Show all your work. Write solutions in the exam booklet without copying the problems. You can use a result (x) of any part of the problem, to show other part of any problem. **Unjustified** answer yields no credit.

**Problem 1.**
(a) Find a basis of the subspace \( U \) of \( \mathbb{R}^4 \) orthogonal to \( x_1 = (1, -2, 3, 4)^T \) and \( x_2 = (3, -5, 7, 8)^T \).
(b) Let \( V \) be an inner product space. Assume that \( U \) is a subspace of \( V \), with an orthonormal basis \( u_1, \ldots, u_m \). Let \( v \in V \). Write down the orthogonal projection \( P_U(v) \) of \( v \) on \( U \) in terms of \( v \) and \( u_1, \ldots, u_m \) and \( v \). Show that for any \( u \in U \) one has the inequality \( \|v - u\| \geq \|v - P_U(v)\| \). Characterize the equality case.

**Problem 2.** Let \( A \in \mathbb{C}^{n \times n} \). Denote by \( R(A) \subseteq \mathbb{C}^m \) the column space of \( A \). Assume that \( b \in \mathbb{C}^m \) but \( b \notin R(A) \).
(a) Show that the system \( Ax = b \) is not solvable.
(b) Let \( x_0 \) be a least square solution: \( A^* Ax_0 = A^* b \). Show that \( Ax_0 \) is the orthogonal projection of \( b \) on \( R(A) \) with respect to the standard inner product in \( \mathbb{R}^m \).
(c) What is the necessary and sufficient condition for \( x_0 \) to be unique?
(d) Assume that the columns of \( A \) are linearly independent. Explain briefly how do you obtain the QR decomposition of \( A \).
(f) Under the assumptions of (d) give a simple formula for \( x_0 \) in terms of \( b, Q, R \).

**Problem 3.** Let \( A = [a_{ij}]_{i,j=1}^n \in \mathbb{C}^{n \times n} \). Assume that \( |a_{ij}| \leq 2 \) for \( i, j = 1, \ldots, n \) and \( n \geq 2 \). Show that \( |\det A| \leq 2^n n^2 \). Can equality hold for some matrix \( A \)?

**Problem 4**
(a) Let \( A(a) := \begin{bmatrix} 2 & -a \sqrt{i} \\ -i & 2 \end{bmatrix} \), where \( i = \sqrt{-1} \) and \( a \) is a real number. For which values of \( a \) is \( A \) normal?
(b) Assume that \( A(1) \) is normal. Find a unitary \( U \) and a diagonal \( D \) such that \( A(1) = U D U^* \).
(c) Assume that \( B \in \mathbb{R}^{n \times n} \) is normal. Suppose furthermore that all the eigenvalues of \( B \) are real. Show that \( A \) is symmetric.

**Problem 5.** Let \( A \in \mathbb{C}^{n \times n} \) be a normal matrix.

1. Is \( A \) diagonalizable? (I.e. \( A \) is similar to a diagonal matrix.) **Justify.**
2. Show that for any polynomial \( p(z) \), \( p(A) \) is a normal matrix. **Justify.**
3. Suppose in addition \( A \) has real entries. Are all eigenvalues of \( A \) have to be real? **Justify.**

**Problem 6.** Let \( x = (x_1, \ldots, x_n)^T \in \mathbb{C}^n \). Denote \( \|x\|_2 := \sqrt{x^* x} \). Let \( A = [a_{ij}]_{i,j=1}^n \in \mathbb{C}^{n \times n} \) and denote by \( r_i := (a_{i1}, \ldots, a_{in})^T \) the \( i \)-th row of \( A \). Show
   1. \( (Ax)_i = r_i^T x \), \( \|Ax\|_1 \leq \|r_i\| \|x\| \).
   2. Let \( M \) be the maximum value of \( \|r_i\|, i = 1, \ldots, n \). Show that \( \|Ax\| \leq \sqrt{n} M \|x\| \).
   3. Let \( \lambda \in \mathbb{C} \) be any eigenvalue of \( A \). Show that \( |\lambda| \leq \sqrt{n} M \).

**Problem 7.** Recall that \( A \in \mathbb{R}^{n \times n} \) is called a permutation matrix if each row and each column contains one entry equal to 1 and all other entries are zero.
1. Show that the set of all permutation matrices $\Pi_n \subset \mathbb{R}^{n \times n}$ form a group under the product of matrices.

2. Show that this group is isomorphic to the group of permutations $S_n$ by exhibiting an isomorphism $\phi : \Pi_n \to S_n$.

3. Express $\text{sign}(\phi(P))$ in terms of some function of $P \in \Pi_n$. 