Problem 1. Consider the following system of linear equations:

\[
\begin{align*}
    x_1 - 3x_2 + x_3 + x_4 &= 1 \\
    3x_1 - 9x_2 + x_3 - x_4 &= 5
\end{align*}
\]

(a) Write down the augmented matrix associated to this system;
(b) Apply the Gauss-Jordan reduction to bring the augmented matrix into the reduced row echelon form;
(c) List all lead and all free variables. Find the set of solutions to the system.

Problem 2. Consider the system

\[
\begin{align*}
    x_1 + 2x_2 + 3x_3 &= 0 \\
    2x_1 + 5x_2 + 4x_3 &= 0 \\
    x_1 + 3x_2 + ax_3 &= 0
\end{align*}
\]

For which values of \(a\) this system:

(a) Is solvable;
(b) Has infinite number of solutions.

Hint: Bring this system to a Row Echelon Form.

Problem 3. True or False. (Justify briefly!):

(a) The set \( S := \{ x = (x_1, x_2) \in \mathbb{R}^2, \ x_1^2 - x_2^2 = 0 \} \) is a subspace of \( \mathbb{R}^2 \).
(b) Any 3 vectors in \( \mathbb{R}^4 \) can not form a basis in \( \mathbb{R}^4 \).

Problem 4. Let \( v_1 := (1,1,1)^T, v_2 = (1,0,1)^T, v_3 = (1,2,1)^T \in \mathbb{R}^3 \). Denote \( V := \text{span}(v_1, v_2, v_3) \).

(a) Is \( (1,0,0)^T \) in \( V \)?
(b) Show that \( v_1, v_2, v_3 \) are linearly dependent by giving a nontrivial linear combination of \( v_1, v_2, v_3 \) which equals to \( 0 \).
(c) Find the dimension of \( V \) and a basis in \( V \).

Problem 5. Let \( \mathcal{P}_4 \) be the set of all polynomials of degree 4 at most. Let \( E, O \subset \mathcal{P}_4 \) be the subspaces of the even, \( p(-x) = p(x) \) for all \( x \), and odd polynomials, \( p(-x) = -p(x) \), respectively.

(a) Find the dimensions of \( E \) and \( O \).
(b) Is \( \mathcal{P}_4 = E \oplus O \)? (Justify)

Problem 6. Let \( v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \). Assume that \( v_1, v_2, v_3 \) form a basis in \( \mathbb{R}^3 \).

(a) Find the transition matrix from the standard basis \( \mathcal{E} = (e_1, e_2, e_3) \) to the basis \( \mathcal{B} = (v_1, v_2, v_3) \).
(b) Find the coordinate of the vector \( v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \) in the basis \( \mathcal{B} \).
Problem 7. Consider the transformation $L : P_2 \rightarrow P_2$, $L(p(x)) = p'(x) + p(1)x$. ($P_2$ is the vector space of all polynomials of degree 2 at most, and $p'$ is the derivative of $p$.)

a. Show that $L$ is a linear transformation.

b. Find the representation matrix of $L$ in the standard basis $\{1, x, x^2\}$ of $P_2$.

c. Is $L$ isomorphism? Justify!