p-metrics on $\mathbf{GL}(n, \mathbb{C})/\mathbf{U}_n$ and their Busemann compactifications

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Abstract

Let $\mathbf{GL}(n, \mathbb{C}) \supset \mathbf{U}_n$ be the group of $n \times n$ complex valued invertible matrices and the subgroup of unitary matrices respectively. In this paper we study Finsler p-metrics on the homogeneous space $\mathbf{X}_n = \mathbf{GL}(n, \mathbb{C})/\mathbf{U}_n$ for $p \in [1, \infty]$, which are induced by Schatten p-norms on the tangent bundle of $\mathbf{X}_n$ and are invariant under the action of $\mathbf{GL}(n, \mathbb{C})$. We show that for $p \in (1, \infty)$ the Busemann p-compactification is the visual compactification. For $p = 1, \infty$ the Busemann p-compactification is not the visual compactification. We give a complete description of Busemann 1-compactification. [1]

1 Introduction

Let $\mathbf{GL}(n, \mathbb{C})$ be the group of $n \times n$ invertible complex valued matrices. Let $\mathbf{M}(n, \mathbb{C})$ be its Lie algebra of $n \times n$ complex valued matrices. For $A \in \mathbf{M}(n, \mathbb{C})$ denote by $\sigma_1(A) \geq ... \geq \sigma_n(A) \geq 0$ the singular values of A. Recall that the Schatten p-norm on $\mathbf{M}(n, \mathbb{C})$ is given by $||A||_p = (\sum_{i=1}^n \sigma_i(A)^p)^{\frac{1}{p}}$. Denote by $\mathbf{H}_n \supset \mathbf{H}_n^+$, $\mathbf{U}_n \subset \mathbf{GL}(n, \mathbb{C})$ be the space of $n \times n$ hermitian matrices, the closed cone of nonnnegative definite hermitian matrices and the subgroup of unitary matrices respectively. Consider the homogeneous space $\mathbf{X}_n = \mathbf{GL}(n, \mathbb{C})/\mathbf{U}_n$. Recall that $\mathbf{X}_n$ can be identified with $\mathbf{H}_n^+ \cap \mathbf{GL}(n, \mathbb{C})$, which is equal to $e^{\mathbf{H}_n} := \{e^A : A \in \mathbf{H}_n\}$. Then there exists a unique Finsler p-metric on the tangent bundle

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of $\mathbf{X}_n$, invariant under the action of $\mathbf{GL}(n,\mathbb{C})$, which is given by the Schatten p-norm on $T_I\mathbf{X}_n = \mathbf{H}_n$. We show that the Finsler p-metric on the tangent bundle of $\mathbf{X}_n$ induces the following metric on $\mathbf{X}_n$:

$$d_p(A,B) := \left(\sum_{i=1}^{n} |\log \sigma_i(A^{-1}B)|^p\right)^{\frac{1}{p}}, \quad A,B \in \mathbf{X}_n,\ p \in [1,\infty]. \tag{1.1}$$

d_2 is the classical Riemannian metric on the homogeneous space $\mathbf{X}_n$. All p-metrics are uniformly Lipschitz equivalent for a fixed value of n. $\mathbf{GL}(n,\mathbb{C})$ acts (from the left) as a subgroup of isometries for each $p \in [1,\infty]$.

The main object of this paper to consider the Busemann functions and the Busemann compactifications for $d_p,\ p \in [1,\infty]$ as in [Bal]. One can view the Busemann p-compactification as a geometric way to add the p-boundary $\partial_p\mathbf{X}_n$ to $\mathbf{X}_n$ using the metric d_p, such that the space $\mathbf{X}_n \cup \partial_p\mathbf{X}_n$ is compact with respect to a suitable topology. For $p \in (1,\infty)$ the Busemann p-compactification is equivalent to the visual compactification, i.e. $\partial_p\mathbf{X}_n$ can be identified as the ends of geodesic rays from a fixed point $o \in \mathbf{X}_n$. For $p = 1,\infty$ the Busemann p-compactification is different from the visual compactification. We give the complete description of $\partial_1\mathbf{X}_n$:

View $\mathbb{C}^n := \{\mathbf{x} := (x_1,...,x_n)^{\mathrm{T}} : \quad x_i \in \mathbb{C},\ i = 1,...,n\}$ as inner product space with standard inner product $< \mathbf{x},\mathbf{y} >= \mathbf{y}^*\mathbf{x}$. Let $\mathbb{C}^n = U_+ \oplus U_0 \oplus U_-$ be a nontrivial orthonormal decomposition of $\mathbb{C}^n$, i.e. $\dim U_0 < n$. Note that each decomposition of $\mathbb{C}^n$ corresponds to the flag $\mathbb{C}^n = U_+ \oplus U_0 \oplus U_- \supset U_0 \oplus U_- \supset U_-$. Let $\mathbf{H}(U_0)$ be the real space of self-adjoint operators $T : U_0 \to U_0$. If $\dim U_0 = 0$ then $\mathbf{H}(U_0)$ has only one element: the complex number 0. Let $(U_+,\mathbf{H}(U_0),U_-) := \{(U_+,T,U_-) : \quad T \in \mathbf{H}(U_0)\}$. If $\dim U_0 = 0$ we identify $(U_+,\mathbf{H}(U_0),U_-)$ with (U_+,U_-). For $A \in \mathbf{H}_n$ and a set $S \subset \mathbb{R}$ let $P_S(A)$ be the orthogonal projection on the invariant subspace of A spanned by the eigenvectors of A corresponding to the eigenvalues in S. If S does not contain an eigenvalue of A then $P_S(A) = 0$. Note that if $A \in \mathbf{H}_n^+$ then the eigenvalues of A coincide with the singular eigenvalues of A.

We show that a sequence $\{e^{A_m}\}_{m=1}^{\infty} \subset e^{\mathbf{H}_n}$ converges to a point in $\partial_1\mathbf{X}_n$ if and only if there exist three nonnegative integers $k_0 < n, k_+, k_-,\ k_+ + k_- + k_0 = n$ with the following properties.

$$\begin{aligned}
&\lim_{m\to\infty} \sigma_i(e^{A_m}) = \infty,\ i = 1,...,k_+,\\
&\lim_{m\to\infty} P_{[\sigma_{k_+}(e^{A_m}),\sigma_1(e^{A_m})]}(e^{A_m})\mathbb{C}^n = U_+,\\
&\lim_{m\to\infty} \sigma_i(e^{A_m}) = 0,\ i = n-k_-+1,...,n,\\
&\lim_{m\to\infty} P_{[\sigma_n(e^{A_m}),\sigma_{n-k_-+1}(e^{A_m})]}(e^{A_m})\mathbb{C}^n = U_-,\\
&\lim_{m\to\infty} \sigma_i(e^{A_m}) = \sigma_i \in (0,\infty),\ i = k_++1,...,k_++k_0,\\
&\lim_{m\to\infty} P_{[\sigma_{k_++k_0}(e^{A_m}),\sigma_{k_++1}(e^{A_m})]}(e^{A_m})\mathbb{C}^n = U_0,\\
&\lim_{m\to\infty} e^{A_m}|P_{[\sigma_{k_++k_0}(e^{A_m}),\sigma_{k_++1}(e^{A_m})]}(e^{A_m})\mathbb{C}^n = e^T \text{ for some } T \in \mathbf{H}(U_0).
\end{aligned} \tag{1.2}$$

Then $\partial_1 \mathbf{X}_n$ is a union of $(U_+, \mathbf{H}(U_0), U_-)$ for all possible nontrivial decompositions of $\mathbb{C}^n$. Let $\mathbf{Gr}(n, k, \mathbb{C})$ be the Grassmannian manifold of all k-dimensional subspaces of $\mathbb{C}^n$. Then compact part of $\partial_1 \mathbf{X}_n$ is $\cup_{k=0}^{n} \mathbf{Gr}(n, k, \mathbb{C})$, which corresponds to all the orthonormal decomposition $\mathbb{C}^n = U_+ \oplus U_-$ and dim $U_+ = k$, for $k = 0, 1, ..., n$.

In [FF1] the authors show that the Busemann 1-compactification of the Siegel upper half plane given as $\mathbf{Sp}(n, \mathbb{R})/\mathbf{K}_n \subset \mathbf{X}_{2n}$, where $\mathbf{Sp}(n, \mathbb{R})$ is the symplectic subgroup of $\mathbf{GL}(2n, \mathbb{R})$ and $\mathbf{K}_n = \mathbf{Sp}(n, \mathbb{R}) \cap \mathbf{U}_{2n}$, is the classical Satake compactification [Sat] as a bounded domain [Hel]. It is of interest to extend these results a larger class of symmetric spaces.

We now survey briefly the contents of this paper. In §2 we discuss briefly the general setting of the Busemann compactification. As an example we consider the Busemann compactification of $\mathbb{R}^n$ with respect to Hölder p-metric. In §3 we show that $d_p(A, B)$ defined in (1.1) is a metric on $\mathbf{X}_n$. In §4 we give describe the Busemann functions corresponding to the boundary points in $\partial_p \mathbf{X}_n$ which are induced by geodesic rays from I. In §5 we show that $\partial_p \mathbf{X}_n$ is the visual boundary of $\mathbf{X}_n$. In §6 we describe the Busemann 1-compactification of $\mathbf{X}_n$.

2 Busemann functions and compactifications

A Finsler manifold is a smooth manifold $\mathbf{X}$ equipped with a continuous function which assigns to each point $x \in \mathbf{X}$ a norm on the tangent space $T_x\mathbf{X}$. The integral of the norm of the tangent vector to a smooth curve γ in X is called the length of γ. The distance $d(x, y)$ between two points $x, y \in \mathbf{X}$ is the infimum of the lengths of the curves connecting x and y. A geodesic is a curve γ which minimizes the distance between any two sufficiently near points on the curve. A complete geodesic space is a Finsler manifold such that it is a complete metric space with respect to d and any two points can be connected by a geodesic. A complete geodesic space is called a Hadamard manifold if any two points are connected by a unique geodesic.

Let $\mathbf{X}$ be a complete geodesic space and locally compact with respect to the metric d. Then $\mathbf{X}$ admits a Busemann compactification defined as $\mathrm{Cl}\, \mathcal{Y}$ in $C(\mathbf{X})$ of the set of functions $\mathcal{Y} := \{b_y(x) : \ y \in \mathbf{X}\}$ where

$$b_y : \mathbf{X} \to \mathbb{R}, \quad b_y(x) = d(y, x) - d(y, o). \tag{2.1}$$

Here o is a base point in $\mathbf{X}$ (the compactification is independent of the choice of o) and the topology on $C(\mathbf{X})$ is given by uniform convergence on compact subsets of $\mathbf{X}$. The map $y \mapsto b_y$ is a homeomorphism of $\mathbf{X}$ and $\mathcal{Y}$. Then $\partial\, \mathrm{Cl}\, \mathcal{Y} := \mathrm{Cl}\, \mathcal{Y} \backslash \mathcal{Y}$ is identified with the Busemann boundary $\partial \mathbf{X}$ [Bal, §II.1]. An unbounded sequence $y_k, k = 1, \ldots$ is said to converge to $\xi \in \partial \mathbf{X}$ if the sequence of functions b_{y_k} converges to a function $b_\xi \in C(\mathbf{X})$ (uniformly on bounded subsets of $\mathbf{X}$).

As an example we consider the Busemann compactification of $\mathbf{X} = \mathbb{R}^n$. Fix $p \in [0, \infty]$ and assume that the norm on $T_\mathbf{x}\mathbb{R}^n$ is given by $||(z_1, ..., z_n)^{\mathrm{T}}||_p := (\sum_{i=1}^{n} |z_i|^p)^{\frac{1}{p}}$. Denote by $\mathbb{R}^n_p$ the corresponding Finsler manifold. Then the distance between $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_p$ is given the Hölder metric $\delta_p(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}||_p$. $\mathbb{R}^n_p$ is a complete geodesic space which is Hadamard if and only if $p \in (1,\infty)$. Let $\partial\mathbb{R}^n_p$ be the Busemann boundary. For $\mathbf{y} \in \mathbb{R}^n_p, \xi \in \partial\mathbb{R}^n_p$ let $b_{\mathbf{y},p}, b_{\xi,p}$ be the corresponding Busemann functions for the distance δ_p with $o = \mathbf{o} = (o_1, ..., o_n)^{\mathrm{T}}$. Set

$$S^{n-1}_p := \{\mathbf{x} \in \mathbb{R}^n : \quad ||\xi||_p = 1\}.$$

Assume first that $p \in (1,\infty)$. Then

$$\lim_{k\to\infty} \mathbf{y}_k \to \xi \iff \lim_{k\to\infty} ||\mathbf{y}_k||_p = \infty \text{ and } \lim_{k\to\infty} \frac{\mathbf{y}_k}{||\mathbf{y}_k||_p} = \xi.$$

Let

$$Q_p(\xi,\mathbf{x}) := -\sum_{i=1}^n \xi_i|\xi_i|^{p-2}x_i, \quad \mathbf{x} = (x_1, ..., x_n)^{\mathrm{T}} \in \mathbb{R}^n,\ \xi = (\xi_1, ..., \xi_n)^{\mathrm{T}} \in \mathbb{R}^n\backslash\{0\}. \tag{2.2}$$

Then

$$b_{\xi,p}(\mathbf{x}) = Q_p(\xi,\mathbf{x}) - Q_p(\xi,\mathbf{o}), \quad \xi \in S^{n-1}_p,\ \mathbf{x} \in \mathbb{R}^n,\ . \tag{2.3}$$

Hence $\partial\mathbb{R}^n_p$ can be identified with S^{n-1}_p, which is diffeomorphic to the Euclidian sphere S^{n-1}_2.

Let $p = 1$ and $< n >:= \{1, 2, \ldots, n\}$. Denote by $2^{<n>}$ all *nonempty* subsets of $< n >$. Fix $\alpha \in 2^{<n>}$. Then $\{1,-1\}^\alpha$ denotes the set of all possible maps of α to $\{1,-1\}$. This set has cardinality $2^{|\alpha|}$, where $|\alpha|$ is the cardinality of the set α. Thus an element $\epsilon \in \{1,-1\}^\alpha$ is a set $\{\epsilon_j\}_{j\in\alpha}$ where $\epsilon_j = \pm1, j \in \alpha$. Let $\mathbb{R}^0$ be a set consisting of one element and $|\emptyset| = 0$.

Lemma 2.1 *The Busemann boundary $\mathbb{R}^n$ with respect to δ_1 has the stratification*

$$\partial\mathbb{R}^n_1 = \cup_{\alpha\in2^{<n>}}\{1,-1\}^\alpha \times \mathbb{R}^{|<n>\backslash\alpha|} \tag{2.4}$$

That is, a sequence $\mathbf{y}_k = (y_{1,k}, ..., y_{n,k})^T, k = 1, \ldots$ converges to $\xi = \{\epsilon_j\}_{j\in\alpha} \times (u_1, ..., u_m)^T$ if the following conditions hold:

$$\begin{aligned}
&\alpha = \{\alpha_1, ..., \alpha_l\}, \quad 1 \le \alpha_1 < \ldots < \alpha_l \le n,\\
&< n > \backslash\alpha = \{\beta_1, \ldots, \beta_m\}, \quad 1 \le \beta_1 < \beta_2 < \ldots < \beta_m \le n, \quad m = n - l,\\
&\lim_{k\to\infty} \epsilon_{\alpha_i} y_{\alpha_i,k} = +\infty, \quad i = 1, ..., l,\\
&\lim_{k\to\infty} y_{\beta_j,k} = u_j, \quad j = 1, ..., m.
\end{aligned} \tag{2.5}$$

For $\mathbf{x} = (x_1, ..., x_n)^T \in \mathbb{R}^n$ and ξ as above let

$$Q_1(\xi,\mathbf{x}) := -\sum_{i=1}^l x_{\alpha_i}\epsilon_{\alpha_i} + \sum_{j=1}^m |u_j - x_{\beta_j}|. \tag{2.6}$$

Then (2.3) holds for $p = 1$.

The proof of the lemma is straightforward. Note that (2.5) implies that the component $\{\rho_j\}_{j\in\gamma}\times\mathbb{R}^{|<n>\setminus\gamma|}$ of the strata $\{1,-1\}^{\gamma}\times\mathbb{R}^{|<n>\setminus\gamma|}$ is a boundary of $\{\epsilon_j\}_{j\in\alpha}\times\mathbb{R}^{|<n>\setminus\alpha|}$ if and only if α is a strict subset of γ and $\epsilon_i=\rho_i$ for $i\in\alpha$.

The stratification of $\partial\mathbb{R}^n_\infty$ is similar to the stratification of $\partial\mathbb{R}^n_1$. One can also define the function $Q_\infty(\xi,\mathbf{x})$ on each strata of $\partial\mathbb{R}^n_\infty$ so that (2.3) holds for $p=\infty$. Note that for $p=1,\infty$ $\partial\mathbb{R}^n_p$ does not correspond to the visual compactification S^{n-1}_p.

3 p-metrics on $\mathbf{X}_n$

Let $\mathbf{M}(m,n,\mathbb{F})$ be the vector space of $m\times n$ matrices over the field $\mathbb{F}$, $\mathbf{M}(n,\mathbb{F}):=\mathbf{M}(n,n,\mathbb{F})$ be the algebra of $n\times n$ matrices and $\mathbf{GL}(n,\mathbb{F})$ be the group of invertible matrices. In this paper $\mathbb{F}=\mathbb{R},\mathbb{C}$ is either the field of real or complex numbers. Denote by $\mathbf{U}_n$, $\mathbf{SU}_n$, $\mathbf{O}_n$ and $\mathbf{SO}_n$ the groups of $n\times n$ unitary, special unitary, real orthogonal and special real orthogonal matrices respectively. Let $A=(a_{pq})_1^n\in\mathbf{M}(n,\mathbb{C})$. Then $\overline{A}=(\overline{a}_{pq})_1^n$, A^{T} is transpose of A and $A^*=\overline{A}^{\mathrm{T}}$. By the spectrum of A we mean the eigenvalues $\lambda_1(A),\dots,\lambda_n(A)$ counted with their multiplicities and arranged in the following order: $\mathrm{Re}\,\lambda_1(A)\ge\cdots\ge\mathrm{Re}\,\lambda_n(A)$. The singular values of A are the eigenvalues of $(AA^*)^{\frac{1}{2}}\in\mathbf{H}_n^+$. Set $\sigma(A):=(\sigma_1(A),...,\sigma_n(A))^{\mathrm{T}}$. For $\mathbf{x}=(x_1,...,x_n)^{\mathrm{T}}\in\mathbb{F}^n$ let $D(\mathbf{x})=\mathrm{diag}(x_1,...,x_n)$ be the diagonal matrix with the diagonal entries $x_1,...,x_n$. Denote by $\mathbf{D}(n,\mathbb{F})\subset\mathbf{M}(n,\mathbb{F})$ the space of all diagonal matrices and let $\mathbf{D}^+(n,\mathbb{R}):=\mathbf{D}(n,\mathbb{R})\cap\mathbf{H}^+(n,\mathbb{C})$. Then

$$A=U\Sigma(A)V,\quad U,V\in\mathbf{U}_n,\quad \Sigma(A)=D(\sigma(A)) \tag{3.1}$$

is called the singular value decomposition (SVD). (It is also called the Cartan decomposition.) If $A\in\mathbf{M}(n,\mathbb{R})$ then the unitary matrices U,V in can be chosen to be orthogonal matrices. Note that $||A||_2=\sigma_1(A)$ is the l_2 norm of A viewed as a linear operator $A:\mathbb{C}^n\to\mathbb{C}^n$. Furthermore $(AA^*)^{\frac{1}{2}}$ is the unique representative of the coset $A\mathbf{U}_n$. Use the singular value decomposition of $A\in\mathbf{GL}(n,\mathbb{C})$ to deduce $\sigma_{n-i+1}(A^{-1})=\sigma_i(A)^{-1},\ i=1,\dots,n$. Observe next that $\sigma_i(A)=1,\ i=1,\dots,n\iff A$ is a unitary matrix.

For $A\in\mathbf{M}(m,n,\mathbb{F})$ and $1\le k\le\min(m,n)$ denote by $\wedge_k A$ the $k-th$ compound matrix. Note that $\wedge_k A\in\mathbf{M}(\binom{m}{k},\binom{n}{k},\mathbb{F})$ and the entries of A are all the $k\times k$ minors of A. ($\wedge_k A$ is the representation matrix of the linear transformation from the k exterior product $\wedge_k\mathbb{F}^n$ to $\wedge_k\mathbb{F}^m$ induced by $A:\mathbb{F}^n\to\mathbb{F}^m$.) The map $\wedge_k:\mathbf{GL}(n,\mathbb{F})\to\mathbf{GL}(\binom{n}{k},\mathbb{F})$ is a homomorphism which commutes with the $*$ involution. If $A\in\mathbf{M}(n,\mathbb{C})$ has complex eigenvalues $\lambda_1(A),\dots,\lambda_n(A)$ then $\wedge_k A$ has the following eigenvalues and singular values, and $\wedge_k e^A$ has the following eigenvalues respectively:

$$\begin{aligned}&\lambda_{i_1}(A)\lambda_{i_2}(A)\cdots\lambda_{i_k}(A),\ \sigma_{i_1}(A)\sigma_{i_2}(A)\cdots\sigma_{i_k}(A),\ e^{\lambda_{i_1}(A)+\lambda_{i_2}(A)+\cdots+\lambda_{i_k}(A)},\\ &1\le i_1<\cdots<i_k\le n.\end{aligned} \tag{3.2}$$

If $A\in\mathbf{H}_n$ ($\mathbf{H}_n^+$) then $\wedge_k A\in\mathbf{H}_{\binom{n}{k}}$ ($\mathbf{H}^+_{\binom{n}{k}}$). See for example [HJ].

The following lemma follows straightforward from SVD.

Lemma 3.1 *Let $(A, B), (C, D) \in \mathbf{X}_n \times \mathbf{X}_n$. Then there exists $T \in \mathbf{GL}(n, \mathbb{C})$ such that*

$$T(A, B) := (TA, TB) = (C, D) \tag{3.3}$$

if and only if

$$\Sigma(A^{-1}B) = \Sigma(C^{-1}D). \tag{3.4}$$

Theorem 3.2 *Let $p \in [1, \infty]$ and assume that $A, B \in \mathbf{GL}(n, \mathbb{C})$. Let $d_p(A, B) = ||\log \sigma(A^{-1}B)||_p$. Then d_p is a metric on the homogeneous space $\mathbf{X}_n$. $\mathbf{X}_n$ is a complete, locally compact, geodesic space with respect to d_p. For $p \in (1, \infty)$ $\mathbf{X}_n$ is Hadamard. Moreover, $\mathbf{GL}(n, \mathbb{C})$ acts (from the left) on $\mathbf{X}_n$ as a subgroup of isometries for d_p.*

Proof. Let $P \in \mathbf{M}(n, \mathbb{C})$. As $\sigma_i(P) = \sigma_i(PU) = \sigma_i(UP)$ for any $U \in \mathbf{U}_n$ we deduce that $d_p(\cdot, \cdot)$ is a nonnegative continuous function defined on $\mathbf{X}_n \times \mathbf{X}_n$. It is straightforward to see that A, B belong to the same left coset of $\mathbf{U}_n$ if and only if $d_p(A, B) = 0$. It is easy to check that $d_p(A, B) = d_p(B, A)$, since $\sigma_j(A^{-1}B) = \sigma_{n-j+1}(B^{-1}A)^{-1}$. We now prove the triangle inequality. As $\sigma_1(P) = ||P||_2$ it follows that $\sigma_1(PQ) \le \sigma_1(P)\sigma_1(Q)$ for any $Q \in \mathbf{M}(n, \mathbb{C})$. Apply the norm inequality to the k-th compound matrix $\wedge_k(PQ)$ to deduce

$$\begin{aligned} &\prod_{i=1}^{k} \sigma_i(PQ) \le \prod_{i=1}^{k} \sigma_i(P) \prod_{i=1}^{k} \sigma_i(Q), \quad k = 1, \ldots, n-1, \\ &\prod_{i=1}^{n} \sigma_i(PQ) = \prod_{i=1}^{n} \sigma_i(P) \prod_{i=1}^{k} \sigma_i(Q). \end{aligned} \tag{3.5}$$

The last equality follows from $|\mathrm{det\ P}| = \prod_{\mathrm{i}=1}^{\mathrm{n}} \sigma_{\mathrm{i}}(\mathrm{P})$. As $A^{-1}C = (A^{-1}B)(B^{-1}C)$ from the above inequalities we obtain

$$\begin{aligned} &\sum_{i=1}^{k} \log \sigma_i(A^{-1}C) \le \sum_{i=1}^{k} \big(\log \sigma_i(A^{-1}B) + \log \sigma_i(B^{-1}C)\big), \quad k = 1, \ldots, n-1, \\ &\sum_{i=1}^{n} \log \sigma_i(A^{-1}C) = \sum_{i=1}^{n} \big(\log \sigma_i(A^{-1}B) + \log \sigma_i(B^{-1}C)\big). \end{aligned} \tag{3.6}$$

Thus $\log \sigma(A^{-1}C)$ is majorized by $\log \sigma(A^{-1}B) + \log \sigma(B^{-1}C)$. As $f(t) = |t|^p$ is a convex function on $\mathbb{R}$ for $p \in [1, \infty)$, the majorization principle [HLP] yields that

$$||\log \sigma(A^{-1}C)||_p^p \le ||\log \sigma(A^{-1}B) + \log \sigma(B^{-1}C)||_p^p, \quad p \in [1, \infty). \tag{3.7}$$

Hence

$$\begin{aligned} &d_p(A, C) \le ||\log \sigma(A^{-1}B) + \log \sigma(B^{-1}C)||_p \le \\ &||\log \sigma(A^{-1}B)||_p + ||\log \sigma(B^{-1}C)||_p = d_p(A, B) + d_p(B, C), \quad p \in [1, \infty). \end{aligned} \tag{3.8}$$

Use the continuity of p at ∞ to obtain the triangle inequality for $p \in [1, \infty]$. It is straightforward to show that $\mathbf{X}_n$ is complete and locally compact for each $d_p, 1 \le p \le \infty$. Clearly, $(CA)^{-1}(CB) = A^{-1}B$. Hence $\mathbf{GL}(n, \mathbb{C})$ acts as a subgroup of isometries on $\mathbf{X}_n$.

Let $C \in \mathbf{H}_n$ and consider the one parameter group e^{tC}. Then for $t_1 \le t_2$ $d_p(e^{t_1 C}, e^{t_2 C}) = (t_2 - t_1)||\sigma(C)||_p$. Hence this one parameter group describes a geodesic with respect to the metric d_p. Since $\mathbf{X}_n$ can be identified with $e^{\mathbf{H}_n}$ it follows that there exists a geodesic between I and any $B \in e^{\mathbf{H}_n}$. As $\mathbf{GL}(n, \mathbb{C})$ acts as a subgroup of isometries on $\mathbf{X}_n$ it follows that there is a geodesic between any $A, B \in \mathbf{X}_n$. Clearly $\lim_{t \searrow 0} \frac{d_p(I, e^{tC})}{t} = ||\sigma(C)||_p$. Since $T_I \mathbf{X}_n = \mathbf{H}_n$ it follows that d_p is induced by the unique Finsler p-norm on the tangent bundle of $\mathbf{X}_n$, which is invariant under the action of $\mathbf{GL}(n, \mathbb{C})$ and is given by the Schatten p-norm on $T_I \mathbf{X}_n = \mathbf{H}_n$. Hence $\mathbf{X}_n$ is a complete, locally compact, geodesic space with respect to d_p.

Observe next that $d_p(e^{D(\mathbf{x})}, e^{D(\mathbf{y})}) = ||\mathbf{x} - \mathbf{y}||_p$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Hence $\mathbb{R}^n$ equipped with the metric δ_p is isometric to and $e^{\mathbf{D}(n,\mathbb{R})}$ equipped with the metrics d_p. Since $\mathbb{R}^n$ is not Hadamard for $p = 1, \infty$ we deduce that $\mathbf{X}_n$ is not Hadamard for $p = 1, \infty$.

Let $p \in (1, \infty)$. We show that there is only one geodesic between $A, B \in \mathbf{X}_n$. Use Lemma 3.1 to deduce that we may assume that $A = I$, $B = e^{D(\mathbf{x})}$ where $\mathbf{x} = (x_1, ..., x_n)^{\mathrm{T}} \in \mathbb{R}^n$ and $x_1 \ge ... \ge x_n$. Let $C \in e^{\mathbf{H}_n}$. Then $\Sigma(C) = e^{D(\log \sigma(C))}$. Suppose that

$$d_p(I, C) + d_p(C, e^{D(\mathbf{x})}) = d_p(I, e^{D(\mathbf{x})}) = ||\mathbf{x}||_p. \tag{3.9}$$

Clearly, $d_p(I, C) = ||\log \sigma(C)||_p$. As $|\mathbf{x}|^p$ is a strictly convex function (3.9) yields equalities in all inequalities in (3.6) [HLP] and equalities in all inequalities in (3.8). Since $\mathbb{R}^n$ is a unique geodesic space the second equality in (3.8) yields that $\log \sigma(C) = t\mathbf{x}$ for some $t \in [0, 1]$. Clearly, we have equalities in (3.5) for all k and $P = B, Q = B^{-1} e^{D(\mathbf{x})}$. Consider first the equality for $k = 1$:

$$||e^{D(\mathbf{x})}||_2 = ||C||_2 ||C^{-1} e^{D(\mathbf{x})}||_2. \tag{3.10}$$

Let $\mathbf{e}^i = (\delta_{1i}, \ldots, \delta_{ni})^{\mathrm{T}}$ for $i = 1, ..., n$. Then

$$||e^{D(\mathbf{x})}||_2 = ||e^{D(\mathbf{x})}\mathbf{e}^1||_2 = ||C(C^{-1}e^{D(\mathbf{x})}\mathbf{e}^1)||_2 \le ||C||_2 ||C^{-1}e^{D(\mathbf{x})}\mathbf{e}^1||_2 \le ||C||_2 ||C^{-1}e^{D(\mathbf{x})}||_2.$$

(3.10) yields

$$||C(C^{-1}e^{D(\mathbf{x})}\mathbf{e}^1)||_2 = ||C||_2 ||C^{-1}e^{D(\mathbf{x})}\mathbf{e}^1||_2, \ ||C^{-1}e^{D(\mathbf{x})}\mathbf{e}^1||_2 = ||C^{-1}e^{D(\mathbf{x})}||_2.$$

Since $C \in \mathbf{H}_n^+$, the first equality implies that $C^{-1}e^{D(\mathbf{x})}\mathbf{e}^1 = e^{x_1} C^{-1}\mathbf{e}^1$ is an eigenvector of C corresponding to the largest eigenvalue $\lambda_1(C) = \sigma_1(C)$. A straightforward calculation shows that $C\mathbf{e}^1 = \lambda_1(C)\mathbf{e}^1$. Repeat the same argument for $k = 2$ in the equality in (3.5) to deduce that $\mathbf{e}^1 \wedge \mathbf{e}^2$ is an eigenvector of $C \wedge C$ for the eigenvalue $\lambda_1(C)\lambda_2(C)$. That is, the subspace spanned by $\mathbf{e}^1, \mathbf{e}^2$ spanned by the two eigenvectors of C corresponding to the eigenvalues $\lambda_1(C), \lambda_2(C)$. Hence $B\mathbf{e}^2 = \lambda_2(B)\mathbf{e}^2$. Repeat this argument for $k = 3, ..., n$ to deduce that $C\mathbf{e}^i = \lambda_i(C)\mathbf{e}^i$, $i = 1, ..., n$. Since $\log \sigma(C) = t\mathbf{x}, t \in [0, 1]$ we deduce that $B = e^{tD(\mathbf{x})}$, i.e. C is a point on the unique geodesic given above. □

Corollary 3.3 *Let the assumptions of Theorem 3.2 hold. Then*

$$d_\infty(A, B) = \max(|\log \sigma_1(A^{-1}B)|, |\log \sigma_1(B^{-1}A)|),$$
$$d_\infty(A, B) \le d_p(A, B) \le (n)^{\frac{1}{p}} d_\infty(A, B).$$

Thus, all the metrics d_p are Lipschitz equivalent. It is straightforward to show that $d_2(A,B)$ is a Riemannian metric on $\mathbf{X}_n$.

4 Busemann functions on $\mathbf{X}_n$

In what follows we identify $\mathbf{X}_n$ with $e^{\mathbf{H}_n}$. Let

$$S_{n,p} := \{A \in \mathbf{H}_n : \quad ||A||_p = 1\}, \quad p \in [1,\infty]$$

be the unit ball in $\mathbf{H}_n$ centered at 0 with radius 1 in Schatten p-norm. Then any $E \in e^{\mathbf{H}_n}\backslash\{I\}$ has the unique form $E = e^{tA}$ for some $A \in S_{n,p}$ and $t > 0$. Let

$$b_{E,p}(C) = d_p(C,E) - d_p(O,E), \quad E,O,C \in e^{\mathbf{H}_n}, \quad p \in [1,\infty], \tag{4.1}$$

be the Busemann p-function with the reference point O. To identify $\partial_p \mathbf{X}_n$ we need to find the conditions under which the sequence $\{b_{e^{t_m A_m},p}\}$ converges, where $\{A_m\} \subset S_{n,p}$ and $\lim_{m\to\infty} t_m = \infty$. In this section we show that if $A_m = A,\ m = 1,...,$ then $lim_{m\to\infty} b_{e^{t_m A_m}} = b_{\xi,p}$ and we identify the point $\xi \in \partial_p \mathbf{X}_n$.

Recall the spectral decomposition of $A \in \mathbf{H}_n$

$$A = UD(\lambda(A))U^*,\ \lambda(A) = (\lambda_1(A),...,\lambda_n(A))^{\mathrm{T}} \in \mathbb{R}^n,\ U = (\mathbf{x}^1,\dots,\mathbf{x}^n) \in \mathbf{U}_n. \tag{4.2}$$

Theorem 4.1 *Let* $0 \neq A \in \mathbf{H}_n$ *satisfy* (4.2) *and*

$$\lambda_1(A) = \cdots = \lambda_{j_1}(A) > \lambda_{j_1+1}(A) = \cdots = \lambda_{j_2}(A) > \cdots > \lambda_{j_{q-1}+1}(A) = \cdots = \lambda_n(A),$$
$$j_0 = 0 < j_1 < \cdots < j_q = n. \tag{4.3}$$

Assume that $i \in [1,q] \cap \mathbb{Z}$ *and* $k \in [j_{i-1}+1, j_i] \cap \mathbb{Z}$. *Let* $\mathbf{V}_k \subset \mathbb{C}^{\binom{n}{k}}$ *be the subspace spanned by* $\mathbf{x}^1 \wedge \mathbf{x}^2 \wedge \ldots \wedge \mathbf{x}^{j_{i-1}} \wedge \mathbf{x}^{l_1} \wedge \mathbf{x}^{l_2} \wedge \ldots \wedge \mathbf{x}^{l_{k-j_{i-1}}}$, *where* $l_1,...,l_{k-j_{i-1}}$ *range over all indices satisfying* $j_{i-1}+1 \le l_1 < \ldots < l_{k-j_{i-1}} \le j_i$. *Denote by* $P_k \in \mathbf{M}(\binom{n}{k},\mathbb{C})$ *the orthogonal projection on* $\mathbf{V}_k$ *for* $k = 1,\dots,n$. *Let* $C \in e^{\mathbf{H}_n}$. *Set*

$$\alpha_0(A,C) = 0,\ \alpha_n(A,C) = \log\det \mathrm{C}^{-1},\ \alpha_\mathrm{k}(\mathrm{A},\mathrm{C}) = \log||(\wedge_\mathrm{k}\mathrm{C}^{-1})\mathrm{P}_\mathrm{k}||_2,\ \mathrm{k} = 1,...,\mathrm{n}-1.$$

Let $A \in S_{n,p}$ *and* $t_m,\ m = 1,\dots,$ *be a sequence of real numbers converging to* ∞. *Then* $b_{e^{t_m A},p}$ *converges to the Busemann function* $b_{\xi,p}$ *for any* $p \in [1,\infty]$. *More precisely, let* $C,O \in e^{\mathbf{H}_n}$. *Then*

$$b_{\xi,\infty}(C) = \alpha_1(A,C) - \alpha_1(A,O),\ \text{if}\ \lambda_1(A) > -\lambda_n(A),$$
$$b_{\xi,\infty}(C) = \alpha_{n-1}(A,C) - \alpha_n(A,C) - \alpha_{n-1}(A,O) + \alpha_n(A,O),\ \text{if}\ \lambda_1(A) < -\lambda_n(A),$$
$$b_{\xi,\infty}(C) = \max(\alpha_1(A,C), \alpha_{n-1}(A,C) - \alpha_n(A,C)) -$$
$$\max(\alpha_1(A,O), \alpha_{n-1}(A,O) - \alpha_n(A,O)),\ \text{if}\ \lambda_1(A) = -\lambda_n(A).$$
$$b_{\xi,1}(C) = \alpha_n(A,C) - \alpha_n(A,O),\ \text{if}\ \lambda_n(A) > 0,$$
$$b_{\xi,1}(C) = -\alpha_n(A,C) + \alpha_n(A,O),\ \text{if}\ \lambda_1(A) < 0,$$

$$b_{\xi,1}(C) = \alpha_{j_{k-1}}(A,C) + \sum_{i=j_{k-1}+1}^{j_k} |\alpha_i(A,C) - \alpha_{i-1}(A,C)| + \alpha_{j_k}(A,C) - \alpha_n(A,C) -$$

$$\alpha_{j_{k-1}}(A,O) - \sum_{i=j_{k-1}+1}^{j_k} |\alpha_i(A,O) - \alpha_{i-1}(A,O)| - \alpha_{j_k}(A,O) + \alpha_n(A,O), \text{if } \lambda_{j_k}(A) = 0,$$

$$b_{\xi,1}(C) = 2\alpha_{j_k}(A,C) - \alpha_n(A,C) - 2\alpha_{j_k}(A,O) + \alpha_n(A,O), \text{ if } \lambda_{j_k}(A) > 0 > \lambda_{j_{k+1}}(A).$$

for $p \in (1,\infty)$ $\quad b_{\xi,p}(B,C) =$

$$(\sum_{i=1}^n |\lambda_i(A)|^p)^{\frac{1-p}{p}} \sum_{i=1}^n \lambda_i(A)|\lambda_i(A)|^{p-2}(\alpha_i(A,C) - \alpha_{i-1}(A,C) - \alpha_i(A,B) + \alpha_{i-1}(A,B)).$$

To prove the theorem we need the standard perturbation techniques for eigenvalues of Hermitian matrices, e.g. [**?**] or [Kat].

Lemma 4.2 *Let $0 \neq A \in \mathbf{H}_n$ satisfy* (4.2) *and* (4.3). *Assume that $C \in e^{\mathbf{H}_n}$. Let $\mu_1(A,C) \geq \ldots \geq \mu_{j_1}(A,C)$ be the eigenvalues of the positive definite matrix F_1:*

$$F_1 := ((\mathbf{x}^l)^* C^{-2} \mathbf{x}^m)_{l,m=1}^{j_1} \in e^{\mathbf{H}_{j_1}}, \ \lambda(F_1) = (\mu_1(A,C), \ldots, \mu_{j_1}(A,C)). \tag{4.4}$$

Then for $t >> 1$

$$\log \sigma_i(C^{-1}e^{At}) = t\lambda_i(A) + \frac{1}{2}\log \mu_i(A,C) + O(e^{-(\lambda_1(A)-\lambda_{j_1+1}(A))t}) =$$

$$t\lambda_1(A) + \frac{1}{2}\log \mu_i(A,C) + O(e^{-(\lambda_1(A)-\lambda_{j_1+1}(A))t}), \quad i = 1,...,j_1. \tag{4.5}$$

In particular

$$\alpha_1(A,C) = \log \sqrt{||F_1||_2} = \frac{1}{2}\log \mu_1(A,C). \tag{4.6}$$

$$\log \sigma_1(C^{-1}e^{At}) = t\lambda_1(A) + \alpha_1(A,C) + O(e^{-(\lambda_1(A)-\lambda_{j_1+1}(A))t}) \quad \text{for } t >> 1, \tag{4.7}$$

$$\sum_{i=1}^{j_1} \log \sigma_i(C^{-1}e^{At}) = t\sum_{i=1}^{j_1} \lambda_i(A) + \frac{1}{2}\log \det \mathrm{F}_1 + \mathrm{O}(\mathrm{e}^{-(\lambda_1(\mathrm{A})-\lambda_{\mathrm{j}_1+1}(\mathrm{A}))\mathrm{t}}), \quad \text{for t} >> 1.$$

Proof. Consider the positive definite matrix $e^{tA}C^{-2}e^{tA}$. By considering the similar Hermitian matrix $U^*e^{tA}U(U^*CU)^{-2}U^*e^{tA}U$ we may assume that $A = D(\lambda(A))$. Let

$$E(t) = e^{-2\lambda_1(A)t}e^{tA}C^{-2}e^{tA}, \ \lim_{t\to\infty} E(t) = E(\infty).$$

Then $E(\infty)$ is a nonnegative definite matrix of rank j_1, which has a block diagonal form $F_1 \oplus 0$. Hence $\mu_1(A,C), ..., \mu_{j_1}(A,C)$ are the nonzero eigenvalues of $E(\infty)$. Clearly

$$E(t) = E(\infty) + O(e^{-at}), \quad a = \lambda_1(A) - \lambda_{j_1+1}(A), \quad t >> 1.$$

Weyl's inequalities [HJ] yield

$$|\lambda_i(E(t)) - \lambda_i(E(\infty))| \le ||E(t) - E(\infty)||_2 = O(e^{-at}), \quad i = 1, ..., n.$$

Clearly

$$\lambda_i(e^{At}C^{-2}e^{At}) = e^{2\lambda_1(A)t}\lambda_i(E(t)), \quad i = 1, ..., n.$$

As singular values of $C^{-1}e^{tA}$ are the positive square roots of the eigenvalues of $e^{tA}C^{-2}e^{tA}$, from the above arguments we deduce (4.5).

Recall from Theorem 4.1 that $\alpha_1(A, C) = \log ||C^{-1}P_1||_2 = \log ||P_1C^{-2}P_1||_2^{\frac{1}{2}}$. As C_1P_1 is a rank one matrix we deduce (4.6) and (4.7) follows. □

Proof of Theorem 4.1. We claim that

$$\log \sigma_k(C^{-1}e^{tA}) = t\lambda_k(A) + \alpha_k(A, C) - \alpha_{k-1}(A, C) + E_k(t), \ \lim_{t\to\infty} E_k(t) = 0, \ k = 1, \ldots, n. \tag{4.8}$$

As in the proof of Lemma 4.2 we may assume that $A = D(\lambda(A))$ and $\mathbf{x}^i = \mathbf{e}^i, \ i = 1, ..., n$. For $k = 1$ (4.8) follows from (4.7). Let $k \in [\max(j_{i-1}, 1) + 1, j_i] \cap \mathbb{Z}$. Consider $\wedge_k e^{tA}$ for $t > 0$. Use (3.2) to deduce that $\mathbf{V}_k$ is the eigenspace corresponding to the maximal eigenvalue $e^{t\sum_{l=1}^k \lambda_l(A)}$ of $\wedge_k e^{tA}$. As $A = D(\lambda(A))$ we deduce that $\lim_{t\to\infty} e^{-t\sum_{l=1}^k \lambda_l(A)} \wedge_k e^{tA} = P_k$. Apply (4.6) to $\wedge_k C^{-1} \wedge_k e^{tA}$ to obtain

$$\log ||\wedge_k C^{-1} \wedge_k e^{tA}|| = \sum_{l=1}^k \log \sigma_l(C^{-1}e^{tA}) = t\sum_{l=1}^k \lambda_i(A) + \alpha_k(A, C) + E^{(k)}(t), \ \lim_{t\to\infty} E^{(k)}(t) = 0.$$

Subtract from the above expression the similar expression for $k - 1$ to deduce (4.8).

Let $p = \infty$. Then $d_\infty(C, e^{t_mA}) = \max(|\log \sigma_1(C^{-1}e^{t_mA})|, |\log \sigma_n(C^{-1}e^{t_mA})|)$. If $-\lambda_n(A) < \lambda_1(A) \Rightarrow \lambda_1(A) > 0$, then for $t_m >> 1$ (4.8) yields

$$d_\infty(C, e^{t_mA}) = \log \sigma_1(C^{-1}e^{t_mA}) = t_m\lambda_1(A) + \alpha_1(A, C) + E_1(t_m).$$

The above equality yields the first case of the formula for $b_{\xi,\infty}$. The case $\lambda_1(A) < -\lambda_n(A)$ yields similarly the second case of the formula for $b_{\xi,\infty}$. Suppose finally that $\lambda_1(A) = -\lambda_n(A)$. Then

$$d_\infty(C, e^{t_mA}) = \max(\log \sigma_1(C^{-1}e^{t_mA}), -\log \sigma_n(C^{-1}e^{t_mA})) =$$
$$t_m\lambda_1(A) + \max(\alpha_1(A, C), -\alpha_n(A, C) + \alpha_{n-1}(A, C)) + E(t_m),$$

and the last case of the formula for $b_{\xi,\infty}$ follows.

Let $p = 1$. Suppose first that $\lambda_n(A) > 0$. Then (4.8) yields that all singular values of $C^{-1}e^{t_mA}$ tend to ∞. Hence

$$d_1(C, e^{t_mA}) = t_m\big(\sum_{i=1}^n \lambda_i(A)\big) + \alpha_n(A, B) + E^{(n)}(t_m),$$

and the first case of the formula for $b_{\xi,1}$ follows. The second case of the formula for $b_{\xi,1}$ follows similarly. Suppose next that $\lambda_{j_k}(A) = 0$. Then all $\sigma_i(C^{-1}e^{t_n A})$ tend to ∞ for $i \le j_{k-1}$ (if $j_{k-1} > 0$), all $\sigma_i(C^{-1}e^{t_m A})$ tend to $-\infty$ for $i > j_k$ (if $j_k < n$), and all $\sigma_i(C^{-1}e^{t_m A})$ are bounded for $j_{k-1} < i \le j_k$. Hence $d_1(C, e^{t_m A})$ equals to

$$t_m \sum_{i=1}^{n} |\lambda_i(A)| + \alpha_{j_{k-1}}(A, C) + \sum_{i=j_{k-1}+1}^{j_k} |\alpha_i(A, C) - \alpha_{i-1}(A, C)| + \alpha_{j_k}(A, B) - \alpha_n(A, C),$$

and the third case of the formula for $b_{\xi,1}$ follows. Similarly one deduces the last case of the formula for $b_{\xi,1}$.

Let $p \in (1, \infty)$. If $\lambda_i(A) \neq 0$ then (4.8) yields:

$$|\log \sigma_i(C^{-1}e^{t_m A})|^p = t_m^p |\lambda_i(A)|^p + p t_m^{p-1} \frac{|\lambda_i(A)|^p}{\lambda_i(A)} (\alpha_i(A, C) - \alpha_{i-1}(A, C)) + o(t_m^{p-1}).$$

If $\lambda_i(A) = 0$ then (4.8) yields that $|\log \sigma_i(C^{-1}e^{t_m A})|^p = O(1)$. Hence

$$d_p(C, e^{t_m A}) = (t_m^p \sum_{i=1}^{n} |\lambda_i(A)|^p + p t_m^{p-1} \sum_{i=1}^{n} \frac{|\lambda_i(A)|^p}{\lambda_i(A)} (\alpha_i(A, C) - \alpha_{i-1}(A, C)) + o(t_m^{p-1}))^{\frac{1}{p}} =$$

$$t_m (\sum_{i=1}^{n} |\lambda_i(A)|^p)^{\frac{1}{p}} + (\sum_{i=1}^{n} |\lambda_i(A)|^p)^{\frac{1-p}{p}} \sum_{i=1}^{n} \frac{|\lambda_i(A)|^p}{\lambda_i(A)} (\alpha_i(A, C) - \alpha_{i-1}(A, C)) + o(1), \qquad (4.9)$$

and the formula for $b_{\xi,p}$ follows. $\square$

5 $\partial_p \mathbf{X}_n$ for $p \in (1, \infty)$

Recall that any $I \neq B \in e^{\mathbf{H}_n}$ has a unique form $B = e^{tA}$, $A \in S_{n,p}$. The visual boundary $\partial_v \mathbf{X}_n$ is identified with $S_{n,p}$ equipped its standard topology. Furthermore, given a sequence $\{t_m\}_1^\infty$ which converges to ∞ and a sequence $\{A_m\}_1^\infty \subset S_{n,p}$ then the sequence $e^{t_m A_m}$ converges to a point in $\partial_v \mathbf{X}_{n,p}$ corresponding to $A \in S_{n,p}$ if and only if $\lim_{m\to\infty} A_m = A$. See for example Karpelivich [Kar] for the Riemannian case $p = 2$.

Theorem 5.1 *Let $p \in (1, \infty)$. Then the Busemann p-boundary $\partial_p \mathbf{X}_n$ can be identified with the visual boundary of $\mathbf{X}_n$.*

To prove this theorem we need the following results:

Lemma 5.2 *Let $0 \neq A \in \mathbf{H}_n$ satisfy (4.2) and (4.3). Then for any $C \in e^{\mathbf{H}_n}$ the following inequalities hold:*

$$\sum_{i=1}^{k} \log \lambda_{n-i+1}(C^{-1}) \le \alpha_k(A, C) \le \sum_{i=1}^{k} \log \lambda_i(C^{-1}),\ k = 1, ..., n-1,$$

$$\alpha_n(A, C) = \sum_{i=1}^{n} \log \lambda_i(C^{-1}). \qquad (5.1)$$

Let $k \in [1, n-1] \cap \mathbb{Z}$ be a fixed integer that satisfies $j_{i-1} < k \le j_i$. Then equality in the right-hand side inequality of (5.1) *holds if and only if the subspace $\mathbf{W}_{j_i}$ spanned by $\mathbf{x}^1, ..., \mathbf{x}^{j_i}$ contains k linearly independent eigenvectors of C^{-1} corresponding to the first k-eigenvalues of C^{-1}. Equality in the left-hand side of* (5.1) *holds if and only if any k-dimensional subspace of $\mathbf{W}_{j_i}$ is a subspace that spanned by last k-eigenvalues of C^{-1}. Furthermore,*

$$\begin{aligned} &\alpha_{j_{k-1}+1}(A,C) - \alpha_{j_{k-1}}(A,C) \ge \alpha_{j_{k-1}+2}(A,C) - \alpha_{j_{k-1}+1}(A,C) \ge \dots \\ &\ge \alpha_{j_k}(A,C) - \alpha_{j_k-1}(A,C),\ k = 1, ..., q. \end{aligned} \tag{5.2}$$

In particular $\alpha_k(A, I) = 0$ for $k = 1, ..., n$.

Proof. Assume that $k = 1$. The maximal characterization of $\lambda_1(C^{-2})$ and the minimal characterization of $\lambda_n(C^{-2})$ and the definition of F_1 in (4.4) yield [**?**]

$$\lambda_n(C^{-2}) \le \mu_{j_1}(A,C) = \lambda_{j_1}(F_1) \le \mu_1(A,C) = \lambda_1(F_1) \le \lambda_1(C^{-2}).$$

Equality in the right-hand side of the above inequality holds if and only if $\mathbf{W}_{j_1}$ contains an eigenvector of C^{-2} corresponding to $\lambda_1(C^{-2})$. Equality $\lambda_n(C^{-2}) = \lambda_1(F)$ yields the equalities $\lambda_n(C^{-2}) = \lambda_{j_1}(F_1) = \ldots = \lambda_1(F)$. These equalities hold if and only if any nonzero vector in $\mathbf{W}_{j_1}$ is an eigenvector of C^{-2} corresponding to $\lambda_n(C^{-2})$. As C^{-1} is a positive definite matrix we deduce that $\lambda_i(C^{-2}) = \lambda_i(C^{-1})^2,\ i = 1, ..., n$. Use (4.6) and the above arguments to deduce the lemma for $k = 1$. To deduce the lemma for $1 < k < n$ one repeats the above arguments for $\wedge_k C^{-2} = (\wedge_k C^{-1})^2$. To deduce the formula for $\alpha_n(A, C)$ observe that $\wedge_n C^{-2}$ is a positive number equal det C^{-2}.

The inequalities (5.2) follow from (4.8), (4.3) and the fact that the singular values of any matrix are arranged in a decreasing order. □

Proof of Theorem 5.1. Fix $p \in (1, \infty)$ and $O = I$. We first show that if A and A' are two distinct points in $S_{n,p}$ then the corresponding induced points $\xi, \xi' \in \partial_p \mathbf{X}_{n,p}$ are distinct. Assume to the contrary that $\xi = \xi'$. The assumption that $\xi = \xi'$ combined with Theorem 4.1 and Lemma 5.2 yield

$$\sum_{i=1}^{n} \lambda_i(A)|\lambda_i(A)|^{p-2}(\alpha_i(A,C) - \alpha_{i-1}(A,C)) = \sum_{i=1}^{n} \lambda_i(A')|\lambda_i(A')|^{p-2}(\alpha_i(A',C) - \alpha_{i-1}(A',C)), \tag{5.3}$$

Observe that the sequence $\{\lambda_i(A)|\lambda_i(A)|^{p-2}\}_1^n$ is a decreasing sequence. Furthermore

$$\begin{aligned} &\sum_{i=1}^{n} \lambda_i(A)|\lambda_i(A)|^{p-2}(\alpha_i(A,C) - \alpha_{i-1}(A,C)) = \\ &\sum_{i=1}^{n-1} \alpha_i(A,C)(\lambda_i(A)|\lambda_i(A)|^{p-2} - \lambda_{i+1}(A)|\lambda_{i+1}(A)|^{p-2}) + \alpha_n(A,C)\lambda_n(A)|\lambda_n(A)|^{p-2}. \end{aligned} \tag{5.4}$$

In (5.3) choose $C = e^{-A'}$. Then Lemma 5.2 yields $\alpha_i(A', C) = \sum_{k=1}^{i} \lambda_k(A')$ for $k = 1, ..., n$. Since $A' \in S_{n,p}$ the right-hand side of (5.3) is equal to 1. Use Lemma 5.2 and (5.4) to deduce that the left-hand side of (5.3) is bounded above by $\sum_{i=1}^{n}\lambda_i(A)|\lambda_i(A)|^{p-2}\lambda_i(A')$. Use the Hölder p-inequality to deduce that the above expression is bounded above by $||A||_p||A'||_p = 1$. Hence $\lambda(A) = \lambda(A')$. Furthermore, the right-hand side inequalities in (5.1) are equalities for $C = e^{-A'}$ whenever $\lambda_i(A) > \lambda_{i+1}(A)$. Lemma 5.2 for $k = j_i$ yields that $\mathbf{W}_{j_i}$ is spanned by the eigenvectors of $e^{A'}$ corresponding to the first j_i eigenvalues of $e^{A'}$ for $i = 1, ..., p-1$. As $\lambda(A) = \lambda(A')$ we deduce that for each eigenvalue $\lambda = \lambda_{j_i}(A) = \lambda_{j_i}(A')$ the eigenspaces of A and A' coincide. Hence $A = A'$ contrary to our assumption.

Let $\{A_m\}_1^\infty \subset S_{n,p}$ be a convergent sequence $\lim_{m\to\infty} A_m = A \in S_{n,p}$. Clearly $\lim_{m\to\infty}\lambda(A_m) = \lambda(A)$. As A may have multiple eigenvalues, the similar statement for the eigenspaces of $\{A_m\}_1^\infty$ is as follows. Assume that A satisfies (4.3). Then the eigenspace $\mathbf{W}_{j_i,m}$, corresponding to the first j_i eigenvalues of A_m, converges to the eigenspace subspace $\mathbf{W}_{j_i}$, corresponding to the first j_i eigenvalues of A, for $i = 1, ..., p$. Hence

$$\lim_{m\to\infty}\alpha_{j_i}(A_m, C) = \alpha_{j_i}(A, C), \quad i = 1, ..., p. \tag{5.5}$$

Let $\lim_{m\to\infty} t_m = \infty$. We have to show that

$$\lim_{m\to\infty} b_p(C, e^{t_m A_m}) = b_{\xi,p}(C), \tag{5.6}$$

where ξ is the limit point of the geodesic ray induced by A. Use (4.8), (5.4) and the equality $\alpha_n(A, C) = \log\det \mathrm{C}^{-1}$ to obtain

$$b_p(C, e^{t_m A_m}) = \sum_{l=1}^{n-1}\alpha_l(A_m, C)(\lambda_l(A_m)|\lambda_l(A_m)|^{p-2} - \lambda_{l+1}(A_m)|\lambda_{l+1}(A_m)|^{p-2})$$
$$+\lambda_n(A_m)|\lambda_n(A_m)|^{p-2}\log\det \mathrm{C} + \mathrm{o}(\frac{1}{\mathrm{t}}). \tag{5.7}$$

Observe that all the numbers $\alpha_l(A_m, C)$ are uniformly bounded for a fixed $C \in e^{\mathbf{H}_n}$. Consider a summand

$$\alpha_l(A_m, C)(\lambda_l(A_m)|\lambda_l(A_m)|^{p-2} - \lambda_{l+1}(A_m)|\lambda_{l+1}(A_m)|^{p-2}) \tag{5.8}$$

appearing in (5.7). We claim that this summand converges to

$$\alpha_l(A, C)(\lambda_l(A)|\lambda_l(A)|^{p-2} - \lambda_{l+1}(A)|\lambda_{l+1}(A)|^{p-2}).$$

For $l = j_i$ this claim follows from (5.5) and the continuity of $\lambda(A)$. For $l \in (j_{i-1}, j_i) \cap \mathbb{Z}$ (5.8) converges to 0. Hence (5.6) holds. □

6 $\partial_1\mathbf{X}_n$

In this Section we show that the structure of $\partial_1\mathbf{X}_n$ is similar in principle to that of $\partial\mathbb{R}_{n,1}$, but more complicated. In what follows we use the notations of §1. For $A \in \mathbf{H}_n$ let

$$U_+(A) := U_{(0,\infty)}(A), \quad U_0(A) := U_{\{0\}}(A), \quad U_-(A) := U_{(-\infty,0)}(A).$$

Then $\mathbb{C}^n = U_+(A) \oplus U_0(A) \oplus U_-(A)$ is an orthonormal decomposition of $\mathbb{C}^n$, with some of the factors may be trivial. Note that $U_-(A)$ is determined by $U_+(A), U_0(A)$. For $A \neq 0$ we denote the above orthonormal decomposition simply as $\mathbb{C}^n = U_+ \oplus U_0 \oplus U_-$, $\dim U_0 < n$.

Lemma 6.1 *The Busemann compactification of the geodesic rays of the form $e^{tA}, A \in S_{n,1}, t > 0$ with respect to the metric d_1 depends only on the eigenspaces $U_+(A), U_0(A), U_-(A)$. Moreover $A, A' \in S_{n,1}$ induce the same point $\xi \in \partial_1 \mathbf{X}_n$ if and only if the eigenspaces of A, A' corresponding to positive, zero and negative eigenvalues coincide respectively.*

Proof. Consider the formulas for $b_{\xi,1}$ in Theorem 4.1. Recall that $\alpha_n(A, C) = \log \det \mathrm{C}^{-1}$. Assume first that $U_0(A) = \{0\}$, i.e. A is nonsingular. Then it is straightforward to see that the Busemann function depends only on $U_+(A)$. Assume now that $U_0(A)$ is a nontrivial subspace. Then $b_{\xi,1}$ is given by the third formula in Theorem 4.1. Clearly, $\alpha_{j_{k-1}}(A, C)$ depends only on $U_+(A)$. The definition of $\alpha_l(A, C)$ for $l \in (j_{k-1}, j_k] \cap \mathbb{Z}$ depends on the choice of an orthonormal basis in $U_+(A)$ and $U_0(A)$. It is straightforward to show that the values of $\alpha_l(A, C), l \in (j_{k-1}, j_k] \cap \mathbb{Z}$ are independent of the choice of these orthonormal bases. (Suffices to note that $\mathbf{x}^1 \wedge \ldots \wedge \mathbf{x}^{j_{i-1}} = \wedge_{j_{i-1}} \mathbf{W}_{j_{i-1}}$.) Hence $b_{\xi,1}$ depends only on $U_+(A), U_0(A)$. It is straightforward to show that different decompositions $\mathbb{C}^n = U_+ \oplus U_0 \oplus U_-$ induce different Busemann functions. (One may take the convenient choice $O = I$.) Hence A, A' induce the same point ξ if and only if the orthogonal decomposition $\mathbb{C}^n$ to the eigenspaces corresponding to positive, zero and negative eigenvalues of A, A' are identical. □

Proposition 6.2 *Let $A \in \mathbf{H}_n$, $B \in \mathbf{GL}(n, \mathbb{C})$. Then*

$$\frac{1}{2} \log \lambda_n(BB^*) \leq \log \sigma_1(Be^A) - \lambda_1(A) \leq \frac{1}{2} \log \lambda_1(BB^*).$$

Proof. Consider the matrix $E = e^{-\lambda_1(A)} Be^A = Be^{A - \lambda_1(A)I}$. Then $BPB^* \leq EE^* \leq BB^*$, where $P := P_{\lambda_1(A)}(A)$. Clearly $\sigma_1(E)^2 = ||EE^*||_2 \leq ||BB^*||_2 = \lambda_1(BB^*)$. Assume that $P\mathbf{u} = \mathbf{u}, ||\mathbf{u}||_2 = 1$. Then

$$\sigma_1(E)^2 \geq ||BPB^*||_2 = ||BP||_2^2 \geq ||BP\mathbf{u}||_2^2 = ||B\mathbf{u}||_2^2 = \mathbf{u}^* B^* B\mathbf{u} \geq \lambda_n(B^*B) = \lambda_n(BB^*).$$

□

Theorem 6.3 *To each nontrivial orthogonal decomposition $\mathbb{C}^n = U_+ \oplus U_0 \oplus U_-$, $\dim U_0 < n$ associate the space $(U_+, \mathbf{H}(U_0), U_-)$. Then the union of all these spaces with respect to all nontrivial orthogonal decomposition of $\mathbb{C}^n$ can be identified with $\partial_1 \mathbf{X}_n$. Let $\{A_m\}_1^\infty \subset \mathbf{H}_n$ be an unbounded sequence. Then $\{e^{A_m}\}_1^\infty$ converges to the point $(U_+, T, U_-), T \in \mathbf{H}(U_0)$ if and only if the conditions* (1.2) *hold.*

Proof. Recall that for $A \in \mathbf{H}_n$ $\sigma_i(e^A) = e^{\lambda_i(A)}$ for $i = 1, ..., n$. For simplicity of the exposition we assume that

$$\dim U_+ = k_+ > 0, \quad \dim U_0 = k_0 > 0, \quad \dim U_- = k_- > 0.$$

We claim that for any $C \in e^{\mathbf{H}_n}$

$$\log \sigma_i(C^{-1}e^{A_m}) = \lambda_i(A_m) + O(1), \quad i = 1, ..., n. \tag{6.1}$$

The case $i = 1$ follows straightforward from Proposition 6.2. Apply Proposition 6.2 to $\wedge_k(C^{-1}e^A)$ for $k > 1$ to deduce $\sum_1^k \log \sigma_i(C^{-1}e^{A_m}) = \sum_{i=1}^k \lambda_i(A_m) + O(1)$. Hence (6.1) holds for any sequence $\{A_m\}_1^\infty \in \mathbf{H}_n$. Assume that (1.2) holds. Then

$$\lim_{m\to\infty} \sigma_i(C^{-1}e^{A_m}) = \infty, \quad i = 1, ..., k_+ \ \lim_{m\to\infty} \sigma_i(C^{-1}e^{A_m}) = -\infty, \quad i = n - k_- + 1, ..., n.$$

Let $A \in S_{n,1}$ such that $U_+(A) = U_+,\ U_0(A) = U_0,\ U_-(A) = U_-$. We claim that

$$\lim_{m\to\infty} \sum_1^{k_+} |\log \sigma_i(C^{-1}e^{A_m})| - \sum_1^{k_+} \lambda_i(A_m) = \alpha_{k_+}(A, C), \tag{6.2}$$

$$\lim_{m\to\infty} \sum_{n-k_-+1}^{n} |\log \sigma_i(C^{-1}e^{A_m})| + \sum_{i=n-k_-+1}^{n} \lambda_i(A_m) = \alpha_n(A, C) - \alpha_{n-k_-}(A, C).$$

The first formula of (6.2) is deduced by considering the norm $|| \wedge_{k_+} C^{-1} \wedge_{k_+} e^{A_m}||_2$, as in the proof of Theorem 4.1. One has to notice that the ratio of a nonmaximal eigenvalue of $\wedge_{k_+} e^{A_m}$ to the maximal eigenvalue $e^{\lambda_1(A_m)+...+\lambda_{k_+}(A_m)}$ of $\wedge_{k_+} e^{A_m}$ converges to 0. The second formula of (6.2) is deduced by using the same arguments for the sequence of the inverse matrices $e^{-A_m}C$.

Assume in addition that for a big enough N

$$\lambda_i(A_m) = 0 \quad \text{for} \quad i = k_+ + 1, ..., k_+ + k_0 \quad \text{and} \quad m > N. \tag{6.3}$$

Repeat the arguments of the proof of Theorem 4.1 for $p = 1$ to deduce that $\{e^{A_m}\}_1^\infty$ converges to ξ, the end of the ray $e^{At}, t > 0$. Note that $T = 0$.

We now consider the general case. Assume that $\lim_{m\to\infty} \lambda_i(A_m) = \theta_i \in (a, b)$ for $i = k_+ + 1, ..., k_+ = k_0$ for some $a < b$. Let

$$E_m := P_{(a,b)}(A_m)A_mP_{(a,b)}(A_m),\ A'_m := A_m - E_m,\ m = 1, ...,.$$

Note that $\lim_{m\to\infty} = E$ and $E|_{U_+\oplus U_-}$ is the zero operator. Let $E|_{U_0} = T \in \mathbf{H}(U_0)$. Then the sequence $\{A'_m\}_1^\infty$ satisfies (6.3). Clearly $A_mE_m = E_mA_m$. Hence

$$d_1(C, e^{A_m}) = d_1(e^{-E_m}C, e^{A'_m}),\ b_{e^{A_m},1}(C) = \hat{b}_{e^{A'_m},1}(e^{-E_m}C),\ m = 1, ...,$$

where $\hat{b}_{e^{A'_m},1}$ is the Busemann function with respect to the new reference point $O_m := e^{-E_m}O$. Note that $\lim_{m\to\infty} O_m = e^{-E}O$. The above arguments show that

$$\lim_{m\to\infty} b_{e^{A_m},1}(C) = \hat{b}_{\xi,1}(e^{-E}C), \tag{6.4}$$

where $\hat{b}_{\xi,1}$ is the Busemann function of the form given by Lemma 6.1 with respect to the reference point $O' = e^{-E}O$. This shows that any sequence $\{A_m\}_1^\infty \subset \mathbf{H}(n, \mathbb{C})$ satisfying the conditions (1.2) converges to a boundary point (U_+, T, U_-). A straightforward argument shows that two different elements $(U_+, T, U_-), (U'_+, T', U'_-)$ induce two different Busemann functions. Hence the above two points in $\partial_1 \mathbf{X}_n$ are distinct. Given a nontrivial decomposition $\mathbb{C}^n = U_+ \oplus U_0 \oplus U_-$ and $T \in \mathbf{H}(U_0)$ it is straightforward to construct a sequence $\{A_m\} \in \mathbf{H}_n$ which satisfies the conditions (1.2) for the given triple (U_+, T, U_-). Hence any allowed triple (U_+, T, U_0) is in $\partial_1 \mathbf{X}_n$. Finally, for a given unbounded sequence $\{e^{B_\ell}\} \subset e^{\mathbf{H}_n}$ there exists a subsequence $\{A_m\}_1^\infty$ satisfying the conditions (1.2). Hence all allowable triples (U_+, T, U_-) form $\partial_1 \mathbf{X}_n$ and $\mathbf{X}_n \cup \partial_1 \mathbf{X}_n$ is compact. □

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