MATH 425, Linear Algebra II, Practice Test 1,
February 18, 2010, Instructor: S. Friedland
Name and e-mail address: 

No books or notes. Show all your work. Write solutions in the exam booklet without copying the problems. You can use a result (x) of any part of the problem, to show other part of any problem. Unjustified answer yields no credit.

Problem 1.
(a) For \( \mathbf{x} = (x_1, x_2) \mathbf{T}, \mathbf{y} = (y_1, y_2) \mathbf{T} \), let
\[
f(\mathbf{x}, \mathbf{y}) = 2x_1y_1 - 3x_1y_2 - 3x_2y_1 + ky_2y_2.
\]
Find the values of \( k \) for which \( f(\mathbf{x}, \mathbf{y}) \) is an inner product on \( \mathbb{R}^2 \).
(b) Find a basis of the subspace \( U \) of \( \mathbb{R}^4 \) orthogonal to \( \mathbf{x}_1 = (1, -2, 3, 4) \mathbf{T} \) and \( \mathbf{x}_2 = (3, -5, 7, 8) \mathbf{T} \).
(c) Let \( V \) be an inner product space. Assume that \( U \) is a subspace of \( V \), with an orthonormal basis \( u_1, \ldots, u_n \). Let \( \mathbf{v} \in V \). Write down the orthogonal projection \( P_U(\mathbf{v}) \) of \( \mathbf{v} \) on \( U \) in terms of \( \mathbf{v} \) and \( u_1, \ldots, u_n \) and \( \mathbf{v} \). Show that for any \( \mathbf{u} \in U \) one has the inequality \( \| \mathbf{v} - \mathbf{u} \| \geq \| \mathbf{v} - P_U(\mathbf{v}) \| \). Characterize the equality case.

Problem 2. Fit the best possible parabola of the form \( y = ax + bx^2 \) to the data \((-2, 0), (1, 1), (0, -1), (1, 1)\).

Problem 3. Let \( A = [a_{ij}]_{i,j=1}^n \in \mathbb{R}^{n \times n} \). Assume that \( a_{ij} \in (0, 2) \) for \( i, j = 1, \ldots, n \) and \( n \geq 2 \). Show that \( |\det A| \leq 2^n n^2 \). Can equality hold for some matrix \( A \)?

Problem 4. Let \( A = \begin{bmatrix} 1 & 1 - i \\ 1 + i & 0 \end{bmatrix} \). Find the eigenvalues, the corresponding eigenvectors, a unitary \( U \) and a diagonal matrix \( D \) such that \( A = UDU^* \).

Problem 5. Let \( A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 5 & -1 & 4 \\ 0 & -1 & -1 & 1 \\ 3 & 4 & 1 & -2 \end{bmatrix} \). Let \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \) be the eigenvalues \( A \).

1. Find the number of positive, negative and zero eigenvalues of \( A \) without computing the eigenvalues of \( A \).
2. Show that \( \lambda_1 \geq 5 \) and \( -2 \geq \lambda_4 \).
3. Show that \( \lambda_1 \geq 3 + \sqrt{8} \) and \( \lambda_2 \geq 3 - \sqrt{8} \). (Use the eigenvalues of the leading \( 2 \times 2 \) matrix and apply eigenvalue characterizations)

Problem 6. Let \( \mathbf{x} = (x_1, \ldots, x_n) \mathbf{T} \in \mathbb{C}^n \). Denote \( \|\mathbf{x}\|_2 := \sqrt{\mathbf{x}^\mathbf{T} \mathbf{x}} \). Let \( A = [a_{ij}]_{i,j=1}^n \in \mathbb{C}^{n \times n} \) and denote by \( r_i := (a_{i1}, \ldots, a_{in}) \mathbf{T} \) the \( i \)-th row of \( A \). Show

1. \( (Ax)_i = r_i^\mathbf{T} \mathbf{x} \), \( |(Ax)_i| \leq \|r_i\| \|\mathbf{x}\| \). 
2. Let \( M \) be the maximum value of \( \|r_i\|, i = 1, \ldots, n \). Show that \( \|Ax\| \leq \sqrt{nM} \|\mathbf{x}\| \).
3. Let \( \lambda \in \mathbb{C} \) be any eigenvalue of \( A \). Show that \( |\lambda| \leq \sqrt{nM} \).
4. Let \( A \) be the matrix given in Problem 5. Show that \( 2\sqrt{46} \geq \lambda_1, \lambda_4 \geq -2\sqrt{46} \). (Apply the inequality in part 3 to \( A \).)