

No books or notes. Show all your work. Write solutions in the exam booklet without copying the problems. You can use a result (x) of any part of the problem, to show other part of any problem. Each problem is worth 25 points. **Unjustified** answer yields no credit.

Problem 1. Let $A \in \mathbb{F}^{n \times n}$ and assume that $A^4 = A^2$.

1. What are all possible eigenvalues of A ?
2. Suppose that A is nonsingular and that $-1 \neq 1$ in \mathbb{F} . Show that A is diagonal.
3. Give an example of a nondiagonal matrix A that satisfies $A^4 = A^2$.

Problem 2. Let $A \in \mathbb{C}^{n \times n}$. Assume that A has only two linearly independent eigenvectors.

1. Show that A has at most two distinct eigenvalues.
2. Assume that $n = 4$. Write down all possible Jordan canonical forms of A .
3. Let $n = 4$ and assume that A^2 is diagonal. What is the Jordan canonical form of A ?

Problem 3. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

- (a) Find the characteristic and the minimal polynomial of A .
- (b) Find the components of A .
- (c) Find the formula for e^A using the components of A .

Problem 4. Let $T : \mathbf{V} \rightarrow \mathbf{V}$ be a linear transformation on a finite dimensional vector space \mathbf{V} over a field \mathbb{F} .

- a. Let \mathbf{U} be a nontrivial subspace of \mathbf{V} , i.e. $0 < \dim \mathbf{U} < \dim \mathbf{V}$. Define the quotient space $\hat{\mathbf{V}} := \mathbf{V}/\mathbf{U}$ and show that it is a vector space over \mathbb{F} of dimension $\dim \mathbf{V} - \dim \mathbf{U}$.
- b. Suppose that \mathbf{U} is T -invariant. Show that T induces a linear transformation $\hat{T} : \hat{\mathbf{V}} \rightarrow \hat{\mathbf{V}}$.
- c. Let $\psi, \hat{\psi}$ be the minimal polynomials of T, \hat{T} respectively. Show that $\hat{\psi}$ divides ψ .

Problem 5. Let T be a linear transformation from a finite dimensional space \mathbf{V} to itself over a field \mathbb{F} .

1. Define a cyclic subspace \mathbf{U} generated by $\mathbf{u} \neq \mathbf{0}$ in \mathbf{V} , (with respect to T).
2. Define the minimal polynomial $\psi_{\mathbf{u}}$ with respect to T .
3. Show that there exists a basis in the above cyclic subspace \mathbf{U} so that T is represented in this basis by a companion matrix.
4. Assume that the minimal polynomial of T is of the form $\phi(z)^l$ where l is a positive integer and $\phi(z)$ is an irreducible polynomial. Show that there exists $\mathbf{u} \in \mathbf{V}$ such that $\psi_{\mathbf{u}} = \phi(z)^l$.

Problem 6. Let $B = [b_{ij}]_{i,j=1}^n \in \mathbb{R}^{n \times n}$ and assume that each entry of B is nonnegative. Assume that there exists a vector $\mathbf{u} = (u_1, \dots, u_n)^\top$ with positive coordinates such that $B\mathbf{u} = \mathbf{u}$.

1. Show that $B^k = [b_{ij}^{(k)}]_{i,j=1}^n$ has nonnegative entries for each positive integer k .

2. Show that $b_{ij}^{(k)} \leq \frac{u_i}{u_j}$ for $i, j = 1, \dots, n$ and $k = 1, 2, \dots$
3. Show that each eigenvalue of B satisfy $|\lambda| \leq 1$.
4. Suppose that λ is an eigenvalue of B and $|\lambda| = 1$. What is the multiplicity of λ in the minimal polynomial of B ?

Problem 7. Let $B = [b_{ij}]_{i,j=1}^n \mathbb{C}^{n \times n}$ be a hermitian matrix. Denote by $A = [b_{ij}]_{i,j=2}^n$ the hermitian matrix obtained from B by deleting the first row and column.

1. Show the Cauchy interlacing inequalities

$$\lambda_i(B) \geq \lambda_i(A) \geq \lambda_{i+1}(B), \text{ for } i = 1, \dots, n-1.$$

2. Show that inequality $\lambda_1(B) + \lambda_n(B) \leq \lambda_1(A) + b_{11}$.

Problem 8. Let $A \in \mathbb{C}^{m \times n}$ and assume that $\text{rank } A = r$. Denote by $\sigma_1(A) \geq \sigma_2(A) \geq \dots \geq 0$ the singular values of A .

1. Show $\sigma_1(A) = \max_{\mathbf{x} \in \mathbb{C}^n, \|\mathbf{x}\|=1} \|\mathbf{Ax}\|$. Here $\|\mathbf{x}\| = \sqrt{\mathbf{x}^* \mathbf{x}}$.
2. Assume that the absolute value of each entry of A is bounded above by a , i.e. $|a_{ij}| \leq a$ for all i, j . Show that $\sigma_1(A) \leq a\sqrt{mn}$. Give an example of A , for which equality holds. **Hint:** Use the Cauchy-Schwarz inequality to estimate $|(\mathbf{Ax})_i|$.
3. Let A^\dagger be the Moore-Penrose inverse of A . Express the positive singular values of A^\dagger in terms of the singular values of A . (**Justify** your formulas!)