) and \( 1 \leq k \leq \min(m, n) \) denote by \( \wedge_k A \) the \( k \)-th compound matrix. Note that \( \wedge_k A \in M(^n_k, ^m_k, \mathbb{F}) \) and the entries of \( A \) are all the \( k \times k \) minors of \( A \). \( \wedge_k A \) is the representation matrix of the linear transformation from the \( k \) exterior product \( \wedge_k \mathbb{F}^n \) to \( \wedge_k \mathbb{F}^m \) induced by \( A : \mathbb{F}^n \rightarrow \mathbb{F}^m \). The map \( \wedge_k : GL(n, \mathbb{F}) \rightarrow GL(^n_k, \mathbb{F}) \) is a homomorphism which commutes with the * involution. If \( A \in M(n, \mathbb{C}) \) has complex eigenvalues \( \lambda_1(A), \ldots, \lambda_n(A) \) then \( \wedge_k A \) has the following eigenvalues and singular values, and \( \wedge_k e^A \) has the following eigenvalues respectively:

\[
\lambda_{i_1}(A)\lambda_{i_2}(A)\cdots\lambda_{i_k}(A), \quad \sigma_{i_1}(A)\sigma_{i_2}(A)\cdots\sigma_{i_k}(A), \quad e^{\lambda_{i_1}(A)+\lambda_{i_2}(A)+\cdots+\lambda_{i_k}(A)},
\]

\[
1 \leq i_1 < \cdots < i_k \leq n.
\]

(3.2)

If \( A \in H_n \) (\( H^+_n \)) then \( \wedge_k A \in H(^m_k, ^n_k) \) (\( H^+_n \)). See for example [HJ].

The following lemma follows straightforward from SVD.
Lemma 3.1 Let \((A, B), (C, D) \in X_n \times X_n\). Then there exists \(T \in \text{GL}(n, \mathbb{C})\) such that
\[
T(A, B) := (TA, TB) = (C, D)
\] (3.3)
if and only if
\[
\Sigma(A^{-1}B) = \Sigma(C^{-1}D).
\] (3.4)

Theorem 3.2 Let \(p \in [1, \infty]\) and assume that \(A, B \in \text{GL}(n, \mathbb{C})\). Let \(d_p(A, B) = ||\log(\sigma^{-1}(A^{-1}B))||_p\). Then \(d_p\) is a metric on the homogeneous space \(X_n\). \(X_n\) is a complete, locally compact, geodesic space with respect to \(d_p\). For \(p \in (1, \infty)\) \(X_n\) is Hadamard. Moreover, \(\text{GL}(n, \mathbb{C})\) acts (from the left) on \(X_n\) as a subgroup of isometries for \(d_p\).

Proof. Let \(P \in M(n, \mathbb{C})\). As \(\sigma_i(P) = \sigma_i(PU) = \sigma_i(UA)\) for any \(U \in U_n\) we deduce that \(d_p(\cdot, \cdot)\) is a nonnegative continuous function defined on \(X_n \times X_n\). It is straightforward to see that \(A, B\) belong to the same left coset of \(U_n\) if and only if \(d_p(A, B) = 0\). It is easy to check that \(d_p(A, B) = d_p(B, A)\), since \(\sigma_i(A^{-1}B) = \sigma_i(B^{-1}A)\). We now prove the triangle inequality. As \(\sigma_1(P) = ||P||_2\) it follows that \(\sigma_1(PQ) \leq \sigma_1(P)\sigma_1(Q)\) for any \(Q \in M(n, \mathbb{C})\). Apply the norm inequality to the \(k\)-th compound matrix \(\wedge_k(PQ)\) to deduce
\[
\prod_{i=1}^k \sigma_i(PQ) \leq \prod_{i=1}^k \sigma_i(P) \prod_{i=1}^k \sigma_i(Q), \quad k = 1, \ldots, n - 1,
\]
\[
\prod_{i=1}^n \sigma_i(PQ) = \prod_{i=1}^n \sigma_i(P) \prod_{i=1}^k \sigma_i(Q).
\] (3.5)
The last equality follows from \(|\det P| = \prod_{i=1}^n \sigma_i(P)\). As \(A^{-1}C = (A^{-1}B)(B^{-1}C)\) from the above inequalities we obtain
\[
\sum_{i=1}^k \log \sigma_i(A^{-1}C) \leq \sum_{i=1}^k (\log \sigma_i(A^{-1}B) + \log \sigma_i(B^{-1}C)), \quad k = 1, \ldots, n - 1,
\]
\[
\sum_{i=1}^n \log \sigma_i(A^{-1}C) = \sum_{i=1}^n (\log \sigma_i(A^{-1}B) + \log \sigma_i(B^{-1}C)).
\] (3.6)
Thus \(\log \sigma(A^{-1}C)\) is majorized by \(\log \sigma(A^{-1}B) + \log \sigma(B^{-1}C)\). As \(f(t) = ||t||_p\) is a convex function on \(\mathbb{R}\) for \(p \in [1, \infty)\), the majorization principle [HLP] yields that
\[
||\log \sigma(A^{-1}C)||_p \leq ||\log \sigma(A^{-1}B) + \log \sigma(B^{-1}C)||_p, \quad p \in [1, \infty).
\] (3.7)
Hence
\[
d_p(A, C) \leq ||\log \sigma(A^{-1}B) + \log \sigma(B^{-1}C)||_p \leq
\]
\[
||\log \sigma(A^{-1}B)||_p + ||\log \sigma(B^{-1}C)||_p = d_p(A, B) + d_p(B, C), \quad p \in [1, \infty).
\] (3.8)
Use the continuity of \(p\) at \(\infty\) to obtain the triangle inequality for \(p \in [1, \infty]\). It is straightforward to show that \(X_n\) is complete and locally compact for each \(d_p\), \(1 \leq p \leq \infty\). Clearly, \((CA)^{-1}(CB) = A^{-1}B\). Hence \(\text{GL}(n, \mathbb{C})\) acts as a subgroup of isometries on \(X_n\).
Let $C \in H_n$ and consider the one parameter group $e^{tC}$. Then for $t_1 \leq t_2$ we have $d_p(e^{t_1C}, e^{t_2C}) = (t_2 - t_1)|\sigma(C)||p|$. Hence this one parameter group describes a geodesic with respect to the metric $d_p$. Since $X_n$ can be identified with $e^{H_n}$ it follows that there exists a geodesic between $I$ and any $B \in e^{H_n}$. As $GL(n, \mathbb{C})$ acts as a subgroup of isometries on $X_n$ it follows that there is a geodesic between any $A, B \in X_n$. Clearly, we have equalities in (3.5) for all $t \in [0, 1]$. We deduce that we may assume that $C$ is a point on the unique geodesic given above.

Observe next that $d_p(e^{D(x)}, e^{D(y)}) = ||x - y||_p$ for any $x, y \in \mathbb{R}^n$. Hence $\mathbb{R}^n$ equipped with the metric $\delta_p$ is isometric to and $e^{D(n, \mathbb{R})}$ equipped with the metrics $d_p$. Since $\mathbb{R}^n$ is not Hadamard for $p = 1, \infty$ we deduce that $X_n$ is not Hadamard for $p = 1, \infty$.

Let $p \in (1, \infty)$. We show that there is only one geodesic between $A, B \in X_n$. Use Lemma 3.1 to deduce that we may assume that $A = I$, $B = e^{D(x)}$ where $x = (x_1, ..., x_n)^T \in \mathbb{R}^n$ and $x_1 \geq ... \geq x_n$. Let $C \in e^{H_n}$. Then $\Sigma(C) = e^{D(\log \sigma(C))}$. Suppose that

$$d_p(I, C) + d_p(C, e^{D(x)}) = d_p(I, e^{D(x)}) = ||x||_p. \quad (3.9)$$

Clearly, $d_p(I, C) = ||\log \sigma(C)||_p$. As $|x|^p$ is a strictly convex function (3.9) yields equalities in all inequalities in (3.6) [HLP] and equalities in all inequalities in (3.8). Since $\mathbb{R}^n$ is a unique geodesic space the second equality in (3.8) yields that $\log \sigma(C) = tx$ for some $t \in [0, 1]$. Clearly, we have equalities in (3.5) for all $k$ and $P = B, Q = B^{-1}e^{D(x)}$. Consider first the equality for $k = 1$:

$$||e^{D(x)}||_2 = ||C||_2|C^{-1}e^{D(x)}||_2. \quad (3.10)$$

Let $e^i = (\delta_{i1}, ..., \delta_{in})^T$ for $i = 1, ..., n$. Then

$$||e^{D(x)}||_2 = ||e^{D(x)}e^i||_2 = ||C^{-1}e^{D(x)}e^i||_2 \leq ||C||_2||C^{-1}e^{D(x)}e^i||_2 \leq ||C||_2||C^{-1}e^{D(x)}||_2. \quad (3.10)$$

Yields

$$||C^i(C^{-1}e^{D(x)}e^i)||_2 = ||C||_2||C^{-1}e^{D(x)}e^i||_2, ||C^{-1}e^{D(x)}e^i||_2 = ||C^{-1}e^{D(x)}||_2.$$  

Since $C \in H^+_n$, the first equality implies that $C^{-1}e^{D(x)}e^i = e^{\sigma(C)}e^i$ is an eigenvector of $C$ corresponding to the largest eigenvalue $\lambda_1(C) = \sigma_1(C)$. A straightforward calculation shows that $Ce^i = \lambda_1(C)e^i$. Repeat the same argument for $k = 2$ in the equality in (3.5) to deduce that $e^i \& e^j$ is an eigenvector of $C$ for the eigenvalue $\lambda_1(C)\lambda_2(C)$. That is, the subspace spanned by $e^1, e^2$ spanned by the two eigenvectors of $C$ corresponding to the eigenvalues $\lambda_1(C), \lambda_2(C)$. Hence $Be^2 = e^2(B)e^2$. Repeat this argument for $k = 3, ..., n$ to deduce that $Ce_i = \lambda_i(C)e^i, i = 1, ..., n$. Since $\log \sigma(C) = tx$, $t \in [0, 1]$ we deduce that $B = e^{tD(x)}$, i.e. $C$ is a point on the unique geodesic given above. $\Box$

**Corollary 3.3** Let the assumptions of Theorem 3.2 hold. Then

$$d_\infty(A, B) = \max(|\log \sigma_1(A^{-1}B)|, |\log \sigma_1(B^{-1}A)|),$$

$$d_\infty(A, B) \leq d_p(A, B) \leq (n)^{\frac{1}{2}}d_\infty(A, B).$$
Thus, all the metrics \( d_p \) are Lipschitz equivalent. It is straightforward to show that \( d_2(A, B) \) is a Riemannian metric on \( X_n \).

4 Busemann functions on \( X_n \)

In what follows we identify \( X_n \) with \( e^{H_n} \). Let

\[
S_{n,p} := \{ A \in H_n : \| A \|_p = 1, \quad p \in [1, \infty] \}
\]

be the unit ball in \( H_n \) centered at 0 with radius 1 in Schatten \( p \)-norm. Then any \( E \in e^{H_n} \setminus \{ I \} \) has the unique form \( E = e^A \) for some \( A \in S_{n,p} \) and \( t > 0 \). Let

\[
b_{E,p}(C) = d_p(C, E) - d_p(O, E), \quad E, O, C \in e^{H_n}, \quad p \in [1, \infty],
\]

be the Busemann \( p \)-function with the reference point \( O \). To identify \( \partial_p X_n \) we need to find the conditions under which the sequence \( \{ b_{t_m, \lambda_n, p} \} \) converges, where \( \{ A_m \} \subset S_{n,p} \) and \( \lim_{m \to \infty} t_m = \infty \). In this section we show that if \( A_m = A, \quad m = 1, \ldots, \) then \( \lim_{m \to \infty} b_{t_m, \lambda_n, m} = b_{\xi, p} \) and we identify the point \( \xi \in \partial_p X_n \).

Recall the spectral decomposition of \( A \in H_n \)

\[
A = UD(\lambda(A))U^*, \quad \lambda(A) = (\lambda_1(A), \ldots, \lambda_n(A))^T \in \mathbb{R}^n, \quad U = (x^1, \ldots, x^n) \in U_n.
\]

**Theorem 4.1** Let \( 0 \neq A \in H_n \) satisfy (4.2) and

\[
\lambda_1(A) = \cdots = \lambda_{j_0}(A) > \lambda_{j_1+1}(A) = \cdots = \lambda_{j_0}(A) > \cdots > \lambda_{j_{n-1}+1}(A) = \cdots = \lambda_n(A),
\]

\[
j_0 = 0 < j_1 < \cdots < j_n = n.
\]

Assume that \( i \in [1, q] \cap \mathbb{Z} \) and \( k \in [j_{i-1} + 1, j_i] \cap \mathbb{Z} \). Let \( V_k \subset C_i(\mathbb{C}) \) be the subspace spanned by \( x^1 \wedge x^2 \wedge \cdots \wedge x^{k-1} \wedge x^{k} \wedge \cdots \wedge x^{k-j_i} \), where \( l_1, \ldots, l_{k-j_i, i} \) range over all indices satisfying \( j_{i-1} + 1 \leq l_1 < \cdots < l_{k-j_i, i} \leq j_i \). Denote by \( P_k \in M_{n,i}(\mathbb{C}) \) the orthogonal projection on \( V_k \) for \( k = 1, \ldots, n \). Let \( C \in e^{H_n} \). Set

\[
\alpha_0(A, C) = 0, \quad \alpha_n(A, C) = \log \det C^{-1}, \quad \alpha_k(A, C) = \log \| (\wedge C^{-1}) P_k \|_2, \quad k = 1, \ldots, n-1.
\]

Let \( A \in S_{n,p} \) and \( t_m, m = 1, \ldots, \), be a sequence of real numbers converging to \( \infty \). Then \( b_{t_m, \lambda_n, p} \) converges to the Busemann function \( b_{\xi, p} \) for any \( p \in [1, \infty] \). More precisely, let \( C, O \in e^{H_n} \). Then

\[
b_{\xi, \infty}(C) = \alpha_1(A, C) - \alpha_1(A, O), \quad \text{if} \quad \lambda_1(A) > -\lambda_n(A),
\]

\[
b_{\xi, \infty}(C) = \alpha_{n-1}(A, C) - \alpha_n(A, C) - \alpha_{n-1}(A, O) + \alpha_n(A, O), \quad \text{if} \quad \lambda_1(A) < -\lambda_n(A),
\]

\[
b_{\xi, \infty}(C) = \max(\alpha_1(A, C), \alpha_{n-1}(A, C) - \alpha_n(A, C)) - \max(\alpha_1(A, O), \alpha_{n-1}(A, O) - \alpha_n(A, O)), \quad \text{if} \quad \lambda_1(A) = -\lambda_n(A),
\]

\[
b_{\xi, 1}(C) = \alpha_n(A, C) - \alpha_n(A, O), \quad \text{if} \quad \lambda_n(A) > 0,
\]

\[
b_{\xi, 1}(C) = -\alpha_n(A, C) + \alpha_n(A, O), \quad \text{if} \quad \lambda_1(A) < 0,
\]

\[
b_{\xi, 1}(C) = -\alpha_n(A, C) + \alpha_n(A, O), \quad \text{if} \quad \lambda_1(A) < 0,
\]

\[
b_{\xi, 1}(C) = -\alpha_n(A, C) + \alpha_n(A, O), \quad \text{if} \quad \lambda_1(A) < 0,
\]
In particular

\[ b_{\xi,1}(C) = \alpha_{j_1-1}(A, C) + \sum_{i=j_1}^{j_k} |\alpha_i(A, C) - \alpha_{i-1}(A, C)| + \alpha_{j_k}(A, C) - \alpha_n(A, C) - \]

\[ \alpha_{j_k-1}(A, O) - \sum_{i=j_k+1}^{j_k+1} |\alpha_i(A, O) - \alpha_{i-1}(A, O)| - \alpha_{j_k}(A, O) + \alpha_n(A, O), \] if \( \lambda_{j_k}(A) = 0, \)

\[ b_{\xi,1}(C) = 2\alpha_{j_k}(A, C) - \alpha_n(A, C) - 2\alpha_{j_k}(A, O) + \alpha_n(A, O), \] if \( \lambda_{j_k}(A) > 0 > \lambda_{j_k+1}(A). \)

To prove the theorem we need the standard perturbation techniques for eigenvalues of Hermitian matrices, e.g. [2] or [Kat].

**Lemma 4.2** Let \( 0 \neq A \in \mathbb{H}_n \) satisfy (4.2) and (4.3). Assume that \( C \in e^{\mathbb{H}_n} \). Let \( \mu_1(A, C) \geq \ldots \geq \mu_{j_1}(A, C) \) be the eigenvalues of the positive definite matrix \( F_1: \)

\[ F_1 := ((x^i C^{-2} x^m_{i,m=1})^t_{i,m=1} \in e^{\mathbb{H}_1}, \lambda(F_1) = (\mu_1(A, C), \ldots, \mu_{j_1}(A, C)). \quad (4.4) \]

Then for \( t >> 1 \)

\[ \log \sigma_i(C^{-1} e^{At}) = t\lambda_i(A) + \frac{1}{2} \log \mu_i(A, C) + O(e^{-(\lambda_i(A) - \lambda_{j_1+1}(A))t}) = \]

\[ t\lambda_1(A) + \frac{1}{2} \log \mu_i(A, C) + O(e^{-(\lambda_i(A) - \lambda_{j_1+1}(A))t}), \quad i = 1, \ldots, j_1. \quad (4.5) \]

In particular

\[ \alpha_1(A, C) = \log \sqrt{\|F_1\|_2} = \frac{1}{2} \log \mu_1(A, C). \quad (4.6) \]

\[ \log \sigma_i(C^{-1} e^{At}) = t\lambda_i(A) + \alpha_i(A, C) + O(e^{-(\lambda_i(A) - \lambda_{j_1+1}(A))t}) \quad \text{for } t >> 1, \quad (4.7) \]

\[ \sum_{i=1}^{j_1} \log \sigma_i(C^{-1} e^{At}) = t \sum_{i=1}^{j_1} \lambda_i(A) + \frac{1}{2} \log \det F_1 + O(e^{-(\lambda_i(A) - \lambda_{j_1+1}(A))t}), \quad \text{for } t >> 1. \]

**Proof.** Consider the positive definite matrix \( e^{tA}C^{-2}e^{tA}. \) By considering the similar Hermitian matrix \( U^*e^{tA}U(U^*CU)^{-2}U^*e^{tA}U \) we may assume that \( A = D(\lambda(A)). \) Let

\[ E(t) = e^{-2\lambda_1(A)t} e^{tA}C^{-2}e^{tA}, \quad \lim_{t \to \infty} E(t) = E(\infty). \]

Then \( E(\infty) \) is a nonnegative definite matrix of rank \( j_1, \) which has a block diagonal form \( F_1 \oplus 0. \) Hence \( \mu_1(A, C), \ldots, \mu_{j_1}(A, C) \) are the nonzero eigenvalues of \( E(\infty). \) Clearly

\[ E(t) = E(\infty) + O(e^{-at}), \quad a = \lambda_1(A) - \lambda_{j_1+1}(A), \quad t >> 1. \]
Weyl's inequalities [HJ] yield
\[ |\lambda_i(E(t)) - \lambda_i(E(\infty))| \leq \|E(t) - E(\infty)\|_2 = O(e^{-at}), \quad i = 1, \ldots, n. \]

Clearly
\[ \lambda_i(e^{At}C^{-2}e^{At}) = e^{2\lambda_i(A)t}\lambda_i(E(t)), \quad i = 1, \ldots, n. \]

As singular values of $C^{-1}e^{tA}$ are the positive square roots of the eigenvalues of $e^{tA}C^{-2}e^{tA}$, from the above arguments we deduce (4.5).

Recall from Theorem 4.1 that $\alpha_1(A, C) = \log ||C^{-1}P_1||_2 = \log ||P_1C^{-2}P_1||_2^{1/2}$. As $C_1P_1$ is a rank one matrix we deduce (4.6) and (4.7) follows. \[ \boxcheck \]

**Proof of Theorem 4.1.** We claim that
\[ \log \sigma_k(C^{-1}e^{tA}) = t\lambda_k(A) + \alpha_k(A, C) - \alpha_{k-1}(A, C) + E_k(t), \quad \lim_{t \to \infty} E_k(t) = 0, \quad k = 1, \ldots, n. \] (4.8)

As in the proof of Lemma 4.2 we may assume that $A = D(\lambda(A))$ and $x^t = e^t, \quad i = 1, \ldots, n$. For $k = 1$ (4.8) follows from (4.7). Let $k \in [\max(j_i-1, 1) + 1, j_i] \cap \mathbb{Z}$. Consider $\land_k e^{tA}$ for $t > 0$. Use (3.2) to deduce that $V_k$ is the eigenspace corresponding to the maximal eigenvalue $e^{t\sum_{i=1}^k \lambda_i(A)}$ of $\land_k e^{tA}$. As $A = D(\lambda(A))$ we deduce that $\lim_{t \to \infty} e^{-t} \sum_{i=1}^k \lambda_i(A) \land_k e^{tA} = P_k$. Apply (4.6) to $\land_k C^{-1} \land_k e^{tA}$ to obtain
\[ \log ||\land_k C^{-1} \land_k e^{tA}|| = \sum_{i=1}^k \log \sigma_i(C^{-1}e^{tA}) = t \sum_{i=1}^k \lambda_i(A) + \alpha_k(A, C) + E^{(k)}(t), \quad \lim_{t \to \infty} E^{(k)}(t) = 0. \]

Subtract from the above expression the similar expression for $k - 1$ to deduce (4.8).

Let $p = \infty$. Then $d_\infty(C, e^{t_m A}) = \max(\log \sigma_1(C^{-1}e^{t_m A})|, |\log \sigma_n(C^{-1}e^{t_m A})|)$. If $-\lambda_n(A) < \lambda_1(A)$, then for $t_m > 1$ (4.8) yields
\[ d_\infty(C, e^{t_m A}) = \log \sigma_1(C^{-1}e^{t_m A}) = t_m \lambda_1(A) + \alpha_1(A, C) + E_1(t_m). \]

The above equality yields the first case of the formula for $b(t, \infty)$. The case $\lambda_1(A) < -\lambda_n(A)$ yields similarly the second case of the formula for $b(t, \infty)$. Suppose finally that $\lambda_1(A) = -\lambda_n(A)$. Then
\[ d_\infty(C, e^{t_m A}) = \max(\log \sigma_1(C^{-1}e^{t_m A}) - \log \sigma_n(C^{-1}e^{t_m A})) = t_m \lambda_1(A) + \max(\alpha_1(A, C), -\alpha_n(A, C) + \alpha_{n-1}(A, C)) + E(t_m), \]

and the last case of the formula for $b(t, \infty)$ follows.

Let $p = 1$. Suppose first that $\lambda_n(A) > 0$. Then (4.8) yields that all singular values of $C^{-1}e^{t_m A}$ tend to $\infty$. Hence
\[ d_1(C, e^{t_m A}) = t_m (\sum_{i=1}^n \lambda_i(A)) + \alpha_n(A, B) + E^{(n)}(t_m), \]

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and the first case of the formula for \( b_{\xi,1} \) follows. The second case of the formula for \( b_{\xi,1} \) follows similarly. Suppose next that \( \lambda_j(A) = 0 \). Then all \( \sigma_i(C^{-1}e^{t-m}A) \) tend to \( \infty \) for \( i \leq j_k \) (if \( j_k > 0 \), all \( \sigma_i(C^{-1}e^{t-m}A) \) tend to \( -\infty \) for \( i > j_k \) (if \( j_k < n \)), and all \( \sigma_i(C^{-1}e^{t-m}A) \) are bounded for \( j_k-1 < i \leq j_k \). Hence \( d_1(C, e^{t-m}A) \) equals to

\[
 t_m \sum_{i=1}^{n} |\alpha_i(A)| + \alpha_{j_k-1}(A, C) + \sum_{i=j_k+1}^{j_k} |\alpha_i(A, C) - \alpha_{i-1}(A, C)| + \alpha_{j_k}(A, B) - \alpha_n(A, C),
\]

and the third case of the formula for \( b_{\xi,1} \) follows. Similarly one deduces the last case of the formula for \( b_{\xi,1} \).

Let \( p \in (1, \infty) \). If \( \lambda_i(A) \neq 0 \) then (4.8) yields:

\[
 |\log \sigma_i(C^{-1}e^{t-m}A)|^p = t_m^p |\lambda_i(A)|^p + pt_m^{p-1} |\lambda_i(A)|^p \lambda_i(A)^{-1} (\alpha_i(A, C) - \alpha_{i-1}(A, C)) + o(t_m^{p-1}).
\]

If \( \lambda_i(A) = 0 \) then (4.8) yields that \( |\log \sigma_i(C^{-1}e^{t-m}A)|^p = O(1) \). Hence

\[
 d_p(C, e^{t-m}A) = (t_m^p \sum_{i=1}^{n} |\lambda_i(A)|^p + pt_m^{p-1} \sum_{i=1}^{n} |\lambda_i(A)|^p \lambda_i(A)^{-1} (\alpha_i(A, C) - \alpha_{i-1}(A, C)) + o(t_m^{p-1}))^{\frac{1}{p}} =
\]

\[
 t_m (\sum_{i=1}^{n} |\lambda_i(A)|^p)^{\frac{1}{p}} + (\sum_{i=1}^{n} |\lambda_i(A)|^p)^{\frac{1}{p}-\frac{1}{p'}} \sum_{i=1}^{n} |\lambda_i(A)|^p \lambda_i(A)^{-1} (\alpha_i(A, C) - \alpha_{i-1}(A, C)) + o(1),
\]

and the formula for \( b_{\xi,p} \) follows. \( \square \)

5 \( \partial_p X_n \) for \( p \in (1, \infty) \)

Recall that any \( I \neq B \in e^{H_n} \) has a unique form \( B = e^{tA}, A \in S_{n,p} \). The visual boundary \( \partial_v X_n \) is identified with \( S_{n,p} \) equipped its standard topology. Furthermore, given a sequence \( \{t_m\}_n^{\infty} \) which converges to \( \infty \) and a sequence \( \{A_m\}_n^{\infty} \subset S_{n,p} \) then the sequence \( e^{t-m}A_m \) converges to a point in \( \partial_v X_n,p \) corresponding to \( A \in S_{n,p} \) if and only if \( \lim_{m \to \infty} A_m = A \). See for example Karpelivich [Kar] for the Riemannian case \( p = 2 \).

**Theorem 5.1** Let \( p \in (1, \infty) \). Then the Busemann p-boundary \( \partial_p X_n \) can be identified with the visual boundary of \( X_n \).

To prove this theorem we need the following results:

**Lemma 5.2** Let \( 0 \neq A \in H_n \) satisfy (4.2) and (4.3). Then for any \( C \in e^{H_n} \) the following inequalities hold:

\[
 \sum_{i=1}^{k} \log \lambda_{n-i+1}(C^{-1}) \leq \alpha_k(A, C) \leq \sum_{i=1}^{k} \log \lambda_i(C^{-1}), \quad k = 1, \ldots, n - 1,
\]

\[
 \alpha_n(A, C) = \sum_{i=1}^{n} \log \lambda_i(C^{-1}). \tag{5.1}
\]
Let \( k \in [1, n - 1] \cap \mathbb{Z} \) be a fixed integer that satisfies \( j_{i-1} < k \leq j_i \). Then equality in the right-hand side inequality of (5.1) holds if and only if the subspace \( W_{j_i} \) contains \( k \) linearly independent eigenvectors of \( C^{-1} \) corresponding to the first \( k \)-eigenvalues of \( C^{-1} \). Equality in the left-hand side of (5.1) holds if and only if any \( k \)-dimensional subspace of \( W_{j_i} \) is a subspace that spanned by last \( k \)-eigenvalues of \( C^{-1} \). Furthermore,

\[
\begin{align*}
\alpha_{jk-1}(A, C) - \alpha_{jk-1}(A, C) & \geq \alpha_{jk-1}(A, C) - \alpha_{jk-1}(A, C) \geq \ldots \\
& \geq \alpha_{jk}(A, C) - \alpha_{jk}(A, C), k = 1, \ldots, q.
\end{align*}
\]  

(5.2)

In particular \( \alpha_k(A, I) = 0 \) for \( k = 1, \ldots, n \).

**Proof.** Assume that \( k = 1 \). The maximal characterization of \( \lambda_1(C^{-2}) \) and the minimal characterization of \( \lambda_n(C^{-2}) \) and the definition of \( F_1 \) in (4.4) yield \( \]

\[
\lambda_n(C^{-2}) \leq \mu_{j_2}(A, C) = \lambda_{j_1}(F_1) \leq \mu_1(A, C) = \lambda_1(F_1) \leq \lambda_1(C^{-2}).
\]

Equality in the right-hand side of the above inequality holds if and only if \( W_{j_1} \) contains an eigenvector of \( C^{-2} \) corresponding to \( \lambda_1(C^{-2}) \). Equality \( \lambda_n(C^{-2}) = \lambda_1(F) \) yields the equalities \( \lambda_n(C^{-2}) = \lambda_{j_1}(F_1) = \ldots = \lambda_{j_1}(F) \). These equalities hold if and only if any nonzero vector in \( W_{j_1} \) is an eigenvector of \( C^{-2} \) corresponding to \( \lambda_n(C^{-2}) \). As \( C^{-1} \) is a positive definite matrix we deduce that \( \lambda_i(C^{-2}) = \lambda_i(C^{-1})^2, i = 1, \ldots, n \). Use (4.6) and the above arguments to deduce the lemma for \( k = 1 \). To deduce the lemma for \( 1 < k < n \) one repeats the above arguments for \( \wedge_k C^{-2} = (\wedge_k C^{-1})^2 \). To deduce the formula for \( \alpha_n(A, C) \) observe that \( \wedge_n C^{-2} \) is a positive number equal \( \det C^{-2} \).

The inequalities (5.2) follow from (4.8), (4.3) and the fact that the singular values of any matrix are arranged in a decreasing order. \( \square \)

**Proof of Theorem 5.1.** Fix \( p \in (1, \infty) \) and \( O = I \). We first show that if \( A \) and \( A' \) are two distinct points in \( S_{n,p} \), then the corresponding induced points \( \xi, \xi' \in \partial_p X_{n,p} \) are distinct. Assume to the contrary that \( \xi = \xi' \). The assumption that \( \xi = \xi' \) combined with Theorem 4.1 and Lemma 5.2 yield

\[
\sum_{i=1}^{n} \lambda_i(\alpha) \lambda_i(A)^{p-2}(\alpha(A, C) - \alpha_{i-1}(A, C)) = \sum_{i=1}^{n} \lambda_i(A') \lambda_i(A')^{p-2}(\alpha_i(A', C) - \alpha_{i-1}(A', C)),
\]

(5.3)

Observe that the sequence \( \{\lambda_i(A) \lambda_i(A)^{p-2}\}_1^n \) is a decreasing sequence. Furthermore

\[
\sum_{i=1}^{n} \lambda_i(A) \lambda_i(A)^{p-2}(\alpha_i(A, C) - \alpha_{i-1}(A, C)) =
\]

(5.4)

\[
\sum_{i=1}^{n-1} \alpha_i(A, C)(\lambda_i(A) \lambda_i(A)^{p-2} - \lambda_{i+1}(A) \lambda_{i+1}(A)^{p-2}) + \alpha_n(A, C) \lambda_n(A) \lambda_n(A)^{p-2}.
\]

In (5.3) choose \( C = e^{-A'} \). Then Lemma 5.2 yields \( \alpha_i(A', C) = \sum_{k=1}^{i} \lambda_k(A') \) for \( k = 1, \ldots, n \). Since \( A' \in S_{n,p} \) the right-hand side of (5.3) is equal to 1. Use Lemma 5.2 and (5.4) to deduce
that the left-hand side of (5.3) is bounded above by \(\sum_{i=1}^{n} \lambda_i(A)|\lambda_i(A)|^{p-2}\lambda_i(A)\). Use the Hölder p-inequality to deduce that the above expression is bounded above by \(\|A\|_p\|A'\|_p = 1\). Hence \(\lambda(A) = \lambda(A')\). Furthermore, the right-hand side inequalities in (5.1) are equalities for \(C = e^{-A}\) whenever \(\lambda_i(A) > \lambda_{i+1}(A)\). Lemma 5.2 for \(k = j_i\) yields that \(\mathcal{W}_{j_i}\) is spanned by the eigenvectors of \(e^{A'}\) corresponding to the first \(j_i\) eigenvalues of \(e^{A'}\) for \(i = 1, \ldots, p - 1\).

As \(\lambda(A) = \lambda(A')\) we deduce that for each eigenvalue \(\lambda = \lambda_{j_i}(A) = \lambda_{j_i}(A')\) the eigenspaces of \(A\) and \(A'\) coincide. Hence \(A = A'\) contrary to our assumption.

Let \(\{A_m\}^1_1 \subset S_{n,p}\) be a convergent sequence \(\lim_{m \to \infty} A_m = A \in S_{n,p}\). Clearly \(\lim_{m \to \infty} \lambda(A_m) = \lambda(A)\). As \(A\) may have multiple eigenvalues, the similar statement for the eigenspaces of \(\{A_m\}^1_1\) is as follows. Assume that \(A\) satisfies (4.3). Then the eigenspace \(\mathcal{W}_{j_i, m}\), corresponding to the first \(j_i\) eigenvalues of \(A_m\), converges to the eigenspace subspace \(\mathcal{W}_{j_i}\), corresponding to the first \(j_i\) eigenvalues of \(A\), for \(i = 1, \ldots, p\).

\[
\lim_{m \to \infty} \alpha_{j_i}(A_m, C) = \alpha_{j_i}(A, C), \quad i = 1, \ldots, p. \tag{5.5}
\]

Let \(\lim_{m \to \infty} t_m = \infty\). We have to show that

\[
\lim_{m \to \infty} b_p(C, e^{t_m A_m}) = b_{\xi, p}(C), \tag{5.6}
\]

where \(\xi\) is the limit point of the geodesic ray induced by \(A\). Use (4.8), (5.4) and the equality 
\[
\alpha_n(A, C) = \log \det C^{-1}
\]

in (5.7) to obtain

\[
b_p(C, e^{t_m A_m}) = \sum_{l=1}^{n-1} \alpha_l(A_m, C)(\lambda_l(A_m)|\lambda_l(A_m)|^{p-2} - \lambda_{l+1}(A_m)|\lambda_{l+1}(A_m)|^{p-2}) + \lambda_n(A_m)|\lambda_n(A_m)|^{p-2} \log \det C + \alpha(\frac{1}{t}). \tag{5.7}
\]

Observe that all the numbers \(\alpha_l(A_m, C)\) are uniformly bounded for a fixed \(C \in e^{H^m}\). Consider a summand

\[
\alpha_l(A_m, C)(\lambda_l(A_m)|\lambda_l(A_m)|^{p-2} - \lambda_{l+1}(A_m)|\lambda_{l+1}(A_m)|^{p-2}) \tag{5.8}
\]

appearing in (5.7). We claim that this summand converges to

\[
\alpha_l(A, C)(\lambda_l(A)|\lambda_l(A)|^{p-2} - \lambda_{l+1}(A)|\lambda_{l+1}(A)|^{p-2}).
\]

For \(l = j_i\), this claim follows from (5.5) and the continuity of \(\lambda(A)\). For \(l \in (j_{i-1}, j_i) \cap Z\) (5.8) converges to 0. Hence (5.6) holds. \(\square\)

6 \(\partial_{1}X_n\)

In this Section we show that the structure of \(\partial_{1}X_n\) is similar in principle to that of \(\partial_{n}R_{n,1}\), but more complicated. In what follows we use the notations of \(\S 1\). For \(A \in H_n\) let

\[
U_+(A) := U_{(0, \infty)}(A), \quad U_0(A) := U_{[0]}(A), \quad U_-(A) := U_{(-\infty, 0)}(A).
\]

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Then \( C^n = U_+(A) \oplus U_0(A) \oplus U_-(A) \) is an orthonormal decomposition of \( \mathbb{C}^n \), with some of the factors may be trivial. Note that \( U_-(A) \) is determined by \( U_+(A), U_0(A) \). For \( A \neq 0 \) we denote the above orthonormal decomposition simply as \( C^n = U_+ \oplus U_0 \oplus U_- \), \( \dim U_0 < n \).

**Lemma 6.1** The Busemann compactification of the geodesic rays of the form \( e^{tA}, A \in S_{n,1}, t > 0 \) with respect to the metric \( \alpha_1 \) depends only on the eigenspaces \( U_+(A), U_0(A), U_-(A) \). Moreover \( A, A' \in S_{n,1} \) induce the same point \( \xi \in \partial_t X_n \) if and only if the eigenspaces of \( A, A' \) corresponding to positive, zero and negative eigenvalues coincide respectively.

**Proof.** Consider the formulas for \( b_{k,1} \) in Theorem 4.1. Recall that \( \alpha_n(A, C) = \log \det C^{-1} \). Assume first that \( U_0(A) = \{0\} \), i.e. \( A \) is nonsingular. Then it is straightforward to see that the Busemann function depends only on \( U_+(A) \). Assume now that \( U_0(A) \) is a nontrivial subspace. Then \( b_{k,1} \) is given by the third formula in Theorem 4.1. Clearly, \( \alpha_{jk,-1}(A, C) \) depends only on \( U_+(A) \). The definition of \( \alpha_{k}(A, C) \) for \( l \in (jk-1, jak) \cap \mathbb{Z} \) depends on the choice of an orthonormal basis in \( U_+(A) \) and \( U_0(A) \). It is straightforward to show that the values of \( \alpha_{k}(A, C), l \in (jk-1, Jk) \cap \mathbb{Z} \) are independent of the choice of these orthonormal bases. (Suffices to note that \( \lambda^1 \wedge \ldots \wedge \lambda^{j-1} = \lambda_{j-1} W_{j-1} \).) Hence \( b_{k,1} \) depends only on \( U_+(A), U_0(A) \). It is straightforward to show that different decompositions \( C^n = U_+ \oplus U_0 \oplus U_- \) induce different Busemann functions. (One may take the convenient choice \( O = I \).) Hence \( A, A' \) induce the same point \( \xi \) if and only if the orthogonal decomposition \( C^n \) to the eigenspaces corresponding to positive, zero and negative eigenvalues of \( A, A' \) are identical. 

\( \Box \)

**Proposition 6.2** Let \( A \in \mathbb{H}_n, B \in \text{GL}(n, \mathbb{C}) \). Then

\[
\frac{1}{2} \log \lambda_n(BB^*) \leq \log \sigma_1(\text{Be}^A) - \lambda_1(A) \leq \frac{1}{2} \log \lambda_1(BB^*).
\]

**Proof.** Consider the matrix \( E = e^{-\lambda_1(A)} \text{Be}^A = \text{Be}^{A-\lambda_1(A)}I \). Then \( BB^* \leq EE^* \leq BB^* \), where \( P := P_{\lambda_1}(A) \). Clearly \( \sigma_1(E)^2 = ||EE^*||_2 \leq ||BB^*||_2 = \lambda_1(BB^*) \). Assume that \( Pu = u, ||u||_2 = 1 \). Then

\[
\sigma_1(E)^2 \geq ||BPB^*||_2 = ||BP||_2^2 \geq ||BPu||_2^2 = ||Bu||_2^2 \geq u^*Bu \geq \lambda_n(B^*B) = \lambda_n(BB^*).
\]

\( \Box \)

**Theorem 6.3** To each nontrivial orthogonal decomposition \( C^n = U_+ \oplus U_0 \oplus U_- \), \( \dim U_0 < n \) associate the space \( (U_+, \mathbb{H}(U_0), U_-) \). Then the union of all these spaces with respect to all nontrivial orthogonal decomposition of \( C^n \) can be identified with \( \partial_t X_n \). Let \( \{A_m\} \subseteq \mathbb{H}_n \) be an unbounded sequence. Then \( \{e^{A_m}\} \rightarrow \) converges to the point \( (U_+, T, U_-), T \in \mathbb{H}(U_0) \) if and only if the conditions (1.2) hold.

**Proof.** Recall that for \( A \in \mathbb{H}_n \) \( \sigma_i(e^A) = e^{\lambda_i(A)} \) for \( i = 1, \ldots, n \). For simplicity of the exposition we assume that

\[
\dim U_+ = k_+ > 0, \quad \dim U_0 = k_0 > 0, \quad \dim U_- = k_- > 0.
\]
We claim that for any $C \in e^H$,
\[
\log \sigma_i(C^{-1}e^{A_m}) = \lambda_i(A_m) + O(1), \quad i = 1, \ldots, n. \tag{6.1}
\]

The case $i = 1$ follows straightforward from Proposition 6.2. Apply Proposition 6.2 to $\wedge_k(C^{-1}e^A)$ for $k > 1$ to deduce $\sum_i \log \sigma_i(C^{-1}e^{A_m}) = \sum_i \lambda_i(A_m) + O(1)$. Hence (6.1) holds for any sequence $\{A_m\}^\infty_1 \subset H(n)$. Assume that (1.2) holds. Then
\[
\lim_{m \to \infty} \sigma_i(C^{-1}e^{A_m}) = \infty, \quad i = 1, \ldots, k_+ \quad \lim_{m \to \infty} \sigma_i(C^{-1}e^{A_m}) = -\infty, \quad i = n - k_+ + 1, \ldots, n.
\]

Let $A \in S_{n,1}$ such that $U_+ (A) = U_+, \quad U_0 (A) = U_0, \quad U_- (A) = U_-$. We claim that
\[
\begin{align*}
\lim_{m \to \infty} \sum_{i=1}^{k_+} |\log \sigma_i(C^{-1}e^{A_m})| = \sum_{i=1}^{k_+} \lambda_i(A_m) = \alpha_{k_+} (A,C), \\
\lim_{m \to \infty} \sum_{i=n-k_-+1}^{n} |\log \sigma_i(C^{-1}e^{A_m})| + \sum_{i=n-k_-+1}^{n} \lambda_i(A_m) = \alpha_n (A,C) - \alpha_{n-k_-} (A,C).
\end{align*}
\tag{6.2}
\]

The first formula of (6.2) is deduced by considering the norm $|| \wedge_k, C^{-1} \wedge_k, e^{A_m} ||_2$, as in the proof of Theorem 4.1. One has to notice that the ratio of a nonmaximal eigenvalue of $\wedge_k, e^{A_m}$ to the maximal eigenvalue $e^{\lambda_1(A_m)} + \ldots + \lambda_{k_+} (A_m)$ of $\wedge_k, e^{A_m}$ converges to 0. The second formula of (6.2) is deduced by using the same arguments for the sequence of the inverse matrices $e^{-A_m} C$.

Assume in addition that for a big enough $N$
\[
\lambda_i (A_m) = 0 \quad \text{for} \quad i = k_+ + 1, \ldots, k_+ + k_0 \quad \text{and} \quad m > N. \tag{6.3}
\]

Repeat the arguments of the proof of Theorem 4.1 for $p = 1$ to deduce that $\{e^{A_m}\}^\infty_1$ converges to $\xi$, the end of the ray $e^{Ak}$, $t > 0$. Note that $T = 0$.

We now consider the general case. Assume that $\lim_{m \to \infty} \lambda_i (A_m) = \theta_i \in (a,b)$ for $i = k_+ + 1, \ldots, k_+ + k_0$ for some $a < b$. Let
\[
E_m := P_{(a,b)} (A_m) A_m P_{(a,b)} (A_m), \quad A'_m := A_m - E_m, \quad m = 1, \ldots.
\]

Note that $\lim_{m \to \infty} E = E$ and $E|_{U_+ \oplus U_-}$ is the zero operator. Let $E|_{U_0} = T \in H(U_0)$. Then the sequence $\{A'_m\}^\infty_1$ satisfies (6.3). Clearly $A_m E_m = E_m A_m$. Hence
\[
d_1 (C, e^{A_m}) = d_1 (e^{-E_m} C, e^{A'_m}), \quad b_{e^{A_m},1} (C) = \hat{b}_{e^{A'_m},1} (e^{-E_m} C), \quad m = 1, \ldots,
\]

where $\hat{b}_{e^{A'_m},1}$ is the Busemann function with respect to the new reference point $O_m := e^{-E_m} O$. Note that $\lim_{m \to \infty} O_m = e^{-E} O$. The above arguments show that
\[
\lim_{m \to \infty} b_{e^{A_m},1} (C) = \hat{b}_{e^{A'_m},1} (e^{-E} C), \tag{6.4}
\]

where $\hat{b}_{e^{A'_m},1}$ is the Busemann function of the form given by Lemma 6.1 with respect to the reference point $O' = e^{-E} O$. This shows that any sequence $\{A_m\}^\infty_1 \subset H(n, \mathbb{C})$ satisfying the
conditions (1.2) converges to a boundary point \((U_+, T, U_-)\). A straightforward argument shows that two different elements \((U_+, T, U_-), (U'_+, T', U'_-)\) induce two different Busemann functions. Hence the above two points in \(\partial_1 X_n\) are distinct. Given a nontrivial decomposition \(\mathbb{C}^n = U_+ \oplus U_0 \oplus U_-\) and \(T \in \mathcal{H}(U_0)\) it is straightforward to construct a sequence \(\{A_m\} \in H_n\) which satisfies the conditions (1.2) for the given triple \((U_+, T, U_-)\). Hence any allowed triple \((U_+, T, U_0)\) is in \(\partial_1 X_n\). Finally, for a given unbounded sequence \(\{e^{B_l}\} \subset e^{H_n}\) there exists a subsequence \(\{A_m\}^\infty_s\) satisfying the conditions (1.2). Hence all allowable triples \((U_+, T, U_-)\) form \(\partial_1 X_n\) and \(X_n \cup \partial_1 X_n\) is compact. \(\Box\)

References


