Positive diagonal scaling of a nonnegative tensor to one with prescribed slice sums

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Abstract

In this paper we give necessary and sufficient conditions on a nonnegative tensor to be diagonally equivalent to a tensor with prescribed slice sums. For matrices these conditions reduce to Menon's conditions.

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1 Introduction

For a positive integer m let $\langle m\rangle$ be the set $\{1,\ldots,m\}$. For positive integers $d,m_1,\ldots,m_d$ denote by $\mathbb{R}^{m_1\times\ldots\times m_d}$ the linear space d-mode tensors $\mathcal{A}=[a_{i_1,i_2,\ldots,i_d}], i_j\in\langle m_j\rangle, j\in\langle d\rangle$. Note that a 1-mode tensor is a vector, and a 2-mode tensor is a matrix. Assume that $d\ge 2$. For a fixed $i_k\in\langle m_k\rangle$ the $(d-1)$-mode tensor $[a_{i_1,\ldots,i_d}], i_j\in\langle m_j\rangle, j\in\langle d\rangle\backslash\{k\}$ is called the (k,i_k) *slice* of $\mathcal{A}$. For $d=2$ the $(1,i)$ slice and the $(2,j)$ slice are the $i-th$ row and the $j-th$ column of a given matrix. Let

$$s_{k,i_k} := \sum_{i_j\in\langle m_j\rangle, j\in\langle d\rangle\backslash\{k\}} a_{i_1,\ldots,i_d},\ i_k\in\langle m_k\rangle, k\in\langle d\rangle \tag{1.1}$$

be the (k,i_k)-slice sum. Denote

$$\mathbf{s}_k := (s_{k,1},\ldots,s_{k,m_k})^\top,\quad k\in\langle d\rangle \tag{1.2}$$

the k-slice vector sum. Note that (k,i_k)-slice sums satisfy the compatibility conditions

$$\sum_{i_1=1}^{m_1} s_{1,i_1} = \ldots = \sum_{i_d=1}^{m_d} s_{d,i_d}. \tag{1.3}$$

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Two d-mode tensors $\mathcal{A} = [a_{i_1,i_2,\dots,i_d}], \mathcal{B} = [b_{i_1,i_2,\dots,i_d}] \in \mathbb{R}^{m_1\times\dots\times m_d}$ are called *positive diagonally* equivalent if there exist $\mathbf{x}_k = (x_{k,1},\dots,x_{k,m_k})^\top \in \mathbb{R}^{m_k}, k \in \langle d\rangle$ such that $a_{i_1,\dots,i_d} = b_{i_1,\dots,i_d} e^{x_{1,i_1}+\dots+x_{d,i_d}}$ for all $i_j \in \langle m_j\rangle$ and $j \in \langle d\rangle$. Denote by $\mathbb{R}_+^{m_1\times\dots\times m_d}$ the cone of nonnegative,(entrywise), d-mode tensors.

In this paper we assume that $\mathcal{B} = [b_{i_1,i_2,\dots,i_d}] \in \mathbb{R}_+^{m_1\times\dots\times m_d}$ is a given nonnegative tensor with no zero slice (k, i_k). Let $\mathbf{s}_k \in \mathbb{R}_+^{m_k}, k \in \langle d\rangle$ are given k positive vectors satisfying the conditions (1.3). Denote by $\mathbb{R}_+^{m_1\times\dots\times m_d}(\mathcal{B},\mathbf{s}_1,\dots,\mathbf{s}_d)$ the set of all nonnegative $\mathcal{A} = [a_{i_1,i_2,\dots,i_d}] \in \mathbb{R}_+^{m_1\times\dots m_d}$ having the same zero pattern as $\mathcal{B}$, i.e. $a_{i_1,\dots,i_d} = 0 \iff b_{i_1,\dots,i_d} = 0$ for all indices $i_1,\dots,i_d$, and satisfying the condition (1.1). The aim of this paper is to give necessary and sufficient conditions on $\mathcal{B}$ so that $\mathbb{R}_+^{m_1\times\dots m_d}(\mathcal{B},\mathbf{s}_1,\dots,\mathbf{s}_d)$ contains a tensor $\mathcal{A}$, which is positively diagonally equivalent to $\mathcal{B}$. For matrices, i.e. $d = 2$, this problem was solved by Menon [6] and Brualdi [2]. See also [7]. For the special case of positive diagonal equivalence to doubly stochastic matrices see [3] and [8]. Our main result is as follows.

Theorem 1.1 *Let $\mathcal{B} = [b_{i_1,i_2,\dots,i_d}] \in \mathbb{R}_+^{m_1\times\dots\times m_d}$, $(d \ge 2)$, be a given nonnegative tensor with no (k, i_k)-zero slice. Let $\mathbf{s}_k \in \mathbb{R}_+^{m_k}, k = 1,\dots,d$ be given positive vectors satisfying (1.3). Then there exists a nonnegative tensor $\mathcal{A} \in \mathbb{R}_+^{m_1\times\dots\times m_d}$, which is positive diagonally equivalent to $\mathcal{B}$ and having each (k, i_k)-slice sum equal to s_{k,i_k}, if and only the following conditions. The system of the inequalities and equalities for $\mathbf{x}_k = (x_{k,1},\dots,\mathbf{x}_{k,m_k})^\top \in \mathbb{R}^{m_k}, k = 1,\dots,d$*

$$x_{1,i_1} + x_{2,i_2} + \dots + x_{d,i_d} \le 0 \text{ if } b_{i_1,i_2,\dots,i_d} > 0, \tag{1.4}$$
$$\mathbf{s}_k^\top \mathbf{x}_k = 0 \text{ for } k = 1,\dots,d, \tag{1.5}$$

imply one of the following equivalent conditions

1. $x_{1,i_1} + x_{2,i_2} + \dots + x_{d,i_d} = 0$ *if* $b_{i_1,i_2,\dots,i_d} > 0$.
2. $\sum_{b_{i_1,i_2,\dots,i_d}>0} x_{1,i_1} + x_{2,i_2} + \dots + x_{d,i_d} = 0$.

In particular, there exists at most one tensor $\mathcal{A} \in \mathbb{R}_+^{m_1\times\dots\times m_d}$ with (k, i_k)-slice sum s_{k,i_k} for all k, i_k, which is positive diagonally equivalent to $\mathcal{B}$.

The above yields the following corollary.

Corollary 1.2 *Let $\mathcal{B} \in \mathbb{R}_+^{m_1\times\dots\times m_d}$, $(d \ge 2)$, be a given nonnegative tensor with no (k, i_k)-zero slice. Let $\mathbf{s}_k \in \mathbb{R}_+^{m_k}, k = 1,\dots,d$ be given positive vectors. Then there exists a nonnegative tensor $\mathcal{C} \in \mathbb{R}_+^{m_1\times\dots\times m_d}$, which is positive diagonally equivalent to $\mathcal{B}$ and each (k, i_k)-sum slice equal to s_{k,i_k}, if and only if there exists a nonnegative tensor $\mathcal{A} = [a_{i_1,i_2,\dots,i_d}] \in \mathbb{R}_+^{m_1\times\dots\times m_d}$, having the same zero pattern as $\mathcal{B}$, which satisfies (1.1).*

For matrices, i.e. $d = 2$, the above corollary is due Menon [6]. Brualdi in [2] gave a nice and simple characterization for the set of nonnegative matrices, with prescribed zero pattern and with given positive row and column sums, to be not empty. It is an open problem to find an analog of Brualdi's results for d-mode tensors, where $d \ge 3$.

Note that the conditions of Theorem 1.1 are stated as a linear programming problem. Hence the existence of a positive diagonally equivalent tensor $\mathcal{A}$ can be determined in a polynomial time. If such $\mathcal{A}$ exists, we show that it can be found by computing the unique minimal point of certain strictly convex functions f. Hence, Newton method can be applied to find the unique minimal point of f and its value very fast. (See §3.)

2 Proof of the main theorem

Identify $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \ldots \times \mathbb{R}^{m_d}$ with $\mathbb{R}^n$, where $n = \sum_{k=1}^d m_k$. We view $\mathbf{y} \in \mathbb{R}^n$ as a vector $(\mathbf{x}_1^\top, \ldots, \mathbf{x}_d^\top)^\top$, where $\mathbf{x}_k \in \mathbb{R}^{m_k}, k \in \langle d \rangle$. Let $\|\mathbf{y}\| := \sqrt{\mathbf{y}^\top \mathbf{y}}$. Define

$$f(\mathbf{y}) = f((\mathbf{x}_1^\top, \ldots, \mathbf{x}_d^\top)^\top) := \sum_{i_j \in \langle m_j \rangle, j \in \langle d \rangle} b_{i_1,\ldots,i_d} e^{x_{1,i_1} + \ldots + x_{d,i_d}}. \tag{2.1}$$

Clearly, f is a convex function on $\mathbb{R}^n$. (Since $e^{x_{1,i_1}+\ldots+x_{d,i_d}}$ is log-convex it follows that f is a log-convex function.) Denote by $\mathbf{U}(\mathbf{s}_1, \ldots, \mathbf{s}_d) \subset \mathbb{R}^n$ the subspace of vectors $(\mathbf{x}_1^\top, \ldots, \mathbf{x}_d^\top)^\top$ satisfying the equalities (1.5).

Lemma 2.1 *Let $\mathcal{B} = [b_{i_1,i_2,\ldots,i_d}] \in \mathbb{R}_+^{m_1 \times \ldots \times m_d}$, $(d \geq 2)$, be a given nonnegative tensor with no (k, i_k)-zero slice. Let $\mathbf{s}_k \in \mathbb{R}_+^{m_k}, k = 1, \ldots, d$ be given positive vectors satisfying (1.3). Then there exists a nonnegative tensor $\mathcal{A} \in \mathbb{R}_+^{m_1 \times \ldots \times m_d}$, which is positive diagonally equivalent to $\mathcal{B}$ and having each (k, i_k)-slice sum equal to s_{k,i_k}, if and only the restriction of f to the subspace $\mathbf{U}(\mathbf{s}_1, \ldots, \mathbf{s}_d)$, $(f|\mathbf{U}(\mathbf{s}_1, \ldots, \mathbf{s}_d)$, has a critical point.*

Proof. Assume first that $f|\mathbf{U}(\mathbf{s}_1, \ldots, \mathbf{s}_d)$ has a critical point. Use Lagrange multipliers, i.e. consider the function $f - \sum_{k=1}^d \mathbf{s}_k^\top \mathbf{x}_k$, to deduce the existence of $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d)^\top$ and $(\boldsymbol{\xi}_1^\top, \ldots, \boldsymbol{\xi}_d^\top)^\top \in \mathbf{U}(\mathbf{s}_1, \ldots, \mathbf{s}_d)$, where $\boldsymbol{\xi}_k = (\xi_{k,1}, \ldots, \xi_{k,i_k})^\top, k \in \langle d \rangle$, satisfying the following conditions

$$\sum_{i_j \in \langle m_j \rangle, j \in \langle d \rangle \setminus \{k\}} b_{i_1,\ldots,i_d} e^{\xi_{1,i_1} + \ldots + \xi_{d,i_d}} = \lambda_k s_{k,i_k}, \ , \ i_k \in \langle m_k \rangle, k \in \langle d \rangle. \tag{2.2}$$

Since $\mathbf{s}_k >$ is a positive vector and the (k, i_k)-slice of $\mathcal{B}$ is not a zero slice we deduce that $\lambda_k > 0$. Summing up the above equation on $i_k = 1, \ldots, m_k$, and using the equalities (1.3) we deduce that $\lambda_1 = \ldots = \lambda_d > 0$. Then $\mathcal{A} = [b_{i_1,\ldots,i_d} e^{(\xi_{1,i_1} - \log \lambda_1) + \xi_{2,i_2} + \ldots + \xi_{d,i_d}}]$.

Vice versa suppose $\mathcal{A} = [b_{i_1,\ldots,i_d} e^{x_{1,i_1} + \ldots + x_{d,i_d}}]$ has (k, i_k)-slice sum equal to s_{k,i_k} for all (k, i_k). Let $\mathbf{1}_m = (1, \ldots, 1)^\top \in \mathbb{R}^m$. Then there exists a unique $t_i \in \mathbb{R}$ such that $\mathbf{s}_k^\top(\mathbf{x}_k - t_k \mathbf{1}_{m_k}) = 0$ for $k \in \langle d \rangle$. Let $\boldsymbol{\xi}_k := \mathbf{x}_k - t_k \mathbf{s}_k, k \in \langle d \rangle$. Then (2.2) holds. □

Denote by $\mathbf{V}(\mathbf{s}_1, \ldots, \mathbf{s}_d) \subset \mathbf{U}(\mathbf{s}_1, \ldots, \mathbf{s}_d)$ the subspace of all vectors $(\mathbf{x}_1^\top, \ldots, \mathbf{x}_d^\top)^\top$ satisfying the condition *1* of Theorem 1.1. Clearly, for each $\mathbf{y} \in \mathbb{R}^n$ the function f has a constant value $f(\mathbf{y})$ on the affine set $\mathbf{y} + \mathbf{V}(\mathbf{s}_1, \ldots, \mathbf{s}_d)$. Hence, if $\boldsymbol{\eta} \in \mathbf{U}(\mathbf{s}_1, \ldots, \mathbf{s}_d)$ is a critical point of $f|\mathbf{U}(\mathbf{s}_1, \ldots, \mathbf{s}_d)$ then any point in $\boldsymbol{\eta} + \mathbf{V}(\mathbf{s}_1, \ldots, \mathbf{s}_d)$ is also a critical of $f|\mathbf{U}(\mathbf{s}_1, \ldots, \mathbf{s}_d)$. Denote by $\mathbf{V}(\mathbf{s}_1, \ldots, \mathbf{s}_d)^\perp \subset \mathbf{U}(\mathbf{s}_1, \ldots, \mathbf{s}_d)$, the orthogonal complement of $\mathbf{V}(\mathbf{s}_1, \ldots, \mathbf{s}_d)$ in $\mathbf{U}(\mathbf{s}_1, \ldots, \mathbf{s}_d)$. Thus, to study the existence of the critical points of $f|\mathbf{U}(\mathbf{s}_1, \ldots, \mathbf{s}_d)$, it is enough to study the existence of the critical points of $f|\mathbf{V}(\mathbf{s}_1, \ldots, \mathbf{s}_d)^\perp$. Since the function e^{at} is strictly convex for $t \in \mathbb{R}$ for any $a \neq 0$, more precisely $(e^{at})'' = a^2 e^{at} > 0$ we deduce the following.

Lemma 2.2 *Let $\mathcal{B} = [b_{i_1,i_2,\ldots,i_d}] \in \mathbb{R}_+^{m_1\times\ldots\times m_d}$, $(d \geq 2)$, be a given nonnegative tensor with no (k, i_k)-zero slice. Let $\mathbf{s}_k \in \mathbb{R}_+^{m_k}, k = 1, \ldots, d$ be given positive vectors satisfying (1.3). Let $\mathbf{U}(\mathbf{s}_1, \ldots, \mathbf{s}_d), \mathbf{V}(\mathbf{s}_1, \ldots, \mathbf{s}_d), \mathbf{V}(\mathbf{s}_1, \ldots, \mathbf{s}_d)^\perp$ be defined as above. Then $f|\mathbf{V}(\mathbf{s}_1, \ldots, \mathbf{s}_d)^\perp$ is strictly convex. More precisely, the Hessian matrix of $f|\mathbf{V}(\mathbf{s}_1, \ldots, \mathbf{s}_d)^\perp$ has positive eigenvalues at each point of $\mathbf{V}(\mathbf{s}_1, \ldots, \mathbf{s}_d)^\perp$.*

Theorem 2.3 *Let $\mathcal{B} = [b_{i_1,i_2,\ldots,i_d}] \in \mathbb{R}_+^{m_1\times\ldots\times m_d}$, $(d \geq 2)$, be a given nonnegative tensor with no (k, i_k)-zero slice. Let $\mathbf{s}_k \in \mathbb{R}_+^{m_k}, k = 1, \ldots, d$ be given positive vectors satisfying (1.3). Then the following conditions are equivalent.*

1. *$f|\mathbf{U}(\mathbf{s}_1, \ldots, \mathbf{s}_d)$ has a global minimum.*
2. *$f|\mathbf{U}(\mathbf{s}_1, \ldots, \mathbf{s}_d)$ has a critical point.*
3. *$\lim f(\mathbf{y}_l) = \infty$ for any sequence $\mathbf{y}_l \in \mathbf{V}(\mathbf{s}_1, \ldots, \mathbf{s}_k)^\perp$ such that $\lim \|\mathbf{y}_l\| = \infty$.*
4. *The only $\mathbf{y} = (\mathbf{x}_1^\top, \ldots, \mathbf{x}_d^\top)^\top \in \mathbf{V}(\mathbf{s}_1, \ldots, \mathbf{s}_k)^\perp$ that satisfies (1.4) is $\mathbf{y} = \mathbf{0}_n$.*

Proof. $1 \Rightarrow 2$. Trivial.
$2 \Rightarrow 3$. Let $\boldsymbol{\beta} \in \mathbf{U}(\mathbf{s}_1, \ldots, \mathbf{s}_d)$ be a critical point of $f|\mathbf{U}(\mathbf{s}_1, \ldots, \mathbf{s}_d)$. Hence any point in $\boldsymbol{\beta} + \mathbf{V}(\mathbf{s}_1, \ldots, \mathbf{s}_d)$ is a critical point of $f|\mathbf{U}(\mathbf{s}_1, \ldots, \mathbf{s}_d)$. Hence $f|\mathbf{U}(\mathbf{s}_1, \ldots, \mathbf{s}_d)$ has a critical point $\boldsymbol{\xi} \in \mathbf{V}(\mathbf{s}_1, \ldots, \mathbf{s}_d)^\perp$. In particular, $\boldsymbol{\xi}$ is a critical point of $f|\mathbf{V}(\mathbf{s}_1, \ldots, \mathbf{s}_d)^\perp$. Let $\mathbf{z} \in \mathbf{V}(\mathbf{s}_1, \ldots, \mathbf{s}_d)^\perp, \|\mathbf{z}\| = 1$. For $t \in \mathbb{R}$ define $g_{\mathbf{z}}(t) := f(\boldsymbol{\xi} + t\mathbf{z})$. So $g_{\mathbf{z}}$ is strictly convex on $\mathbb{R}$ and $g'_{\mathbf{z}}(0) = 0$. Let $\mathrm{H}(f)(\mathbf{y})$ be the Hessian matrix of $f|\mathbf{V}(\mathbf{s}_1, \ldots, \mathbf{s}_d)^\perp$ at $\mathbf{y} \in \mathbf{V}(\mathbf{s}_1, \ldots, \mathbf{s}_d)^\perp$, i.e. the symmetric matrix of the second derivatives of $f|\mathbf{V}(\mathbf{s}_1, \ldots, \mathbf{s}_d)^\perp$ at $\mathbf{y} \in \mathbf{V}(\mathbf{s}_1, \ldots, \mathbf{s}_d)^\perp$. Lemma 2.2 implies that the smallest eigenvalue $\alpha(\mathbf{y})$ of $\mathrm{H}(f)(\mathbf{y})$ is positive. Clearly, $\mathrm{H}(\mathbf{y})$ and hence $\alpha(\mathbf{y})$ are continuous on $\mathbf{V}(\mathbf{s}_1, \ldots, \mathbf{s}_d)^\perp$. Hence $\min_{\|\mathbf{y}-\boldsymbol{\xi}\|\leq 1} \alpha(\mathbf{y}) = 2a > 0$. Therefore, $g''_{\mathbf{z}}(t) \geq a$ for $t \in [-1, 1]$. In particular $g'_{\mathbf{z}}(t) \geq 2at$ and $g_{\mathbf{z}}(t) \geq f(\boldsymbol{\xi}) + at^2$ for any $t \in [0, 1]$. So $g_{\mathbf{z}}(1) \geq f(\boldsymbol{\xi}) + a$. Since $g'_{\mathbf{z}}(t)$ increases on $\mathbb{R}$ it follows that $g'_{\mathbf{z}}(t) \geq 2a$ for $t \geq 1$. Hence $g_{\mathbf{z}}(t) \geq f(\boldsymbol{\xi}) + a + 2a(t-1) = f(\boldsymbol{\xi}) + a(2t-1)$ for $t \geq 1$. Thus $f(\boldsymbol{\xi} + \mathbf{u}) \geq a(2\|u\| - 1)$ for any $\mathbf{u} \in \mathbf{V}(\mathbf{s}_1, \ldots, \mathbf{s}_d), \|\mathbf{u}\| \geq 1$. Hence 3 holds.
$3 \Rightarrow 1$. Since $f = \infty$ on $\partial\mathbf{V}(\mathbf{s}_1, \ldots, \mathbf{s}_d)^\perp$ it follows that $f|\mathbf{V}(\mathbf{s}_1, \ldots, \mathbf{s}_d)^\perp$ achieves its minimum at $\boldsymbol{\xi} \in \mathbf{V}(\mathbf{s}_1, \ldots, \mathbf{s}_d)^\perp$. Clearly, for any point $\mathbf{y} \in \mathbf{U}(\mathbf{s}_1, \ldots, \mathbf{s}_d)$ there exits $\mathbf{z} \in \mathbf{V}(\mathbf{s}_1, \ldots, \mathbf{s}_d)^\perp$ such that $\mathbf{y} \in \mathbf{z} + \mathbf{V}(\mathbf{s}_1, \ldots, \mathbf{s}_d)$. Recall that $f(\mathbf{y}) = f(\mathbf{z}) \geq f(\boldsymbol{\xi})$. Hence $f(\boldsymbol{\xi})$ is the minimum of $f|\mathbf{U}(\mathbf{s}_1, \ldots, \mathbf{s}_d)$.
$3 \Rightarrow 4$. Assume to the contrary that there exists $\mathbf{0} \neq \mathbf{y} = (\mathbf{x}_1^\top, \ldots, \mathbf{x}_d^\top)^\top \in \mathbf{V}(\mathbf{s}_1, \ldots, \mathbf{s}_d)^\perp$ which satisfies (1.4). Hence, there exists $i_j \in \langle m_j \rangle, j \in \langle d \rangle$ such that $b_{i_1,\ldots,i_d} > 0$ and $x_{1,i_1} + \ldots + x_{d,i_d} < 0$. Thus, there exist $\alpha_1, \ldots, \alpha_p < 0$ and $\beta_1, \ldots, \beta_l > 0$ such that $f(t\mathbf{y}) = \gamma + \sum_{l=1}^p \beta_l e^{t\alpha_l}$. (Each α_q is equal to some $x_{1,i_1} + \ldots + x_{d,i_d} < 0$, where $b_{i_1,\ldots,i_d} > 0$, and each β_q is a sum of corresponding $b_{i_1,\ldots,i_d} > 0$.) Hence, $\lim_{t\to\infty} f(t\mathbf{y}) = \gamma$, which contradicts 3.
$4 \Rightarrow 3$. Let $\mathbf{y} \in (\mathbf{x}_1^\top, \ldots, \mathbf{x}_d^\top)^\top \in \mathbf{V}(\mathbf{s}_1, \ldots, \mathbf{s}_d)^\perp, \|\mathbf{y}\| = 1$. Then

$$h(\mathbf{y}) := \max_{b_{i_1,\ldots,i_d}>0} x_{1,i_1} + \ldots + x_{d,i_d} > 0.$$

The continuity of $h(\mathbf{y})$ on the unit sphere in $\mathbf{V}(\mathbf{s}_1, \ldots, \mathbf{s}_d)^\perp$ implies that

$$\min_{\mathbf{y}\in\mathbf{V}(\mathbf{s}_1,\ldots,\mathbf{s}_d)^\perp, \|\mathbf{y}\|=1} h(\mathbf{y}) = \alpha > 0.$$

Let $\beta = \min_{b_{i_1,\ldots,i_d}>0} b_{i_1,\ldots,i_d} > 0$. Hence, for any $\mathbf{y} \in (\mathbf{x}_1^\top, \ldots, \mathbf{x}_d^\top)^\top \in \mathbf{V}(\mathbf{s}_1, \ldots, \mathbf{s}_d)^\perp$, $\|\mathbf{y}\| = 1$ and $t > 0$ we have that $f(t\mathbf{y}) \geq \beta e^{\alpha t}$. This inequality yields 3. □

Proof of Theorem 1.1. Assume first that there exists a nonnegative tensor $\mathcal{A} \in \mathbb{R}_+^{m_1 \times \ldots \times m_d}$, which is positive diagonally equivalent to $\mathcal{B}$ and having each (k, i_k)-slice sum equal to s_{k,i_k}. Lemma 2.1 yields that $f|\mathbf{U}(\mathbf{s}_1, \ldots, \mathbf{s}_d)$ has a critical point, i.e. the condition *2* of Theorem 2.3 holds. Since $f|\mathbf{V}(\mathbf{s}_1, \ldots, \mathbf{s}_d)^\perp$ is strictly convex, it has a unique critical point $\boldsymbol{\xi} \in \mathbf{V}(\mathbf{s}_1, \ldots, \mathbf{s}_d)^\perp$. Hence all critical points of a convex $f|\mathbf{U}(\mathbf{s}_1, \ldots, \mathbf{s}_k)$ must be of the form $\boldsymbol{\xi} + \mathbf{V}(\mathbf{s}_1, \ldots, \mathbf{s}_d)$. The proof of Lemma 2.1 yields that $\mathcal{A}$ is unique.

Theorem 2.3 implies the condition *4*. Hence the conditions (1.4) and (1.5) yield the conditions *1* and *2* of Theorem 1.1.

Assume that the conditions (1.4) and (1.5) hold. Clearly the conditions *1* and *2* of Theorem 1.1 are equivalent. Suppose now that the conditions (1.4) and (1.5) imply the condition *1* of Theorem 1.1. Hence the condition *4* of Theorem 2.3 holds. Use the the condition *2* of Theorem 2.3 and Lemma 2.1 to deduce the existence of a nonnegative tensor $\mathcal{A} \in \mathbb{R}_+^{m_1 \times \ldots \times m_d}$, which is positive diagonally equivalent to $\mathcal{B}$ and having each (k, i_k)-slice sum equal to s_{k,i_k}.

□

Proof of Corollary 1.2. We prove the nontrivial part of the corollary. Suppose that there exists a nonnegative tensor $\mathcal{A} = [a_{i_1,i_2,\ldots,i_d}] \in \mathbb{R}_+^{m_1 \times \ldots \times m_d}$, having the same zero pattern as $\mathcal{B}$, which satisfies (1.1). Clearly, $\mathcal{A}$ is positively diagonally equivalent to $\mathcal{A}$ and has each (k, i_k)-sum slice equal to s_{k,i_k}. Apply Theorem 1.1 to $\mathcal{A}$ to deduce that the set of inequalities $x_{1,i_1} + x_{2,i_2} + \ldots + x_{d,i_d} \leq 0$ if $a_{i_1,i_2,\ldots,i_d} > 0$, together with the equalities (1.5) yields the condition $\sum_{a_{i_1,i_2,\ldots,i_d}>0} x_{1,i_1} + x_{2,i_2} + \ldots + x_{d,i_d} = 0$. Since $a_{i_1,\ldots,i_d} > 0 \iff b_{i_1,\ldots,i_d} > 0$ we deduce that the conditions (1.4) and (1.5) of Theorem 1.1 yield the condition *2* of Theorem 1.1. Hence there exists a nonnegative tensor $\mathcal{C} \in \mathbb{R}_+^{m_1 \times \ldots \times m_d}$, which is positive diagonally equivalent to $\mathcal{B}$ and has (k, i_k)-sum slices equal to s_{k,i_k}. □

3 Remarks

As we observed in the Introduction, the necessary and sufficient conditions for the existence of a nonnegative tensor $\mathcal{A} \in \mathbb{R}_+^{m_1 \times \ldots \times m_d}$, which is positive diagonally equivalent to $\mathcal{B}$ and has (k, i_k)-sum slices equal to s_{k,i_k}, given by Theorem 1.1 are stated in terms of linear programming. Hence by the results of [5, 4] one can verify this condition in a polynomial time. The proof of Theorem 2.3, combined Lemma 2.1, shows that to find $\mathcal{A}$ we need to find the minimum of the strict convex function $f|\mathbf{V}(\mathbf{s}_1, \ldots, \mathbf{s}_d)^\perp$. There are many numerical methods to to find the unique minimum, e.g. [1]. It is clear, that since the Hessian at the critical point has positive eigenvalues, the best strategy to switch to the Newton method when one is close enough to the minimum, to obtain the quadratic convergence.

In the special case of diagonal equivalence to doubly stochastic matrices, one can performs the Sikhorn algorithm [8], which converges linearly. Hence, after a several iterations of the Sinkhorn algorithm it would be better to switch to the Newton method, which converges quadratically.

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