

No books or notes. Show all your work. Write solutions in the exam booklet without copying the problems. You can use a result (x) of any part of the problem, to show other part of any problem. **Unjustified** answer yields no credit.

Problem 1. Let $A = \begin{bmatrix} 3 & -1 & 1 \\ 7 & -5 & 1 \\ 6 & -6 & 2 \end{bmatrix}$

- Show that the characteristic and the minimal polynomial of A are equal to $(z-2)^2(z+4)$.
- Find the components of A .
- Find the formula for A^l for any integer l using the components of A .
- Find the general solution for the system $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$.

Problem 2. Suppose that $A \in \mathbb{C}^{n \times n}$ and the minimal polynomial of A is $(z - \alpha)(z - \beta)^2(z - \gamma)^3$ where α, β, γ are three distinct complex numbers.

- Write down the general form of the characteristic polynomial of A .
- What is the condition for A to be power stable, i.e. $\lim_{l \rightarrow \infty} A^l = 0$.
- What is the condition for A to be power convergent, i.e. $\lim_{l \rightarrow \infty} A^l = B$.

Problem 3.

- For $\mathbf{x} = (x_1, x_2)^\top, \mathbf{y} = (y_1, y_2)^\top$, let

$$f(\mathbf{x}, \mathbf{y}) = x_1y_1 - 2x_1y_2 - 2x_2y_2 + kx_2y_2.$$

Find the values of k for which $f(\mathbf{x}, \mathbf{y})$ is an inner product.

- Find a basis of the subspace \mathbf{U} of \mathbb{R}^4 orthogonal to $\mathbf{x}_1 = (1, -2, 3, 4)^\top$ and $\mathbf{x}_2 = (3, -5, 7, 8)^\top$.
- Let \mathbf{V} be an inner product space. Assume that \mathbf{U} is a subspace of \mathbf{V} , with an orthonormal basis $\mathbf{u}_1, \dots, \mathbf{u}_m$. Let $\mathbf{v} \in \mathbf{V}$. Write down the orthogonal projection $P_{\mathbf{U}}(\mathbf{v})$ of \mathbf{v} on \mathbf{U} in terms of \mathbf{v} and $\mathbf{u}_1, \dots, \mathbf{u}_m$ and \mathbf{v} . Show that for any $\mathbf{u} \in \mathbf{U}$ one has the inequality $\|\mathbf{v} - \mathbf{u}\| \geq \|\mathbf{v} - P_{\mathbf{U}}(\mathbf{v})\|$. Characterize the equality case.

Problem 4. Let $A \in \mathbb{R}^{m \times n}$. Denote by $R(A) \subset \mathbb{R}^m$ the column space of A . Assume that $\mathbf{b} \in \mathbb{R}^m$ but $\mathbf{b} \notin R(A)$.

- Show that the system $A\mathbf{x} = \mathbf{b}$ is not solvable.
- Let \mathbf{x}_0 be a least square solution: $A^\top A\mathbf{x}_0 = A^\top \mathbf{b}$. Show that $A\mathbf{x}_0$ is the orthogonal projection of \mathbf{b} on $R(A)$ with respect to the standard inner product in \mathbb{R}^m .
- What is the necessary and sufficient condition for \mathbf{x}_0 to be unique?
- Assume that the columns of A are linearly independent. Explain briefly how do you obtain the QR decomposition of A .
- Under the assumptions of (d) give a simple formula for \mathbf{x}_0 in terms of \mathbf{b}, Q, R .

Problem 5. Let $A = [a_{ij}]_{i,j=1}^n \in \mathbb{R}^{n \times n}$. Assume that $a_{ij} \in (0, 2]$ for $i, j = 1, \dots, n$ and $n \geq 2$. Show that $|\det A| \leq 2^n n^{\frac{n}{2}}$. Can equality hold for some matrix A ?