

No books or notes. Show all your work. Write solutions in the exam booklet without copying the problems. **Unjustified** answer yields no credit.

**Problem 1.**

- (a) Display all subsets of  $S = \{a, b, c\}$ ;
- (b) How many subsets has a set with  $n \geq 1$  elements?
- (c) Assume that  $A$  has 7 elements,  $B$  has 5 elements and  $A \cap B$  has 2 elements. How many elements  $A \cup B$  has?

**Problem 2.** In the long run are the following positive or negative? Why?

- (a)  $n^2 - \log n$ ;
- (b)  $n^2 - \frac{n^3}{\log n}$ ,
- (c)  $n^2 - (1.1)^n$ .

**Problem 3.**

- a. Which of the following statements are logically equivalent? Why: (i)  $x = 3$ . (ii)  $x^2 = 9$ . (iii)  $x^3 = 27$ . (iv)  $x$  is the smallest odd prime. (v)  $|x| = 9$ .
- b. State the converse of the following: If a computer program is correct, it will terminate.
- c. Is this statement is correct? Why? "To show that  $ABCD$  is rhombus it is sufficient to show that it is a square".

**Problem 4.**

- a. Apply the Euclid algorithm to find the greatest common divisor of 440 and 946.
- b. Explain why the Euclid algorithm applied to two positive integers  $m, n$  will terminate at most in  $2 \log_2 n + 1$  steps.

**Problem 5.** Consider the game of Towers of Hanoi with 3 poles and  $n$  rings. Assume that a minimal number of steps to transfer  $n$  rings from pole  $i$  to any other pole  $j$  is  $h_n$ .

- a. Describe briefly how to move the rings in a recursive way.
- b. What is the recursive relation between  $h_n$  and  $h_{n-1}$ ?
- c. What is the formula for  $h_n$ ?
- d. Prove by induction this formula.

**Problem 6.**

- a. Consider the round-robin tournament between  $n$  teams where there are no ties. ( $n \geq 2$ .) Show that one can always number the teams  $t_1, t_2, \dots, t_n$  so that  $t_1$  beat  $t_2$ ,  $t_2$  beat  $t_3$ , and so on, through  $t_{n-1}$  beat  $t_n$ .
- b. Consider the sequence  $a_1, a_2, \dots$ , where  $a_n = a_{n-1} + a_{n-2}$  for  $n \geq 3$ . Show
  - (b1) if  $a_1$  and  $a_2$  integers then each  $a_n$  is an integer.
  - (b2) Assume that  $a_1 = 1, a_2 = 2$ . Show that  $a_n \leq 2^{n-1}$ .