REVISITING THE SIEGEL UPPER HALF PLANE I

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Abstract

In the first part of the paper we show that the Busemann 1-compactification of the Siegel upper half plane of rank n: $\mathbf{SH}_n = \mathbf{Sp}(n, \mathbb{R})/\mathbf{K}_n$ is the compactification as a bounded domain. In the second part of the paper we study certain properties of discrete groups Γ of biholomorphisms of $\mathbf{SH}_n$. We show that the set of accumulation points of the orbit $\Gamma(Z)$ on the Shilov boundary of $\mathbf{SH}_n$ is independent of Z, and denote this set by $\Lambda(\Gamma)$. We associate with Γ the standard class of Patterson-Sullivan p-measures. For p-regular Γ these measures are supported on $\Lambda(\Gamma)$. For 1-regular Γ Patterson-Sullivan 1-measures are conformal densities. For Γ, with $\Lambda(\Gamma) \neq \emptyset$, we give a modified version of the class of Patterson-Sullivan measures, which are always supported on $\Lambda(\Gamma)$. [1]

1 Introduction

In the past thirty years there have been a great deal of mathematical activity on Fuchsian and Kleinian groups. One of the most important notions is the Patterson-Sullivan (PS) measures. This class of conformal measures was introduced by Patterson [**?**] for Fuchsian groups. The construction of conformal measures were extended by Sullivan [**?**] to hyperbolic groups acting on n-dimensional hyperbolic spaces $\mathbf{H}^n$ in general and in particular for Kleinian groups ($n = 3$). A good summary of the ideas and results on this subject are in [**?**]. One way to extend these results is to consider PS measures for discrete groups in higher rank symmetric spaces G/K as was done by Albuquerque [**?**]. In that case the boundary of G/K

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consists of several strata. The important stratum is the Furstenberg boundary [**?**]. Using recent results of Benoist [**?**], Albuquerque shows that there are families of Zariski dense groups for which the PS measures are supported on the Furstenberg boundary.

In his fundamental paper [**?**] Siegel introduced a special symmetric space $\mathbf{SH}_n$ of rank n for $n = 1, ...,$, which is called now the $(n - th)$ Siegel upper half plane. $\mathbf{SH}_1$ is the hyperbolic upper half plane $\mathbf{H}^2$. $\mathbf{SH}_n$ is formally defined as the subset of $n \times n$ complex symmetric matrices $\mathbf{Sym}(n, \mathbb{C})$ whose imaginary part is a positive definite matrix. In fact, the origin of $\mathbf{SH}_n$ can be traced to Riemann, who defined the Riemann matrix $A \in \mathbf{SH}_n$ corresponding to a compact Riemann surface of genus n, endowed with a specific complex structure. $\mathbf{SH}_n$ is the homogeneous space corresponding to the symplectic group $\mathbf{Sp}(n, \mathbb{R}) \subset \mathbf{SL}(2n, \mathbb{R})$ quotient by the maximal compact subgroup $\mathbf{K}_n := \mathbf{Sp}(n, \mathbb{R}) \cap \mathbf{SO}(2n, \mathbb{R})$. $\mathbf{SH}_n$ is a complex manifold of complex dimension $\frac{n(n+1)}{2}$. $\mathbf{Sp}(n, \mathbb{R})$ is the biholomorphism group of $\mathbf{SH}_n$. Of special interest is the lattice $\mathbf{Sp}(n, \mathbb{Z})$, which is called the Siegel modular group. Siegel upper half plane and Siegel modular group have many applications to modular forms [**?**]. The natural compactification of $\mathbf{SH}_n$ is the compactification as a bounded domain $\mathbf{SD}_n := \{Z \in \mathbf{Sym}(n, \mathbb{C}) : \ ||Z||_2 < 1\}$. We show that this compactification is equivalent to Busemann 1-compactification of $\mathbf{Sp}(n, \mathbb{R})/\mathbf{K}_n$ viewed as a submanifold of $\mathbf{GL}(2n, \mathbb{C})/\mathbf{U}_{2n}$ with respect to the metric d_1 introduced in [**?**] Recall that $\mathbf{SD}_n$ is biholomorphic to $\mathbf{SH}_n$. The compact strata of the boundary of $\mathbf{SD}_n$ is the Shilov boundary of $\mathbf{SD}_n$. It is the set of $n \times n$ unitary symmetric matrices $\mathbf{USym}(n)$, which is a manifold of real dimension $\frac{n(n+1)}{2}$. $\mathbf{USym}(2)$ fibers over the circle with the fibre S^2.

The object of this paper is to study certain problems for a discrete groups $\Gamma \subset \mathbf{Sp}(n, \mathbb{R})$: the appropriate definitions of the limit set of Γ and the appropriate constructions of the PS measures. These problems are closely related to $\frac{n(n+1)}{2}$ dimensional complex manifolds whose universal cover is $\mathbf{SH}_n$. As we show, there are many common features of discrete groups $\Gamma \subset \mathbf{Sp}(n, \mathbb{R})$ for $n > 1$ with the classical Fuchsian groups ($n = 1$). Of course there are still many differences with Fuchsian groups, and more generally with discrete subgroups in rank one symmetric spaces. It will be apparent to the reader that the discrete groups $\Gamma \subset \mathbf{Sp}(n, \mathbb{R})$ posses remarkable properties, some of which we were able to expose. The most promising case is $n = 2$. Here a discrete group $\Gamma \subset \mathbf{Sp}(2, \mathbb{R})$ acts on $\mathbf{USym}(2)$, which seems to be a natural generalization of the action of the Kleinian group on the Riemann sphere. That is why we study in detail various compactifications of $\mathbf{SH}_2$ and the action of a single element $\gamma \in \mathbf{Sp}(2, \mathbb{R})$ on $\mathbf{SH}_2$ in [**?**]. There is an overlap between some of our results on the fixed points of the action of elements $M \in \mathbf{Sp}(n, \mathbb{R})$ on $\mathbf{SH}_n$ and the forms of special representatives of the conjugacy class of M in $\mathbf{Sp}(n, \mathbb{R})$, and the two papers of Gottschling [**?**] and [**?**]. In his papers Gottschling considered only the fixed points of $M \in \mathbf{Sp}(n, \mathbb{Z})$ and the forms of special representative of the conjugacy class M in $\mathbf{Sp}(n, \mathbb{Z})$. He was not concerned with the exact location of the fixed points with respect to the stratification of $\partial\mathbf{SD}_n$.

We now outline briefly the main results of our paper. In §2 we recall the known models of $\mathbf{SH}_n$. In §3 we consider the compactifications of these models as bounded domains. In §4 we discuss the properties of symplectic matrices needed here. In §5 we discuss discrete subgroups $\Gamma \subset \mathbf{Sp}(n, \mathbb{R})$ which act on $\mathbf{SH}_n$. We define the limit set $\Lambda(\Gamma)$ as the set of accumulation points of the orbit $\Gamma(Z)$ on the Shilov boundary for some $Z \in \mathbf{SH}_n$. Our main result is that $\Lambda(\Gamma)$ is independent of the choice of $Z \in \mathbf{SH}_n$, as in the classical case of Fuchsian and Kleinian groups. In §6 we define the notion of the Patterson-Sullivan measure. We give conditions for this measure to be supported on the Shilov boundary of $\mathbf{SH}_n$. In §7 we discuss briefly a modified definition of the Patterson-Sullivan measure for discrete groups Γ satisfying $\Lambda(\Gamma) \neq \emptyset$. In §8 we discuss briefly the notion of the critical exponent.

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2 The Siegel upper half plane

We recall the well known facts about $\mathbf{Sp}(n,\mathbb{R}), \mathbf{SH}_n, \mathbf{SD}_n$ which can be found in [?], [?],[?] and [?]. We use the notations of [?] given in the §1 and §3. For $\mathbb{F} = \mathbb{R}, \mathbb{C}$ denote by $\mathbf{Sym}(n, F) \subset \mathbf{M}(n, F)$ the subspace of $n \times n$ symmetric matrices. The *symplectic group* $\mathbf{Sp}(n, F)$ is defined as

$$\mathbf{Sp}(n,\mathbb{F}) := \{M \in \mathbf{GL}(2n, F) : \ M^{\mathrm{T}} J_n M = J_n\}, \ J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in \mathbf{SL}(2n,\mathbb{R}).$$

Equivalently

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbf{Sp}(n, F) \iff M^{-1} = \begin{pmatrix} D^{\mathrm{T}} & -B^{\mathrm{T}} \\ -C^{\mathrm{T}} & A^{\mathrm{T}} \end{pmatrix} \iff$$
$$A^{\mathrm{T}} C \text{ and } B^{\mathrm{T}} D \text{ are symmetric and } A^{\mathrm{T}} D - C^{\mathrm{T}} B = I_n. \tag{2.1}$$

Let $\mathbf{Y}_n := \mathbf{Sp}(n, \mathbf{R})/\mathbf{K}_n$. Then $\mathbf{Y}_n$ is a submanifold of $\mathbf{X}_{2n} = \mathbf{GL}(2n, \mathbb{C})/\mathbf{U}_{2n}$. Furthermore $\mathbf{Y}_n$ is a complete metric space with respect to d_p for $p \in [1, \infty]$ [?]. Recall that $\mathbf{Sp}(n,\mathbb{R})$ acts on $\mathbf{SH}_n$ as follows: For M of the form given in (??) $M(Z) := (AZ + B)(CZ + D)^{-1}$. We will call these maps *generalized Möbius transformations*. The matrices M and $-M$ have the same action on $\mathbf{SH}_n$. Then $\mathbf{PSp}(n,\mathbb{R}) := \mathbf{Sp}(n,\mathbb{R})/\{\pm I_{2n}\}$ is equal to the group of biholomorphisms of $\mathbf{SH}_n$. Furthermore, $\mathbf{PSp}(n,\mathbb{R})$ acts as a subgroup of isometries with respect to the Siegel metric $ds(\cdot,\cdot)$ on $\mathbf{SH}_n$.

It is straightforward to show that the map $\phi_1 : \mathbf{SH}_n \to \mathbf{Sp}(n,\mathbb{R})$ given by

$$\phi_1(X + \sqrt{-1}Y) := \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \begin{pmatrix} \sqrt{Y} & 0 \\ 0 & \sqrt{Y^{-1}} \end{pmatrix} = \begin{pmatrix} \sqrt{Y} & X\sqrt{Y^{-1}} \\ 0 & \sqrt{Y^{-1}} \end{pmatrix}, \tag{2.2}$$

Induces a bijection $\Phi_1 \ : \mathbf{SH}_n \to \mathbf{Sp}(n,\mathbb{R})/\mathbf{K}_n$ given by $Z \mapsto \phi_1(Z)\mathbf{K}_n$. It is not difficult to show that $ds(Z,W) = \sqrt{2} d_2(\Phi_1(Z), \Phi_1(B))$ for any $Z, W \in \mathbf{SH}_n$.

The next model for $\mathbf{SH}_n$ is the $\mathbf{SD}_n$, which a bounded domain in $\mathbf{Sym}(n,\mathbb{C})$. There are two complex symplectic maps connecting these two models:

$$\begin{aligned} \Phi_2 \ &: \mathbf{SH}_n \to \mathbf{SD}_n, \quad Z \mapsto (Z - \sqrt{-1}I_n)(Z + \sqrt{-1}I_n)^{-1}, \\ \Phi_2^{-1} \ &: \mathbf{SD}_n \to \mathbf{SH}_n, \quad Z \mapsto \sqrt{-1}(I_n + Z)(I_n - Z)^{-1}. \end{aligned}$$

Let

$$\mathbf{SU}(n,n) := \{M \in \mathbf{SL}(2n,\mathbb{C}) : \quad M^* \operatorname{diag}(I_n, -I_n) M = \operatorname{diag}(I_n, -I_n)\}.$$

Then all biholomorphisms of $\mathbf{SD}_n$ are given by generalized Möbius transformation induced by the subgroup $\mathbf{Sp}(n,\mathbb{R})' := \mathbf{Sp}(n,\mathbb{C}) \cap \mathbf{SU}(n,n)$. Let $\operatorname{Stab}(0) := \{M \in \mathbf{Sp}(n,\mathbb{R})' : M(0) = 0\}$. Then $\operatorname{Stab}(0)$ is isomorphic to $\mathbf{U}_n$:

$$M \in \operatorname{Stab}(0) \quad \Longleftrightarrow \quad M(Z) = UZU^{\mathrm{T}}, \quad U \in \mathbf{U}_n, \quad Z \in \mathbf{SD}_n. \tag{2.3}$$

The classical result of Schur [**?**] (see also [**?**]) states:

Lemma 2.1 *Let $Z \in \mathbf{Sym}(n,\mathbb{C})$. Then there exists a unitary $U \in \mathbf{U}_n$ so that $Z = U\Sigma(Z)U^{\mathrm{T}}$.*

Corollary 2.2 *Let $W_1, W_2 \in \mathbf{SD}_n$. Then there exists $M' \in \mathbf{Sp}(n,\mathbb{R})'$ such that*

$$M'(W_1) = 0, \quad M'(W_2) = D(\mathbf{y}), \quad y = (y_1, ..., y_n)^{\mathrm{T}} \in \mathbb{R}^n, \quad 1 > y_1 \geq \ldots \geq y_n \geq 0.$$

Suppose furthermore that $Z_1, Z_2 \in \mathbf{SH}_n$. Then there exists $M \in \mathbf{Sp}(n,\mathbb{R})$ such that $M(Z_1) = \sqrt{-1}I_n$, $M(Z_2) = \sqrt{-1}D(\mathbf{x}))$, where $\mathbf{x} = (x_1, ..., x_n)^{\mathrm{T}}$, $x_1 \geq \ldots \geq x_n \geq 1$.

We now consider the projective model of $\mathbf{SH}_n$. Consider the Grassmannian $\mathbf{Gr}(2n, n, \mathbb{F}$, which is the variety of all n-dimensional subspaces of $\mathbb{F}^{2n}$. Denote by $\mathbf{M}_f(2n,n,\mathbb{F}) \subset \mathbf{M}(2n,n,\mathbb{F})$ the subset all $2n \times n$ matrices of maximal rank n. Let $A \in \mathbf{M}_f(2n,n,\mathbb{F})$ and view the columns of A as a basis of a subspace of $\mathbb{F}^{2n}$. Denote by $[A]$ the n-dimensional subspace spanned by the columns of A. Note $[B] = [A]$ if and only if $B \in A\mathbf{GL}(n,\mathbb{F})$. Hence $\mathbf{Gr}(2n,n,\mathbb{F}) = \mathbf{M}_f(2n,n,\mathbb{F})/\mathbf{GL}(n,F)$. Let $S^o(2n,n,\mathbb{F})$ be the following quasiprojective variety in $\mathbf{Gr}(2n,n,\mathbb{F})$: $S^o(2n,n,\mathbb{F}) := \{[A] : \quad A = \begin{pmatrix} Z \\ I_n \end{pmatrix}, \quad Z \in \mathbf{Sym}(n,F).\}$ The projective model for $\mathbf{SH}_n$ is the set of all n dimensional subspaces of $S^o(2n,n,\mathbb{C})$ that admit a representative a matrix A of the above type with $Z \in \mathbf{SH}_n$. We denote this set by $\mathbf{SPH}_n$. This embedding intertwines the action of $\mathbf{Sp}(n,\mathbb{R})$ on $\mathbf{SH}_n$ with the restriction to the subgroup of $\mathbf{Sp}(n,\mathbb{R})$ of the standard action of $\mathbf{GL}(2n,\mathbb{C})$ on $\mathbf{Gr}(2n,n,\mathbb{C})$:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{bmatrix} Z \\ I_n \end{bmatrix} = \begin{bmatrix} AZ+B \\ CZ+D \end{bmatrix} = \begin{bmatrix} (AZ+B)(CZ+D)^{-1} \\ I_n \end{bmatrix}.$$

The map connecting $\mathbf{SH}_n$ to $\mathbf{SPH}_n$ is $\Phi_3 \; : \mathbf{SH}_n \to \mathbf{SPH}_n$ given by $Z \mapsto \begin{bmatrix} Z \\ I_n \end{bmatrix}$. which is a 1-1 map. This model and the action are studied in a more general setting in [**?**].

Finally consider another related projective model. Let $\wedge_n \mathbf{SPH}_n := \{\wedge_n W : [W] \in \mathbf{SPH}_n\}$, where we identify $v = u$ if and only if there exists a nonzero complex number z such that $v = uz$. This is a subset of the projective space $\mathbb{CP}^{N-1}$, $N = \binom{2n}{n}$. The action is defined as left multiplication by $\wedge_n M$: for $M \in \mathbf{Sp}(n,\mathbb{R})$ and $v \in \wedge_n \mathbf{SPH}_n$, the action is $[v] \mapsto [\wedge_n M v]$. It is straightforward to show that one obtains a well defined map from $\mathbf{SPH}_n$ to $\wedge_n \mathbf{SPH}_n$ given by $[V] \mapsto [\wedge_n V]$.

3 Compactifications

The closure of $\mathbf{SD}_n$ is $\{Z \in \mathbf{Sym}(n,\mathbb{C}) : ||Z||_2 \le 1\}$ which is denoted by $\mathrm{Cl}(\mathbf{SD}_n)$. Observe that $\partial\mathbf{SD}_n$ has the following stratification :$\partial_k\mathbf{SD}_n = \{Z \in \partial\mathbf{SD}_n : \text{ rank } (I - Z\overline{Z}) = n-k\}$ for $k = 1,...,n$. Note that $\partial_k\mathbf{SD}_n = \{Z \in \mathbf{Sym}(n,\mathbb{C}) : \sigma_1(Z) = \ldots = \sigma_k(Z) = 1 > \sigma_{k+1}(Z)\}$ for $k \le n-1$, and $\partial_n\mathbf{SD}_n = \mathbf{USym}(n) := \mathbf{U}_n \cap \mathbf{Sym}(n,\mathbb{C})$. The group acting on $\mathbf{SD}_n$ is $\mathbf{Sp}(n,\mathbb{R})'$. The quotient of this group by the subgroup $\{\pm I_{2n}\}$ is the biholomorphism group of $\mathbf{SD}_n$. The action of $\mathbf{Sp}(n,\mathbb{R})'$ extends to $\mathrm{Cl}(\mathbf{SD}_n)$. It is known that each stratum of $\partial\mathbf{SD}_n$ is an orbit for the action of $\mathbf{Sp}(n,\mathbb{R})'$ [**?**, p. 200].

It is useful to consider a similar compactification $\mathbf{SH}_n$. For $A \in \mathbf{M}(m,n,\mathbb{C})$ let $\mathrm{Re}\, A, \mathrm{Im}\, A \in \mathbf{M}(m,n,\mathbb{R})$ be the unique matrices such that $A = \mathrm{Re}\, A + \sqrt{-1}\, \mathrm{Im}\, A$. For $A, B \in \mathbf{H}_n$ $A \ge B \iff A - B \in \mathbf{H}_n^+$. Let $\mathrm{Cl}(\mathbf{SH}_n)$ be the closure of the Siegel upper half plane in $\mathbf{Sym}(n,\mathbb{C})$, which consists of all $Z \in \mathbf{Sym}(n,\mathbb{C})$, $\mathrm{Im}\, Z \ge 0$. Call the boundary of $\mathrm{Cl}(\mathbf{SH}_n)$ as a *finite* boundary of $\mathbf{SH}_n$ and denote it by fin, which consists of all $Z \in \mathbf{Sym}(n,\mathbb{C})$ such that $\mathrm{Im}\, Z \ge 0$ and rank $\mathrm{Im}\, Z < n$. Clearly, we have the following stratification of the finite boundary: $\mathrm{fin}(\partial_k\mathbf{SH}_n)$ consists of all $Z \in \mathbf{Sym}(n,\mathbb{C})$ such that $\mathrm{Im}\, Z \ge 0$ and rank $\mathrm{Im}\, Z = n-k$ for $k = 1,...,n$. A straightforward calculation [**?**] shows: $\Phi_2(\mathrm{fin}(\partial_k\mathbf{SH}_n)) \subset \partial_k\mathbf{SD}_n$ for $k = 1,...,n$. The complete compactification of $\mathbf{SH}_n$ by considering $\mathrm{Cl}(\mathbf{SPH}_n)$ in the compact manifold $\mathbf{G}(2n,n,\mathbb{C})$. We identify $\partial\mathbf{SPH}_n$ with $\partial\mathbf{SH}_n$. The following lemma follows straightforward [**?**]

Lemma 3.1 *The compactification of* $\mathbf{SPH}_n$ *is equivalent to the compactification of* $\mathbf{SD}_n$ *as a bounded domain. Furthermore, the finite boundary of* $\mathbf{SH}_n$ *correspond to the set of all equivalence classes that admit a representative of the type* $\begin{pmatrix} Z \\ I \end{pmatrix}$ *with* Z *and* $\mathrm{Im}\, Z \ge 0$*, and such a representative is unique. Moreover, let* Z_1, Z_2 *be points in the finite boundary of* $\mathbf{SH}_n$ *such that* $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{bmatrix} Z_1 \\ I \end{bmatrix} = \begin{bmatrix} Z_2 \\ I \end{bmatrix}$. *Then* $CZ_1 + D$ *is invertible.*

The infinite boundary of $\mathbf{SH}_n$ corresponds to the set $\{Z \in \partial\mathbf{SD}_n : \ \det\,(\mathrm{Z} - \mathrm{I}) = 0\}$, where Φ_2^{-1} is not defined. The following proposition follows straightforward:

Proposition 3.2 *Let* G *be a subgroup of* 2×2 *block upper triangular matrices in* $\mathbf{Sp}(n,\mathbb{R})$*. Then* G *is generated by translations and congruencies:*

$$Z \mapsto T(Z) = Z + B, \quad B \in \mathbf{Sym}(\mathbb{R},n), \quad T = \begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix},$$

$$Z \mapsto Q(Z) = AZA^{\mathrm{T}}, \quad A \in \mathbf{GL}(n,\mathbb{R}), \quad Q = \begin{pmatrix} A & 0 \\ 0 & (A^{\mathrm{T}})^{-1} \end{pmatrix}.$$

G *stabilizes each* $\mathrm{fin}(\partial_k\mathbf{SH}_n)$*. Furthermore,* G *acts transitively on* $\mathbf{SH}_n$ *and on each* $\mathrm{fin}(\partial_k\mathbf{SH}_n)$*.*

Recall that $\mathbf{SD}_n$ is a complex manifold. The Shilov boundary of $\mathbf{SD}_n$ is the minimal closed subset of $S \subset \partial\mathbf{SD}_n$ with the following property: The maximum modulus of any continuous complex valued function f on $\mathrm{Cl}(\mathbf{SD}_n)$, which is analytic on $\mathbf{SD}_n$, is achieved on S. It is well known that $\mathbf{USym}_n$ is the Shilov boundary of $\mathbf{SD}_n$, e.g. [**?**]. One can show that $\mathbf{USym}(n)$ can be presented as a homogeneous spaces $\mathbf{U}_n/\mathbf{O}(n,\mathbb{R})$ and $\mathbf{K}_n/\tilde{\mathbf{O}}(n,\mathbb{R})$, where $\tilde{\mathbf{O}}(n,\mathbb{R}) \subset \mathbf{Sp}(n,\mathbb{R})$ is the group matrices of the form $diag(Q,Q)$ with $Q \in \mathbf{O}(n,\mathbb{R})$ [?]. The main result of this section is:

Theorem 3.3 *The compactification of $\mathbf{SH}_n$ as a bounded domain is equivalent to the compactification of $\mathbf{Y}_n$ with respect to the Busemann function d_1*

The proof of this theorem is given at the end of the next section.

4 Properties of symplectic matrices and applications

Let $\mathbb{F}$ be a field of characteristic 0. Let $\mathbf{W}$ be a vector field over $\mathbb{F}$ of dimension $2n$. Let $(\mathbf{u},\mathbf{v})$ be a skew form on W. That is $(\mathbf{v},\mathbf{u}) = -(\mathbf{u},\mathbf{v})$. $(\cdot,\cdot)$ is called nondegenerate if the linear functional $f : \mathbf{W} \to \mathbb{F}$, given by $f(\mathbf{x}) = (\mathbf{x},\mathbf{u}),\ 0 \neq \mathbf{u} \in \mathbf{W}$, is a nonzero functional. A symplectic basis $(\mathbf{e}^1,\ldots,\mathbf{e}^n,\mathbf{f}^1,\ldots,\mathbf{f}^n)$ in $\mathbf{W}$ satisfies

$$(\mathbf{e}^j,\mathbf{f}^k) = -(\mathbf{f}^k,\mathbf{e}^j) = \delta_{jk}, \quad (\mathbf{e}^j,\mathbf{e}^k) = (\mathbf{f}^j,\mathbf{f}^k) = 0 \text{ for all } j,k = 1,\ldots,n.$$

A subspace $\mathbf{V} \subset \mathbf{W}$ is isotropic if for all $\mathbf{u},\mathbf{v}$ in the subspace, $(\mathbf{u},\mathbf{v}) = 0$. An isotropic subspace is called Lagrangian if it has the maximal dimension n. Clearly, $\mathrm{span}(\mathbf{e}^1,...,\mathbf{e}^n)$ and $\mathrm{span}(\mathbf{f}^1,...,\mathbf{f}^n)$ are Lagrangian subspaces. A nontrivial subspace $\mathbf{U}$ is called nondegenerate if the restriction of the form $(\cdot,\cdot)$ to $\mathbf{U}$ is nondegenerate. Two subspaces $\mathbf{U},\mathbf{V} \subset \mathbf{W}$ are called skew orthogonal if $(\mathbf{u},\mathbf{v}) = 0$ for every $\mathbf{u} \in \mathbf{U},\mathbf{v} \in \mathbf{V}$. A decomposition $\mathbf{W} = \oplus_{i=1}^k \mathbf{U}_i$ is called to an orthoskew decomposition if any two distinct subspaces $\mathbf{U}_i,\mathbf{U}_j$ are orthogonal with respect to the given skew form. The following lemma is well known:

Lemma 4.1 *Let $(\cdot,\cdot)$ be a nondegenerate skew form on a vector space W of dimension $2n$. Then the following are equivalent:*
(a) $\mathbf{W} = \oplus_{i=1}^k \mathbf{U}_i$ *is an orthoskew decomposition with* $\mathbf{U}_i \neq \{0\}$ *for* $i = 1,...,k$*;*
(b) *There exists a symplectic basis* $(\mathbf{e}^1,\ldots,\mathbf{e}^n,\mathbf{f}^1,\ldots,\mathbf{f}^n)$ *of* $\mathbf{W}$ *and* $k+1$ *integers* $j_0 = 0,\ 1 \leq j_1 < j_2 < \ldots < j_k = n$ *such that* $\mathbf{U}_i = \mathrm{span}(\mathbf{e}^{j_{i-1}+1},\ldots,\mathbf{e}^{j_i},\mathbf{f}^{j_{i-1}+1},\ldots,\mathbf{f}^{j_i})$ *for* $i = 1,\ldots,k$.

On $\mathbb{F}^{2n} \times \mathbb{F}^{2n}$ define a skew (symplectic) form as $(\mathbf{u},\mathbf{v}) := \mathbf{u}^{\mathrm{T}} J\mathbf{v}$. Note that this skew form is nondegenerate. Then $M \in \mathbf{GL}(2n,\mathbb{F})$ is symplectic if and only if $(M\mathbf{u},M\mathbf{v}) = (\mathbf{u},\mathbf{v})$ for all $\mathbf{u},\mathbf{v} \in \mathbb{F}^{2n}$. Furthermore, $M \in \mathbf{GL}(2n,\mathbb{F})$ is symplectic if and only if it is a change of basis matrix from one symplectic basis to another one. Let $A \in \mathbf{M}(m,\mathbb{C})$. Denote by $\mathrm{spec}(A) \subset \mathbb{C}$ the spectrum of A: $\mathrm{spec}(A) := \{\lambda \in \mathbb{C} :\ \det(\lambda I_m - A) = 0\}$. In what follows we need the following subsets of $\mathrm{spec}(A)$:

$$\mathrm{spec}_{1^+}(A) := \{\lambda \in \mathrm{spec}(A) :\ |\lambda| > 1\},\ \mathrm{spec}_{1^-}(A) := \{\lambda \in \mathrm{spec}(A) :\ |\lambda| < 1\},$$
$$\mathrm{spec}_1(A) := \{\lambda \in \mathrm{spec}(A) :\ |\lambda| = 1\},\ \mathrm{spec}_q(A) := \{\lambda \in \mathrm{spec}(A) :\ |\lambda| \leq 1,\ \mathrm{Im}\,\lambda \geq 0\}.$$

For any set $L \subset \mathbb{C}$ let $P_L(A) \in \mathbf{M}(m,\mathbb{C})$ be the spectral projection on the generalized eigenspace of A associated with $L \cap \mathrm{spec}(A)$. Note that if $L \cap \mathrm{spec}(A) = \emptyset$ then $P_L(A) = 0$.

Furthermore, if $L = \overline{L}$ and $A \in \mathbf{M}(m, R)$ then $P_L(A) \in \mathbf{M}(m, \mathbb{R})$ and $\mathbb{C} \otimes P_L(A)\mathbb{R}^m = P_L(A)\mathbb{C}^m$. Suppose that $L \subset \mathbb{C}\backslash\{0\}$. Then $L^{-1} := \{z \in \mathbb{C} : \quad z^{-1} \in L\}$. Denote by $P_{1^+}(A), P_{1^-}(A), P_1(A)$ the spectral projections on $\mathrm{spec}_{1^+}(A), \mathrm{spec}_{1^-}(A), \mathrm{spec}_1(A)$ respectively. On $\mathbb{C}^m$ define a symmetric form $< \mathbf{u}, \mathbf{v} >= \mathbf{u}^{\mathrm{T}}\mathbf{v}$ any $\mathbf{u}, \mathbf{v}$ in $\mathbb{C}^m$. Note that on $\mathbb{R}^m$ this symmetric form is positive definite. The following claims can be deduced straightforward:

Proposition 4.2 *Let $A \in \mathbf{M}(m, \mathbb{C})$, $L, L' \in \mathbf{C}, L \cap L' = \emptyset$. Then $\mathbf{u}, \mathbf{v} >= 0$ for $\mathbf{u} \in P_L(A)\mathbb{C}^m$ and $\mathbf{v} \in P_{L'}(A^{\mathrm{T}})\mathbb{C}^m$.*

Proposition 4.3 *Let $M \in \mathbf{Sp}(n, \mathbb{C})$. Let $L, L_1 \subset \mathbb{C}\backslash\{0\}$ such that $L_1 \cap L^{-1} = \emptyset$. Then $P_L(M)\mathbb{C}^{2n}, P_{L_1}(M)\mathbb{C}^{2n}$ are skew orthogonal. Assume furthermore that $L \cap L^{-1} = \emptyset$. Then $P_L(M)\mathbb{C}^{2n}$ is an isotropic subspace. Suppose furthermore that $M \in \mathbf{Sp}(n, \mathbb{R})$ and $L = \overline{L}$. Then $P_L(M)\mathbb{R}^{2n}$ is an isotropic subspace of $\mathbb{R}^{2n}$.*

Corollary 4.4 *Let $M \in \mathbf{Sp}(n, \mathbb{R})$. Then $P_{1^+}(M)\mathbb{R}^{2n}$ and $P_{1^-}(M)\mathbb{R}^{2n}$ are isotropic subspaces.*

Lemma 4.5 *Let $M \in \mathbf{Sp}(n, \mathbb{R})$. Then*

$$\mathbb{R}^{2n} = \sum_{\lambda \in \mathrm{spec}_q(M)} \oplus P_{\{\lambda, \bar{\lambda}, \lambda^{-1}, \overline{\lambda^{-1}}\}}(M)\mathbb{R}^{2n} \tag{4.1}$$

is an orthoskew decomposition of $\mathbb{R}^{2n}$. Assume that $\lambda \in \mathrm{spec}_q(M)\backslash\mathrm{spec}_1(M)$. Then

$$P_{\{\lambda, \bar{\lambda}, \lambda^{-1}, \overline{\lambda^{-1}}\}}(M)\mathbb{R}^{2n} = P_{\{\lambda, \bar{\lambda}\}}(M)\mathbb{R}^{2n} \oplus P_{\{\lambda^{-1}, \overline{\lambda^{-1}}\}}(M)\mathbb{R}^{2n} \tag{4.2}$$

is a decomposition of $P_{\{\lambda, \bar{\lambda}, \lambda^{-1}, \overline{\lambda^{-1}}\}}(M)\mathbb{R}^{2n}$ to a direct sum of its two Lagrangian subspaces.

Let $\mathbf{Sp}(n, \mathbb{R})^+ := \mathbf{Sp}(n, \mathbb{R}) \cap \mathbf{H}^+(2n, \mathbb{C})$. Then

Corollary 4.6 *Assume that $A \in \mathbf{Sp}(n, \mathbb{R}) \cap \mathbf{Sym}(2n, \mathbb{R})$. Then*

$$A = OBO^T, \quad O \in \mathbf{K}_n, \quad B \in \mathbf{Sp}(n, \mathbb{R}) \cap \mathbf{D}(2n, \mathbb{R}). \tag{4.3}$$

Furthermore any $A \in \mathbf{Sp}(n, \mathbb{R})$ has the SVD:

$$A = O_1(D \oplus D^{-1})O_2, \quad O_1, O_2 \in \mathbf{K}_n, \quad D = \mathrm{diag}(d_1, ..., d_n), \quad 0 < d_1 \leq \ldots \leq d_n \leq 1. \tag{4.4}$$

In particular any $A \in \mathbf{Sp}(n, \mathbb{R})^+$ has the above form with $O_2 = O_1^{\mathrm{T}}$.

Use the above results to deduce the analog of [**?**, Lemma 3.1] for $\mathbf{Y}_n$:

Corollary 4.7 *Let $(A, B), (C, D) \in \mathbf{Y}_n \times \mathbf{Y}_n$. Then there exists $T \in \mathbf{Sp}(n, \mathbb{R})$ such that $T(A, B) = (C, D)$ if and only if $\Sigma(A^{-1}B) = \Sigma(C^{-1}D)$. In particular, for any pair $(A, B) \in \mathbf{Y}_n \times \mathbf{Y}_n$ there exits $T \in \mathbf{Sp}(n, \mathbb{R})$ such that $T(A, B) = (I_{2n}, D \oplus D^{-1})$, where D satisfies (**??**).*

An equivalent statement in the above Corollary is due to Siegel [**?**].

Proof of Theorem ??. Corollary **??** yields that $\mathbf{Y}_n$ can be presented by $\mathbf{Sp}(n,\mathbb{R})^+$. Let $\Phi_1^{-1}: \mathbf{Sp}(n,\mathbb{R})^+ \to \mathbf{SH}_n$ be the inverse map to $\Phi_1: \mathbf{SH}_n \to \mathbf{Sp}(n,\mathbb{R})/\mathbf{K}_n$. Consider a sequence of matrices $D_m \oplus D_m^{-1} \in \mathbf{Sp}(n,\mathbb{R})^+,\ m = 1,\ldots,$ where each D_m is of the form given in (**??**). [**?**, Theorem 6.3] yields that the sequence $\{D_m \oplus D_m^{-1}\}_1^\infty$ converges to a point in $\partial_1\mathbf{Y}_n$ if and only if

$$\begin{aligned}
&D_m = diag(d_{1,m},\ldots,d_{n,m}), \quad 0 \le d_{1,m} \le \ldots \le d_{n,m} \le 1, \quad m = 1,\ldots,\\
&\lim_{m\to\infty} D_m = \Delta := diag(\delta_1,\ldots\delta_n),\\
&0 = \delta_1 = \ldots = \delta_{i_1} < \delta_{i_1+1} = \ldots = \delta_{i_2} < \ldots < \delta_{i_{l-1}+1} = \ldots = \delta_{i_l} \le 1,\\
&0 = i_0 < i_1 < i_2 < \ldots < i_l = n.
\end{aligned} \tag{4.5}$$

Note that $i_1 = n$ if and only if $\Delta = 0$. Thus, $\lim_{m\to\infty} \Phi_1^{-1}(D_m \oplus D_m^{-1}) = \sqrt{-1}\Delta^2$. Hence $\sqrt{-1}\Delta^2 \in \mathrm{fin}(\partial_{i_1}\mathbf{SH}_n)$. Assume that the limit point in $\partial_1\mathbf{Y}_n$, given by the sequence (**??**), corresponds to the boundary point $\sqrt{-1}\Delta^2$. Let $\{B_m\}_1^\infty \subset \mathbf{Sp}(n,\mathbb{R})^+$ be a sequence of points converging to a point $\eta \in \mathbf{Y}_{n,1}(\infty)$. Corollary **??** yields that $B_m = O_m(D_m \oplus D_m^{-1})O_m^{\mathrm{T}}$, where $O_m \in \mathbf{K}_n$ and D_m is of the above form. As $\{B_m\}_1^\infty$ converges to η [**?**, Theorem 6.3] yields that (**??**) holds. Pick up a subsequence $\{O_{m_l}\}_{l=1}^\infty$ which converges to $O \in \mathbf{K}_n$. Let $\{B_{m_l}\}$ correspond to a boundary point $C = O(\sqrt{-1}\Delta^2)$ in the finite or infinite boundary of $\mathbf{SH}_n$. Our first claim is that C does not depend on the subsequence $\{B_{m_l}\}$, i.e. $C = C(\eta)$. By considering the sequence $\{PB_mP^{\mathrm{T}}\}_1^\infty$ for a suitable $P \in \mathbf{K}_n$, to prove the first claim we may assume that η corresponds to the limit point given by the sequence (**??**). Use [**?**, Theorem 6.3] to deduce that O has the block diagonal form:

$$O = \sum_{j=1}^{2l} \oplus O_j, \quad O_j, O_{2l-j+1} \in \mathbf{O}(i_j - i_{j-1},\mathbb{R}), \quad j = 1,...,l. \tag{4.6}$$

As $O \in \mathbf{K}_n$ we have the additional equalities $O_{2l-j+1} = O_j$ for $j = 1,...,l$. Thus $O(\sqrt{-1}\Delta^2) = \sqrt{-1}\Delta^2$ and the first claim is proved. Our second claim that $C(\eta) = O(\sqrt{-1}\tilde{\Delta}^2)$ gives any point on the finite or infinite boundary of $\mathbf{SH}_n$, for a suitable choice of Δ and $O \in \mathbf{K}_n$. (We can assume that $O_m = O,\ m = 1,....$) Observe that

$$-\Phi_2(\sqrt{-1}\Delta^2) = (I - \Delta^2)(I + \Delta^2)^{-1} = \Sigma((I - \Delta^2)(I + \Delta^2)^{-1}).$$

Use Schur's lemma **??** to deduce that any $B \in \partial_k\mathbf{SD}_n$ is of the form $U(I - \Delta^2)(I + \Delta^2)U^{\mathrm{T}}$ for some $U \in \mathbf{U}_n$, and a corresponding Δ. The second claim is established. Our third claim is that for $\xi, \eta \in \partial_1\mathbf{Y}_n,\ \xi \ne \eta$ we have $C(\eta) \ne C(\xi)$. Let η be given by $\{D_m \oplus D_m^{-1}\}_1^\infty$, where each $\tilde{D}_m$ is of the form (**??**). Assume that $B_m = O_m(\tilde{D}_m \oplus \tilde{D}_m^{-1})O_m^{\mathrm{T}}\ O_m \in \mathbf{K}_n$, where each $\tilde{D}_m$ is of the form (**??**), converges to ξ. The above arguments show that we can assume $\lim_{m\to\infty} \tilde{D}_m = \tilde{\Delta}$ and $\lim_{m\to\infty} O_m = O \in \mathbf{K}_n$. Then $C(\xi) = O(\sqrt{-1}\tilde{\Delta}^2)$. Assume to the contrary that $C(\xi) = C(\eta) = \sqrt{-1}\Delta^2$. We claim that $\tilde{\Delta} = \Delta$ and O is of the form (**??**). A simple way to show this claim is to consider the equality $-\Phi_2(C(\xi)) = -\Phi_2(C(\eta))$:

$$U\tilde{\Sigma}U^T = \Sigma,\ \Sigma = (I - \Delta^2)(I + \Delta^2)^{-1},\ \tilde{\Sigma} = (I - \tilde{\Delta}^2)(I + \tilde{\Delta}^2)^{-1},\ U \in \mathbf{U}_n.$$

Schur's lemma **??** yields that $\tilde{\Sigma} = \Sigma$. Hence $\tilde{\Delta} = \Delta$. Use the original arguments of Schur [**?**] (or [**?**, Lemma 2]) to deduce that the above equality implies $U_n = \oplus_{j=1}^{l} O_j$ where $O_j \in \mathbf{O}(i_j - i_{j-1}, \mathbb{R})$ for $j = 1, ..., l$. It is straightforward to show that the above equality yields that O is of the form (**??**). From the arguments of the proof of our first claim it follows that $\eta = \xi$, contrary to our assumption. $\square$

Proposition 4.8 *Let ξ be a point in the compact strata of the Busemann 1-boundary of $\mathbf{Sp}(n, \mathbb{R})/\mathbf{K}_n$. Then ξ is uniquely presented by the Lagrangian subspace $\Xi \subset \mathbf{R}^{2n}$. Identify ξ with a unit vector in the one dimensional subspace $\wedge_n \Xi$. Then with respect to the reference point X_0:*

$$b_{1,\xi}(X) = 2 \log ||(\wedge_n X)\xi||_2 - 2 \log ||(\wedge_n X_0)\xi||_2.$$

Proof. A point ξ in the compact strata of the Busemann 1-boundary of $\mathbf{Sp}(n, \mathbb{R})/\mathbf{K}_n$ corresponds to the two flag

$$P_{1^+}(A)\mathbb{R}^{2n} \subset P_{1^+}(A)\mathbb{R}^{2n} \oplus P_{1^-}(A)\mathbb{R}^{2n} = \mathbb{R}^{2n}, \quad A \in \mathbf{Sp}(n, \mathbb{R})^+.$$

Hence A is hyperbolic and $\Xi := P_{1^+}(A)\mathbb{R}^{2n}$ is a Lagrangian subspace. Clearly, $\Xi^\perp = P_{1^-}(A)\mathbb{R}^{2n}$. [**?**, Theorem 6.3] yields that ξ is determined uniquely by $U_+ = \Xi$, $U_- = \Xi^\perp$ ($\mathbf{H}(U_0) = 0$). Vice versa, assume that Ξ is a Lagrangian subspace. Use Corollary **??** to find $A \in \mathbf{Sym}(2n, \mathbb{R})$, $e^A \in \mathbf{Sp}(n, \mathbb{R})^+$, such that $P_{1^+}(e^A)\mathbb{R}^{2n} = \Xi$. Note that $\lambda_n(A) > 0$. Then $e^{tA} \to \xi$ as $t \to \infty$.

We use [**?**, Theorem 4.1] to calculate $b_{\xi,1}(X)$. Observe that $\lambda_n(A) > 0 > \lambda_{n+1}(A)$, i.e. $j_k = n$. As $\mathbf{Sp}(n, \mathbb{R}) \subset \mathbf{SL}(2n, \mathbb{R})$ we deduce that $b_{\xi,1}(X) = 2\alpha_n(A, X) - 2\alpha_n(A, X_0)$. Use the definition $\alpha_n(A, X)$ and the fact that $\wedge_n \Xi$ is a one dimensional subspace to obtain the proposition. $\square$

Corollary 4.9 *The compact strata of the Busemann 1-boundary of $\mathbf{SH}_n$, denoted by $\partial_n \mathbf{SH}_n$, is equivalent to the Shilov boundary $\mathbf{USym}_n$.*

Proof. Let

$$B = diag(b_1, ..., b_n) \in \mathbf{D}(n, \mathbb{R}),\ b_1 < b_2 < \ldots < b_n < 0,\ C = B \oplus -B. \tag{4.7}$$

Then C represents a Weyl chamber in $\mathbf{Sp}(n, \mathbb{R})/\mathbf{K}_n$. The geodesic ray $e^{tC}, t > 0$ converges to the point ξ on the compact strata of the Busemann 1-boundary of $\mathbf{SH}_n$. Clearly ξ corresponds to $0 \in fin(\partial_n \mathbf{SH}_n)$. As the compact strata of the Busemann 1-boundary is given by the limit of the geodesic rays $Oe^{tC}O^{\mathrm{T}}$, $O \in \mathbf{K}_n$, we deduce the corollary. $\square$

5 Discrete subgroups of $\mathbf{Sp}(n, \mathbb{R})$

In the rest of this paper we always assume that Γ is a discrete subgroup of $\mathbf{Sp}(n, \mathbb{R})$. Assume that Γ is torsion free. As $\mathbf{PSp}(n, \mathbb{R})$ is the group of biholomorphisms of $\mathbf{SH}_n$ it follows that $\mathbf{SH}_n/\Gamma = \Gamma \backslash \mathbf{Sp}(n, \mathbb{R})/\mathbf{K}_n$ is a complex manifold of dimension $\frac{n(n+1)}{2}$, whose universal cover is $\mathbf{SH}_n$. Assume that Γ has torsion. According to Selberg [**?**] Γ has a subgroup Γ_0 of finite index in such that Γ_0 is torsion free. Hence the manifold $\mathbf{SH}_n/\Gamma_0$ is a finite cover of the orbifold $\mathbf{SH}_n/\Gamma$. Therefore $\mathbf{SH}_n/\Gamma$ is a complex space [?]. The case when Γ is a lattice in $\mathbf{Sp}(n,\mathbb{R})$ is very closely related to modular forms and algebraic geometry [?], [?]. (In many known cases $\mathbf{SH}_n/\Gamma$ is a quasiprojective variety.) As $\mathbf{Sp}(n,\mathbb{R})$ is a simple Lie group of rank n, for $n>1$ the study of Γ falls into category of discrete subgroups in higher rank groups. Some aspects of such discrete subgroups, in particular the Patterson-Sullivan theory, is treated in Albuquerque [?]. For $n=1$ Γ is a Fuchsian group. The modern treatment of Fuchsian and Kleinian groups can be found in [?]. To compare the properties of Γ (for $n>1$) with the properties of Fuchsian groups it is useful to note that $\mathbf{SL}(2,\mathbb{R})^n := \mathbf{SL}(2,\mathbb{R})\times\ldots\times\mathbf{SL}(2,\mathbb{R})$ is isomorphic to a subgroup of $\mathbf{Sp}(n,\mathbb{R})$:

$$\Theta : \mathbf{SL}(2,\mathbb{R})^n \to \mathbf{Sp}(n,\mathbb{R}), \quad \Theta(M_1\times\ldots\times M_n) = M_1\odot\ldots\odot M_n,$$

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \odot\ldots\odot \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} = \left(\begin{array}{ccc|ccc} a_1 & & & b_1 & & \\ & \ddots & & & \ddots & \\ & & a_n & & & b_n \\ \hline c_1 & & & d_1 & & \\ & \ddots & & & \ddots & \\ & & c_n & & & d_n \end{array}\right).$$

Note that the action of $\mathbf{SL}(2,\mathbb{R})^n$ on $(\mathbf{H}^2)^n$ is isomorphic to the action of $\Theta(\mathbf{SL}(2,\mathbb{R})^n)$ on $\mathbf{DH}_n := \mathbf{D}(n,\mathbb{C})\cap\mathbf{SH}_n$. Namely $M_1\odot\ldots\odot M_m(\mathrm{diag}(z_1,...,z_n)) = \mathrm{diag}(M(z_1),...,M(z_n))$ for $z_1,...,z_n\in\mathbf{H}^2$.

For any set $T\subset\mathbf{SH}_n$ denote by $\mathrm{BCl}(T)$ the closure of T with respect to Busemann 1-compactification of $\mathbf{SH}_n$. Note that $\mathrm{Cl}(\mathbf{SH}_n)\subset\mathrm{BCl}(\mathbf{SH}_n)$. To define the limit set of Γ we need the following theorem:

Theorem 5.1 *Let $\gamma_k\in\mathbf{Sp}(n,\mathbb{R})$, $k=1,...,$ be a given sequence. Assume that for $Z\in\mathbf{SH}_n$ the sequence $\gamma_k(Z)$, $k=1,...,$ converges to a point $P\in\partial_n\mathbf{SH}_n$. Then for any $W\in\mathbf{SH}_n$ the sequence $\gamma_i(W)$, $k=1,...,$ converges to P.*

Proof. Since $\mathbf{Sp}(n,\mathbb{R})$ acts transitively on $\partial_n\mathbf{SH}_n$, there exists $\gamma\in\mathbf{Sp}(n,\mathbb{R})$ so that $\lim_{i\to\infty}\gamma\gamma_i(Z) = 0\in fin(\partial_n\mathbf{SH}_n)$. Hence to prove the lemma it is enough to consider the case $P=0$. Write $\gamma_k(Z) = X_k+\sqrt{-1}Y_k$, $\gamma_k(W) = U_k+\sqrt{-1}V_k$ for $k=1,...$ Let $A\mathbf{K}_n$ and $B\mathbf{K}_n$ be the cosets in $\mathbf{Sp}(n,\mathbb{R})/\mathbf{K}_n$ corresponding to Z and W respectively. The isomorphism $\Phi_1:\mathbf{SH}_n\to\mathbf{Y}_n$, given by (**??**), implies that $\gamma_k A\mathbf{K}_n$ and $\gamma_k B\mathbf{K}_n$ have the representatives $A_k = \begin{pmatrix} Y_k^{\frac12} & X_kY_k^{-\frac12} \\ 0 & Y_k^{-\frac12}\end{pmatrix}$ and $B_k = \begin{pmatrix} V_k^{\frac12} & U_kV_k^{-\frac12} \\ 0 & V_k^{-\frac12}\end{pmatrix}$ respectively. Clearly $||B^{-1}A||_2 = \sigma_1(B^{-1}A) = \sigma_1(B_k^{-1}A_k) = ||B_k^{-1}A_k||_2$. Furthermore $B_k^{-1}A_k = \begin{pmatrix} V_k^{-\frac12}Y_k^{\frac12} & V_k^{-\frac12}X_kY_k^{-\frac12} - V_k^{-\frac12}U_kY_k^{-\frac12} \\ 0 & V_k^{\frac12}Y_k^{-\frac12}\end{pmatrix}$. We claim that $||B^{-1}A||_2 = ||B_k^{-1}A_k||_2 \ge ||V_k^{\frac12}Y_k^{-\frac12}||_2$. The last inequality follows from the standard inequalities on l_2 norms of matrices as follows. For any $C\in\mathbf{M}(2n,\mathbb{R})$, its operator norm is given by $||C||_2 =$

$\max_{||x||_2=||y||_2=1} |y^{\mathrm{T}}Cx|$. Hence $||C||_2 \ge \max_{i,j=1,2} ||C_{ij}||_2$ for $C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$, $C_{ij} \in \mathbf{M}(n,\mathbb{R}),\ i,j=1,2$. Observe next

$$||V_k^{\frac{1}{2}}||_2 = ||(V_k^{\frac{1}{2}}Y_k^{-\frac{1}{2}})Y_k^{\frac{1}{2}}||_2 \le ||V_k^{\frac{1}{2}}Y_k^{-\frac{1}{2}}||_2||Y_k^{\frac{1}{2}}||_2 \Rightarrow ||V_k^{\frac{1}{2}}Y_k^{-\frac{1}{2}}||_2 \ge ||V_k^{\frac{1}{2}}||_2||Y_k^{\frac{1}{2}}||_2^{-1}.$$

Hence $||B^{-1}A||_2 \ge ||V_k||_2^{\frac{1}{2}}||Y_k||_2^{-\frac{1}{2}} \ \Rightarrow\ ||V_k||_2 \le ||B^{-1}A||_2^2||Y_k||_2$. As $||Y_k||_2 \to 0$ we deduce that $||V_k|| \to 0$. Using the above arguments for $(1,2)$ block of $B_k^{-1}A_k$ we obtain

$$||B^{-1}A||_2 \ge ||V_k^{-\frac{1}{2}}(X_k-U_k)Y_k^{-\frac{1}{2}}||_2 \ge ||V_k^{-\frac{1}{2}}(X_k-U_k)||_2\ ||Y_k^{\frac{1}{2}}||_2^{-1} \ge$$
$$||V_k^{\frac{1}{2}}||_2^{-1}||(X_k-U_k)||_2\ ||Y_k^{\frac{1}{2}}||_2^{-1}.$$

Thus $||X_k-U_k||_2 \le ||B^{-1}A||_2||Y_k||_2^{\frac{1}{2}}||V_k||_2^{\frac{1}{2}}$ for $k=1,\ldots$ Since $X_k, Y_k, V_k \to 0$, we deduce that $\gamma_k(W) = U_k + \sqrt{-1}V_k \to 0$. □

We remark that Theorem **??** does not hold if $\gamma_k(Z) \to P \in \partial_m\mathbf{SH}_n := \Phi_2^{-1}(\partial_m\mathbf{SD}_n)$ for any $m \in [1,n-1]\cap\mathbb{Z}$. Indeed, let $M = \mathrm{diag}(\frac{1}{2},2) \in \mathbf{SL}(2,\mathbb{R})$ and define $\gamma_k = \underbrace{M^k \odot \ldots \odot M^k}_{m \text{ times}} \odot \underbrace{I_2 \odot \ldots \odot I_2}_{n-m \text{ times}} \in \mathbf{Sp}(n,\mathbb{R})$ for $k=1,\ldots$ Then $\lim_{k\to\infty}\gamma_k(\mathrm{diag}(z_1,\ldots,z_n)) = \mathrm{diag}(0,\ldots,0,z_{m+1},\ldots,z_n)$ for any $\mathrm{diag}(z_1,\ldots,z_n) \in \mathbf{DH}_n$..

Corollary 5.2 *Let $\gamma_k \in \mathbf{Sp}(n,\mathbb{R}),\ k=1,\ldots,$ be a given sequence. Then for any $Z \in \mathbf{SH}_n$ all the accumulation points of the sequence $\{\gamma_k(Z)\}_1^\infty$ lie in the Shilov boundary of $\mathbf{SH}_n$ if and only if*

$$\lim_{k\to\infty} \sigma_i(\gamma_k) = \infty, \quad i=1,\ldots,n. \tag{5.1}$$

Proof. Corollary **??** implies that $\gamma_k = O_{1,k}(D_k \oplus D_k^{-1})O_{2,k}$, where $O_{1,k}, O_{2,k} \in \mathbf{K}_n,\ D_k = \mathrm{diag}(\sigma_{2n}(\gamma_k),\ldots,\sigma_{n+1}(\gamma_k))$, for $k=1,\ldots$ Let $Z = \sqrt{-1}I_n$. Then Z is represented by the coset I_{2n} in $\mathbf{Sp}(n,\mathbb{R})/\mathbf{K}_n$ Suppose that a subsequence $\gamma_{k_i}(Z) \to P$. Pick up a subsequence $\{k_i'\}_{i=1}^\infty$ such that $O_{1,k_i'} \to O \in \mathbf{K}_n$. Then $O_{k_i'}\gamma_{k_i'}(Z) \to O(P)$. Clearly $O_{1,k}^{\mathrm{T}}\gamma_k(Z) = \sqrt{-1}\mathrm{diag}(\sigma_{2n}(\gamma_k)^2,\ldots,\sigma_{n+1}(\gamma_k)^2)$ for $k \in \mathbb{N}$. Thus $\gamma_{k_i'}(Z) \to P \iff \lim_{i\to\infty}\sigma_j(\gamma_{k_i'}) = \delta_{2n-j+1} \le 1,\ j=1,\ldots,n$. Then $P \in \partial_n\mathbf{SH}_n$ if and only if $\delta_{n+1}=0$. Thus all the accumulation points of the sequence $\{\gamma_k(Z)\}_1^\infty$ lie in $\partial_n\mathbf{SH}_n$ if and only if and only if (**??**) holds. Use Theorem **??** to deduce the corollary. □

Definition 5.3 *Let Γ be a discrete group of $\mathbf{Sp}(n,\mathbb{R})$. Then the limit set $\Lambda(\Gamma)$ is given by the set $\mathrm{BCl}(\Gamma(Z)) \cap \partial_n\mathbf{SH}_n$ for some $Z \in \mathbf{SH}_n$.*

Theorem **??** implies that the definition of $\Lambda(\Gamma)$ is independent of the choice of $Z \in \mathbf{SH}_n$ as in the case of Fuchsian groups. Corollary **??** gives a necessary and sufficient conditions for Γ so that $\Lambda(\Gamma) \ne \emptyset$. For a Fuchsian group Γ the limit set $\Lambda(\Gamma) \ne \emptyset$ if and only if Γ is infinite. For $n>1$ there exist infinite Γ for which $\Lambda(\Gamma)=\emptyset$. Indeed, let $\Gamma_1,\ldots,\Gamma_n$ be Fuchsian groups, where Γ_1 is finite and $\Gamma_2,\ldots,\Gamma_n$ infinite. Then the above arguments show that $\Lambda(\Gamma_1 \odot \ldots \odot \Gamma_n) = \emptyset$.

An element $\gamma \in \mathbf{Sp}(n,\mathbb{R})$ is called *hyperbolic* if it does not have eigenvalues on the unit circle, i.e. $\mathrm{spec}_1(\gamma)=\emptyset$.

Proposition 5.4 *Let $\gamma \in \mathbf{Sp}(n,\mathbb{R})$ be hyperbolic. Then γ is conjugate in $\mathbf{Sp}(n,\mathbb{R})$ to*

$$\tilde\gamma = \begin{pmatrix} C & 0 \\ 0 & (C^{\mathrm{T}})^{-1} \end{pmatrix}, \ C \in \mathbf{GL}(n,\mathbb{R}),\ \mathrm{spec}_{1^-}(\tilde\gamma)=\mathrm{spec}(C). \tag{5.2}$$

Proof. Corollary **??** yields that $P_{1^-}(\gamma)\mathbb{R}^{2n}$ and $P_{1^+}(\gamma)\mathbb{R}^{2n}$ are Lagrangian subspaces. Pick bases $\mathbf{e}^1,...,\mathbf{e}^n$ and $\mathbf{f}^1,...,\mathbf{f}^n$ in the above Lagrangian subspaces such that such that $\mathbf{e}^1,...,\mathbf{e}^n,\mathbf{f}^1,...,\mathbf{f}^n$ is a symplectic base of $\mathbb{R}^{2n}$. Let $\mathbf{e}^1,...\mathbf{e}^n,\mathbf{f}^1,...,\mathbf{f}^n$ be the columns of T. Then $T \in \mathbf{Sp}(n,\mathbb{R})$ and $\tilde\gamma$ has the block diagonal form $\mathrm{diag}(C,C')$, where $C,C' \in \mathbf{M}(n,\mathbb{R})$. As $\mathrm{diag}(C,C')$ is symplectic we deduce that $C'=(C^{\mathrm{T}})^{-1}$. As C represents the restriction of γ to $P_{1^-}(M)\mathbb{R}^{2n}$. Hence the last equality of (**??**) holds. □

Lemma 5.5 *Let $\gamma \in \mathbf{Sp}(n,\mathbb{R})$ be hyperbolic. Then there exist two distinct fixed points $\xi_+(\gamma),\xi_-(\gamma) \in \partial_n\mathbf{SH}_n$ of γ such that*

$$\lim_{k\to\infty}\gamma^k(Z)=\xi_+(\gamma),\ \lim_{k\to\infty}\gamma^{-k}(Z)=\xi_-(\gamma),\ \text{for all}\ \ Z \in \mathbf{SH}_n. \tag{5.3}$$

Proof. Without loss of generality we may assume that γ is equal to $\tilde\gamma$ given in (**??**). Then $\tilde\gamma^k(Z)=C^kZ(C^{\mathrm{T}})^k$ for $k \in \mathbb{Z}$. As all the eigenvalues of C are in the open unit disk, we deduce $\lim_{k\to\infty}C^k=0$. Hence the first equality of (**??**) holds with $\xi_+(\tilde\gamma)=0$. Observe next that $\tilde\gamma^{-1}=J_n\tilde\gamma^{\mathrm{T}}J_n^{-1}$. Hence the second equality of (**??**) holds with $\xi_-(\gamma)=J_n(0)$. □

For $n=1$ the hyperbolic element $\gamma \in \mathbf{SL}(2,\mathbb{R})$ has exactly two fixed points in the closure of $\mathbf{H}^2$ which are located on the boundary. For $n>1$ a hyperbolic element can have more then two fixed points in $\partial\,\mathrm{BCl}(\mathbf{SH}_n)$. Indeed, let $\gamma_1,...,\gamma_n \in \mathbf{SL}(2,\mathbb{R})$ be n hyperbolic elements so that the set $\{\xi_+(\gamma_1),\xi_-(\gamma_1),\ldots,\xi_+(\gamma_n),\xi_-(\gamma_n)\}$ is a set of $2n$ distinct real points. Let $\gamma=\gamma_1\odot\ldots\odot\gamma_n \in \mathbf{Sp}(n,\mathbb{R})$. Then the following 2^n points are fixed points of γ: $\mathrm{diag}(\xi_\pm(\gamma_1),\ldots,\xi_\pm(\gamma_n)) \in fin(\partial_n\mathbf{SH}_n)$. With some effort one can show that such γ has exactly 2^n fixed points in $\mathrm{BCl}(\mathbf{SH}_n)$. Note that

$$\xi_+(\gamma)=\mathrm{diag}(\xi_+(\gamma_1),\ldots,\xi_+(\gamma_n)),\quad \xi_-(\gamma)=\mathrm{diag}(\xi_-(\gamma_1),\ldots,\xi_-(\gamma_n)).$$

It is possible to show that a hyperbolic transformation has at most 2^n isolated fixed points in $\mathrm{BCl}(\mathbf{SH}_n)$. It may happen that a hyperbolic transformation has less than 2^n isolated points. In [**?**] we show that for $n=2$ any hyperbolic transformation has either 2, 3 or 4 isolated fixed points in $\partial_2\mathbf{SH}_2$ or two isolated fixed points and a closed connected real 1-dimensional variety of fixed points $\sim S^1$ in $\partial_2\mathbf{SH}_2$.

Denote by Γ_h the set of all hyperbolic elements in Γ. Assume that $\gamma \in \Gamma_h$. Then $\xi_\pm(\gamma) \in \Lambda(\Gamma)$. As $\alpha\gamma\alpha^{-1} \in \Gamma_h$ for any $\alpha \in \Gamma$ it follows that $\alpha(\xi_\pm(\gamma)) \in \Lambda(\Gamma)$.

Definition 5.6 *Let Γ be a discrete subgroup of $\mathbf{Sp}(n,\mathbb{R})$. Then*

$$\Lambda_h(\Gamma)=\mathrm{BCl}(\cup_{\gamma\in\Gamma_h}\{\xi_+(\gamma),\xi_-(\gamma)\}).$$

$\Lambda_h(\Gamma)$ is a closed Γ-invariant subset of $\Lambda(\Gamma)$. If Γ is a nonelementary Fuchsian group then $\Lambda_h(\Gamma) = \Lambda(\Gamma)$. Moreover $\Lambda_h(\Gamma)$ is an uncountable perfect set [**?**]. An analog of a nonelementary Fuchsian group is a discrete Zariski dense subgroup $\Gamma \subset \mathbf{Sp}(n, \mathbb{R})$. Since $\mathbf{Sp}(n, \mathbb{R})$ is a simple Lie group, the results of Goldsheid-Margulis [**?**] yields that any Zariski dense subgroup of $\mathbf{Sp}(n, \mathbb{R})$ contains hyperbolic elements. The following theorem is closely related to the Lemma in [**?**, 3.6]:

Theorem 5.7 *Let $\Gamma \subset \mathbf{Sp}(n, \mathbb{R})$ be a discrete Zariski dense subgroup. Let T be a closed Γ-invariant subset of* $\mathrm{BCl}(\mathbf{SH}_n)$*. Then T contains $\Lambda_h(\Gamma)$. Furthermore, $\Lambda_h(\Gamma)$ is a perfect set.*

Proof. By conjugating Γ by an element in $\mathbf{Sp}(n, \mathbb{R})$, we may assume that Γ has an element $\tilde{\gamma}$ of the form (**??**). Let $W \in \mathrm{BCl}(\mathbf{SH}_n) \backslash \mathrm{fin}(\partial \mathbf{SH}_n)$. Consider the projective model $\mathbf{SPH}_n$. Then W is presented by the following representative: $W_1 = \begin{pmatrix} A \\ B \end{pmatrix}$, $\det B = 0$. That is, W_1 is located on algebraic variety of $\mathbf{G}(2n, n, \mathbb{R})$. As Γ is Zariski dense in $\mathbf{Sp}(n, \mathbb{R})$, there exists $\alpha \in \Gamma$ such that $V = \alpha(W) \in \mathrm{Cl}(\mathbf{SH}_n)$. Assume that T is Γ-invariant set. The above argument show that there exists $V \in T \cap \mathrm{Cl}(\mathbf{SH}_n)$. Then $\tilde{\gamma}^k(V) \to 0 = \xi_+(\tilde{\gamma})$. Since T is closed $0 \in T$. Hence $\xi_\pm(\beta) \in T$ for any $\beta \in \Gamma_h$. Thus $T \supset \Lambda_h(\Gamma)$. To show that $\Lambda_h(\Gamma)$ is a perfect set we must show that $\Lambda_h(\Gamma)$ does not contain isolated points. Assume to the contrary that $\eta \in \Lambda_h(\Gamma)$ is an isolated point. From the definition of $\Lambda_h(\Gamma)$ it follows that $\eta = \xi_+(\alpha)$ for some $\alpha \in \Gamma_h$. Without a loss of generality we may assume that $\eta = \xi_+(\tilde{\gamma}) = 0$. As Γ Zariski dense, there exists $\beta \in \Gamma$ such that $0 \neq \beta(0) \in \mathrm{fin}(\partial_n \mathbf{SH}_n)$. Then $\tilde{\gamma}^k(\beta(0)), k = 1, ...,$ is a sequence of pairwise distinct points in $\Lambda_h(\Gamma)$ which converges to 0, contrary to our assumption. $\square$

We do not know if $\Lambda(\Gamma) = \Lambda_h(\Gamma)$ for any Zariski dense subgroup Γ and $n > 1$. Let $\Omega(\Gamma)$ be the open set of the Shilov boundary of $\mathbf{SH}_n$ on which Γ acts properly discontinuously. ($\Omega(\Gamma)$ may be an empty set.)

Definition 5.8 *Let Γ be a discrete subgroup of $\mathbf{Sp}(n, \mathbb{R})$. Denote by $\Lambda_d(\Gamma)$ the smallest closed set in the Shilov boundary of $\mathbf{SH}_n$ such that Γ acts properly discontinuously on the complement of $\Lambda_d(\Gamma)$ in the Shilov boundary of $\mathbf{SH}_n$* $(\Omega(\Gamma))$.

$\Lambda_d(\Gamma)$ is a closed Γ-invariant set of $\partial_n \mathbf{SH}_n$. For a Fuchsian (Kleinian) group $\Lambda_d(\Gamma) = \Lambda(\Gamma)$.

Lemma 5.9 *Let $\Gamma \subset \mathbf{Sp}(n, \mathbb{R})$ be a discrete group. Then $\Lambda(\Gamma) \subset \Lambda_d(\Gamma)$.*

Proof. . Clearly, it is enough to consider the case where $\Lambda(\Gamma) \neq \emptyset$ Assume that $\gamma_k(Z) \to P \in \Lambda(\Gamma)$, where $\gamma_k \in \Gamma,\ k = 1, ...$ and $Z \in \mathbf{SH}_n$. Assume that γ_k is of the form as in the proof of Corollary **??**. By choosing a subsequence of $\gamma_k,\ k = 1, ...,$ we may assume that $O_{1,k} \to O_1, O_{2,k} \to O_2$. Use the proof of Corollary **??** to deduce that $P = O_1(0)$. Let $W = O_2^{-1}(0) \in \partial_n \mathbf{SH}_n$. Use the proof of Corollary **??** again to conclude that $\gamma_k(O_2^{-1}(0)) \to O_1(0)$. $\square$

It is not difficult to find simple examples for which $\Lambda(\Gamma) \neq \Lambda_d(\Gamma)$. Let $\alpha = \mathrm{diag}(\frac{1}{2}, 2) \in \mathbf{SL}(2, \mathbb{R})$, $\gamma = \alpha \odot \alpha \in \mathbf{Sp}(2, \mathbb{R})$ and $\Gamma =< \gamma >$. Then $\Lambda(\Gamma) = \{0, J_2(0)\}$. In [**?**] it is shown that γ has a curve of fixed points $F \subset \partial_2 \mathbf{SH}_2$, which belongs to $\Lambda_d(\Gamma)$. Hence $\Lambda(\Gamma)$ is strictly contained in $\Lambda_d(\Gamma)$.

The structure of $\Omega(\Gamma)$ is closely related to the fundamental domains of Γ in $\mathbf{Y}_n = \mathbf{Sp}(n,\mathbb{R})/\mathbf{K}_n$. We consider here the Dirichlet domains. Fix $p \in [0,\infty]$ and $A \in \mathbf{Y}_n$. Let

$$D_p(A,\Gamma) := \{B \in \mathbf{Y}_n : \quad d_p(B,\gamma A) - d_p(B,A) \geq 0, \quad \gamma \in \Gamma\}. \tag{5.4}$$

Let $\tilde{D}_p(A,\Gamma) := \mathrm{BCl}(D_p(A,\Gamma)) \cap \partial_n \mathbf{SH}_n$. If $\tilde{D}_p(A,\Gamma)$ has an open interior (relative to the Shilov boundary) then it belongs to $\Omega(\Gamma)$. Since the compact strata of the Busemann 1-boundary is the Shilov boundary it is natural to choose $p = 1$. In (**??**) let B converge to ξ in the compact strata of the Busemann 1-boundary. Use the definition of Busemann 1-function to obtain with the reference point A

$$\tilde{D}_1(A,\Gamma) = \{\xi \in \partial_n \mathbf{SH}_n : \quad b_{\xi,1}(\gamma A) \geq 0, \quad \gamma \in \Gamma\}. \tag{5.5}$$

See Proposition **??** for the simple formula for $b_{\xi,1}(B)$.

6 Patterson-Sullivan measures

Let $\mathcal{S} \subset \mathbf{Sp}(n,\mathbb{R})$ be a countable discrete set. ($\mathcal{S}$ has no accumulation points in $\mathbf{Sp}(n,\mathbb{R})$.) Assume furthermore that $\mathcal{S}$ is symmetric, i.e. $\gamma \in \mathcal{S} \iff \gamma^{-1} \in \mathcal{S}$. (We assume that an empty set is a symmetric set.) Fix $A, B \in \mathbf{Sp}(n,\mathbb{R})/\mathbf{K}_n$ and $p \in [1,\infty]$. For $r > 0$ let

$$N_p(\mathcal{S}, r, A, B) = \#\{\gamma \in \mathcal{S} : \quad d_p(A,\gamma B) < r\}$$

be the p-orbital counting function [**?**]. Taking in account the properties of $d_p(\cdot,\cdot)$ given in [**?**], the fact that $||x||_p,\ x \in \mathbf{R}^m$ is a decreasing function of $p, p \geq 1$, we deduce in a straightforward manner that

$$N_p(\mathcal{S}, r, A, B) = N_p(\mathcal{S}, r, B, A),\ N_p(\mathcal{S}, r, A, B) \leq N_p(\mathcal{S}, r + d_p(A,C), C, B),$$
$$N_{\mathcal{S},p_1}(r, A, B) \leq N_{p_2}(\mathcal{S}, r, A, B),\ 1 \leq p_1 \leq p_2,\ N_\infty(\mathcal{S}, r, A, B) \leq N_p(\mathcal{S}, (2n)^{\frac{1}{p}} r, A, B).$$

Hence, the p-Poincaré exponent: $\delta_p(\mathcal{S}) := \limsup_{r\to\infty} \frac{\log N_p(\mathcal{S},r,A,B)}{r}$, is independent of the choices A, B. Note that $\delta_p(\emptyset) = -\infty$ and $\delta_p(\mathcal{S}) = 0$ if $\mathcal{S}$ is a nonempty finite set for all $p \in [1,\infty]$. The associated Poincaré series is

$$g_{s,p}(\mathcal{S}, A, B) := \sum_{\gamma \in \mathcal{S}} e^{-s d_p(A,\gamma B)}, \quad s > 0. \tag{6.1}$$

Assume that $\mathcal{S}$ is infinite. Then $\delta_p(\mathcal{S}) \geq 0$ and $g_{0,p}(\mathcal{S}, A, B) = \infty$. Assume that $0 < \delta_p(\mathcal{S}) < \infty$. It is straightforward to show that the Poincaré series converges for $s > \delta_p(\mathcal{S})$ and diverges for $s < \delta_p(\mathcal{S})$ [**?**]. If the Poincaré series diverges for $s = \delta_p(\mathcal{S})$ then $\mathcal{S}$ is called of p-divergence type. Otherwise, $\mathcal{S}$ is called of p-convergence type. (The divergence (convergence) type of $\mathcal{S}$ depends only on the value of p.) The construction of the family of

PS measures is straightforward for infinite discrete symmetric sets $\mathcal{S}$ of divergence type. In what follows B is kept fixed while A may vary. Let

$$\mu_{\mathcal{S},s,A,p} = \frac{1}{g_{s,p}(\mathcal{S}, B, B)} \sum_{\gamma \in \mathcal{S}} e^{-sd_p(A,\gamma B)} \Delta_{\gamma B}, \quad s > 0. \tag{6.2}$$

Here Δ_B denote the Dirac measure on $\mathbf{Y}_n$ at the point B. Then $\mu_{\mathcal{S},s,A,p}$ is a finite measure on $\mathbf{Y}_n$. Identify $\mathbf{Y}_n$ with $\mathbf{SH}_n$. We view $\mu_{\mathcal{S},s,A,p}$ as a finite measure on $\mathrm{BCl}(\mathbf{SH}_n)$. Let $\{s_m\}_1^\infty$ be a strictly decreasing sequence which converges to $\delta_p(\Gamma)$. The Helly selection principle states that we can find a subsequence $\{m_k\}_{k=1}^\infty$ so that the sequence of measures $\mu_{\mathcal{S},s_{m_k},A,p}$ converges weakly to a finite measure $\mu_{\mathcal{S},A,p}$. The assumption that $\mathcal{S}$ was of divergence type implies straightforward

$$\mathrm{supp}\ \mu_{\mathcal{S},A,p} \subset \mathrm{BCl}(\mathcal{S}(B)) \cap \partial\, \mathrm{BCl}(\mathbf{SH}_n). \tag{6.3}$$

Let $\mathcal{M}_{\mathcal{S},A,p}$ be the family of all measures $\mu_{\mathcal{S},A,p}$ obtained by considering all weakly convergent subsequences of $\{\mu_{\mathcal{S},s_m,A,p}\}_{m=1}^\infty$. If $\mathcal{S}$ is of p-convergent type, then one has to modify the definition of the Poincaré series (**??**) and induced measures (**??**) as in [**?**].

Lemma 6.1 *Let $\mathcal{S} \subset \mathbf{Y}_n$ be an infinite discrete set. Assume that $0 < \delta_p(\mathcal{S}) < \infty$ and $\mathcal{S}$ be of p-convergent type. Then there exists a continuous nondecreasing function $h_p : [0, \infty) \to [0, \infty)$ with the following properties:*
(a) *For any $A, B \in \mathbf{Y}_n$ the series $\sum_{\gamma\in\mathcal{S}} e^{-sd_p(A,\gamma B)} h_p(e^{d_p(A,\gamma B)})$ converges for $s > \delta_p(\mathcal{S})$ and diverges for $s = \delta_p(\mathcal{S})$.*
(b) *For a given $\epsilon > 0$ there exists $r_\epsilon > 0$ so that for $r > r_\epsilon$, $t > 1$ $h_p(rt) < t^\epsilon h_p(r)$.*

Proof. Fix $A, B \in \mathbf{Y}_n$. Use the construction of h in [**?**, Lemma 3.1.1] to construct h_p, using the metric $d_p(\cdot,\cdot)$, which satisfies properties (a) and (b). Use property (b) to deduce that the convergence and the divergence of the series in (a) do not depend on the choice of $A, B \in \mathbf{Y}_n$. $\square$

Let $\mathcal{S}$ be an infinite discrete set of p-convergent type. Let $h_p(\cdot)$ be the function defined in Lemma **??**. Set

$$g^*_{s,p}(\mathcal{S}, A, B) := \sum_{\gamma\in\mathcal{S}} e^{-sd_p(A,\gamma B)} h_p(d_p(A, \gamma B)), \quad s > 0, \tag{6.4}$$

$$\mu_{\mathcal{S},s,A,p} = \frac{1}{g^*_{s,p}(\mathcal{S}, B, B)} \sum_{\gamma\in\mathcal{S}} e^{-sd_p(A,\gamma B)} h_p(d_p(A, \gamma B))\Delta_{\gamma B}, \quad s > 0.$$

Then $\mathcal{M}_{\mathcal{S},A,p}$ is the family of all measures $\mu_{\mathcal{S},A,p}$ obtained by considering all weakly convergent subsequences $\{\mu_{\mathcal{S},s_m,A,p}\}_{m=1}^\infty$, where $\{s_m\}_1^\infty$ is a strictly decreasing sequence which converges to $\delta_p(\Gamma)$. Clearly, (**??**) holds. Note that if a subsequence $\{\mu_{\mathcal{S},s_m,A,p}\}_{m=1}^\infty$ converges weakly for $A = A_0$ then this sequence converges weakly for any $A \in \mathbf{Sp}(n, \mathbb{R})/\mathbf{K}_n$ [**?**]. Hence each $\mu_{\mathcal{S},A,p}$ represents a family of measures, which depends on a parameter A.

For a measurable set $T \subset \mathbf{Y}_n$ denote by $\mathrm{vol}(T)$ the volume of T with respect to the Haar measure on $\mathbf{Y}_n$, induced by the by the Riemannian metric $||A||_2$ on $T_I\mathbf{Y}_n$. Let

$\Gamma \subset \mathbf{Sp}(n,\mathbb{R})$ be an infinite discrete group. Then $\mathcal{M}_{\Gamma,A,p}$ is the set of PS measures. We claim that $\delta_p(\Gamma) \le v_{n,p} < \infty$ for any $p \in [1,\infty]$. The constant $v_{n,p}$ is the *volume growth* of p-balls in $\mathbf{Y}_n$: Let $B_{n,p}(A,r) = \{B \in \mathbf{Y}_n : \ d_p(A,B) < r\}$ be the open p-ball of radius $r > 0$ for any $p \in [1,\infty]$. Then

$$v_{n,p} := \limsup_{r\to\infty} \frac{\log \mathrm{vol}(B_{n,p}(A,r))}{r}, \quad p \in [1,\infty].$$

Clearly, $v_{n,p}$ is independent of $A \in \mathbf{Y}_n$. In what follows we use the standard notation $f \asymp g$, for two positive functions $f(r), g(r)$ defined on (t,∞), if

$$0 < \liminf_{r\to\infty} \frac{f(r)}{g(r)} \le \limsup_{r\to\infty} \frac{f(r)}{g(r)} < \infty.$$

Proposition 6.2 *For $n > 1$, there exists a constant $\kappa_n > 0$, such that for $p \in [1,\infty]$ and $r > 0$*

$$\mathrm{vol}(B_{n,p}(I,r)) = \kappa_n \int_{\log y \in \Theta_{n,p}(r)} \prod_{1\le i<j\le n} \frac{(y_i y_j - 1)(y_i - y_j)}{y_i y_j} \prod_{1\le i\le n} \frac{(y_i^2-1)}{y_i^2} dy_1 \ldots dy_n,$$

$$\Theta_{n,p}(r) := \{x = (x_1,...,x_n) \in \mathbb{R}^n : \quad 0 \le x_n \le \ldots \le x_1, \quad ||x||_p < 2^{\frac{p-1}{p}} r\}. \tag{6.5}$$

In particular

$$v_{n,1} = n \le v_{n,p} \le v_{n,\infty} = n(n+1). \tag{6.6}$$

Proof. Recall that $\mathrm{Stab}(\sqrt{-1}I_n) = \mathbf{K}_n$. Furthermore, for each $Z \in \mathbf{SH}_n$ there exists $O \in \mathbf{K}_n$ such that $O(Z) = \sqrt{-1}\mathrm{diag}(y_1,...,y_n)$ with $y_1 \ge y_2 \ge \ldots \ge y_n \ge 1$ such that $\sigma_i(\phi_1(O(Z))) = \sigma_i(\phi_1(Z)) = \sqrt{y_i}$ for $i = 1,...,n$. Assume that $y_1 > y_2 \ldots > y_n > 1$. Then the stabilizer of $\phi_1(\sqrt{-1}\mathrm{diag}(y_1,...,y_n))$ is a finite group of diagonal matrices $\mathcal{D}_{2n} \cap \mathbf{K}_n$. Let $\mathbb{R}^n_{\ge} := \{y = (y_1,...,y_n)^{\mathrm{T}} \in \mathbb{R}^n : \ y_1 \ge y_2 \ge \ldots \ge y_n \ge 1\}$. Then up to a zero measure we have the decomposition

$$\begin{aligned} &\mathbf{SH}_n \sim \mathbf{K}_n/(\mathcal{D}_{2n} \cap \mathbf{K}_n) \times \mathbb{R}^n_{\ge}, \\ &U + \sqrt{-1}V = O(\mathrm{diag}(y_1,...,y_n)), \ O \in \mathbf{K}_n, \quad (y_1,...,y_n) \in \mathbb{R}^n_{\ge}. \end{aligned} \tag{6.7}$$

With respect to the above decomposition the ball $B_{n,p}(I,r)$ is identified with $\{y = e^x : x \in \Theta_{n,p}(r)\}$. Recall [?] that Riemannian metric on the tangent bundle of $\mathbf{SH}_n$ is given by

$$ds^2 = \mathrm{trace}(V^{-1}dUV^{-1}dU + V^{-1}dVV^{-1}dV), \quad U, V \in \mathbf{Sym}(n,\mathbb{R}), \ U + \sqrt{-1}V \in \mathbf{SH}_n.$$

We compute ds^2 for $U = 0, V = Y = \mathrm{diag}(y_1,...,y_n)$ using (**??**). Recall that the Lie algebra of $\mathbf{K}_n$ is given by

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \quad -A^{\mathrm{T}} = A = (a_{ij})_1^n, \ B^{\mathrm{T}} = B = (b_{ij})_1^n.$$

A straightforward computation shows

$$dU = dB - Y(dB)Y, \quad dV = (dA)Y - YdA + dY,$$
$$ds^2 = 2\sum_{1\le i<j\le n} \frac{(y_iy_j-1)^2(db_{ij})^2 + (y_i-y_j)^2(da_{ij})^2}{y_iy_j} + \sum_{1\le i\le n} \frac{(y_i^2-1)^2(db_{ii})^2 + (dy_i)^2}{y_i^2}.$$

Hence the volume element is

$$d\omega = 2^{n(n+1)} \prod_{1\le i<j\le n} \frac{(y_iy_j-1)(y_i-y_j)}{y_iy_j} \prod_{1\le i\le n} \frac{y_i^2-1}{y_i^2} \prod_{1\le i<j\le n} da_{ij} \prod_{1\le i\le j\le n} db_{ij} \prod_{1\le i\le n} dy_i.$$

Integrate the above expression over $\mathbf{K}_n/(\mathcal{D}_{2n}\cap \mathbf{K}_n)\times \Theta_{n,p}(r)$ to deduce (**??**). As $||x||_p$ is a decreasing function of p we deduce that $B_{n,p}(I,r)$ are increasing set in p for any fixed values of n and r. Hence $v_{n,p}$ are increasing functions in $p\in[1,\infty]$ for any integer $n>1$. We first estimate $v_{n,\infty}$ from above. Clearly

$$\prod_{1\le i<j\le n} \frac{(y_iy_j-1)(y_i-y_j)}{y_iy_j} \prod_{1\le i\le n} \frac{y_i^2-1}{y_i^2} < \prod_{1\le i<j\le n}(y_i-y_j) < \prod_{1\le i\le n} y_i^{n-i}, \quad y\in \mathbb{R}^n_{\ge},$$
$$\int_{1\le y_n\le y_{n-1}\le\ldots\le y_1\le e^{2r}} \prod_{1\le i\le n} y_i^{n-i} dy_1\ldots dy_n \asymp e^{n(n+1)r}.$$

The definition of $v_{n,\infty}$, (**??**) and the above inequalities yield $v_{n,\infty}\le n(n+1)$. Fix $\epsilon>0$ and let $\Theta_{n,p,\epsilon}(r) := \{x\in\Theta_{n,\infty}(r):\ \epsilon\le x_n,\ \ x_i+\epsilon\le x_{i+1},\ \ i=1,...,n-1\}$. A straightforward argument show that

$$\int_{\log y\in\Theta_{n,\infty,\epsilon}(r)} \prod_{1\le i<j\le n} \frac{(y_iy_j-1)(y_i-y_j)}{y_iy_j} \prod_{1\le i\le n} \frac{(y_i^2-1)}{y_i^2} dy_1\ldots dy_n \asymp$$
$$\int_{\log y\in\Theta_{n,\infty,\epsilon}(r)} \prod_{1\le i\le n} y_i^{n-i} dy_1\ldots dy_n \asymp e^{n(n+1)r} \asymp e^{n(n+1)r}.$$

Hence $v_{n,\infty}=n(n+1)$. Similar arguments show that $v_{n,1}=n$. □

Recall that $\Gamma\subset \mathbf{Sp}(n,\mathbb{R})$ is called a lattice if Γ is discrete and $\mathrm{vol}(\mathbf{SH}_n/\Gamma)<\infty$. Siegel modular group $\mathbf{Sp}(n,\mathbb{Z})$ is a lattice. The volume estimate for discrete groups and lattices [**?**], [**?**], [**?**] combined with Proposition **??** yield:

Theorem 6.3 *Let $\Gamma\subset \mathbf{Sp}(n,\mathbb{R})$ be a discrete group. Then $\delta_p(\Gamma)\le v_{n,p}$. Assume that Γ is a lattice. Then $\delta_p(\Gamma)=v_{n,p}$ and Γ is of divergence type.*

Definition 6.4 *Let $\Gamma < \mathbf{Sp}(n,\mathbb{R})$ be a discrete group and $n>1$. Then Γ is called p-regular if for any $\mu\in\mathcal{M}_{\Gamma,A,p}$* supp $\mu\subset\Lambda(\Gamma)$.

We are interested in conditions which insure that for a given $p\in[1,\infty]$ Γ is p-regular. For a fixed $t\ge 0$ let

$$\mathbf{Sp}(n,\mathbb{R})_t := \{\gamma\in\mathbf{Sp}(n,\mathbb{R}):\quad \sigma_n(\gamma)\le e^t\},\quad \Gamma_t := \Gamma\cap\mathbf{Sp}(n,\mathbb{R})_t.$$

Definition 6.5 *Let $\Gamma < \mathbf{Sp}(n, \mathbb{R})$ be a discrete group. Then Γ is called p-strongly regular if for any $t \geq 0$: $\delta_p(\Gamma_t) < \delta_p(\Gamma)$.*

Lemma 6.6 *Let $\Gamma \subset \mathbf{Sp}(n, \mathbb{R})$ be a p-strongly regular discrete subgroup of $\mathbf{Sp}(n, \mathbb{R})$ for some $p \in [1, \infty]$. Then $\delta_p(\Gamma) > 0$, $\Lambda(\Gamma) \neq \emptyset$, and* supp $\mu \subset \Lambda(\Gamma)$ *for every $\mu \in \mathcal{M}_{\Gamma,A,p}$.*

Proof. Since $I \in \Gamma$ it follows that $\Gamma_t \neq \emptyset$. Hence $0 \leq \delta_p(\Gamma_t) < \delta_p(\Gamma)$. Moreover, Γ contains a sequence $\{\gamma_k\}_1^\infty$ which satisfies the condition (**??**). Hence $\Lambda(\Gamma) \neq \emptyset$. Assume first that Γ is of divergence type. Fix $t > 0$. Let

$$\mu_{\Gamma,s,A,p,t} = \frac{1}{g_{s,p}(\Gamma, B, B)} \sum_{\gamma \in \Gamma_t} e^{-sd_p(A,\gamma B)} \Delta_{\gamma B}, \quad s > \delta_p(\Gamma).$$

Then for any sequence $s_m \searrow \delta_p(\Gamma)$ $\mu_{\Gamma,s_m,p,t} \to 0$. Corollary **??** and Theorem **??** yield that for any $\mu \in \mathcal{M}_{\Gamma,A,p}$ supp $\mu \subset \Lambda(\Gamma)$. Similar arguments apply if Γ is of p-convergence type. □

Lemma 6.7 *Let Γ be a lattice in $\mathbf{Sp}(n, \mathbb{R})$. Then Γ is strongly regular.*

Proof. Fix $t > 0$. Let $v_{n,p,t}$ be the volume growth of $B(A, r) \cap \mathbf{Sp}(n, \mathbb{R})_t$. Observe that $B(A, r) \cap \mathbf{Sp}(n, \mathbb{R})_t$ has the decomposition (**??**) with $\{y = e^x : \ x \in \Theta_{n,p}(r), \ x_n \leq t\}$. Use the arguments of the proof of Proposition **??** to deduce that $v_{n,p,t} < v_{n,p}$. As Γ is a lattice the volume estimates yield $\delta(\Gamma_t) = v_{n,p,t} < v_{n,p} = \delta_p(\Gamma)$. □

Note that our results for lattices are analogous to the results of [**?**]. As in [**?**, §4], recent results of Benoist [**?**] imply the existence of many discrete Zariski dense subgroups Γ of $\mathbf{Sp}(n, \mathbb{R})$ which are p-regular for any $p \in [1, \infty]$. Let $H(\Gamma) \subset \mathbb{R}^{2n}$ be the set of rays spanned by all limit directions of the sequences

$$\frac{\log \sigma(\gamma_k)}{||\log \sigma(\gamma_k)||_2}, \quad \gamma_k \in \Gamma, \ \ k = 1, ..., \quad \lim_{k \to \infty} ||\log \sigma(\gamma_k)||_2 = \infty.$$

As $\log \sigma(\gamma) = -\log \sigma(\gamma)$ for any $\gamma \in \mathbf{Sp}(n, \mathbb{R})$ we deduce the $-H(\Gamma) = H(\Gamma)$. It is shown in [**?**] that if Γ is a Zariski dense subgroup in $\mathbf{Sp}(n, \mathbb{R})$ then $H(\Gamma)$ is a closed convex cone in $\mathbb{R}^{2n}$. Clearly, this cone can be identified with a subcone of $\mathbb{R}^n_+$ (the cone of all nonnegative vectors in $\mathbb{R}^n$). Benoist shows that for any closed convex cone $\mathcal{K} \subset \mathbb{R}^n_+$ there exists a Zariski dense subgroup $\Gamma \subset \mathbf{Sp}(n, \mathbb{R})$ such that $PH(\Gamma) = \mathcal{K}$, where $P : \mathbb{R}^{2n} \to \mathbb{R}^n$ is the projection given by $P(x_1, ..., x_{2n})^{\mathrm{T}} = (x_1, ..., x_n)^{\mathrm{T}}$.

Definition 6.8 *A discrete subgroup $\Gamma \subset \mathbf{Sp}(n, \mathbb{R})$ is called generic if Γ is Zariski dense in $\mathbf{Sp}(n, \mathbb{R})$ and any nonzero vector $x \in H(\Gamma)$ has nonzero coordinates.*

Proposition 6.9 *Let Γ be a generic subgroup of $\mathbf{Sp}(n, \mathbb{R})$. The Γ is p-regular for any $p \in [1, \infty]$.*

Proof. As Γ is generic, we easily deduce that the set Γ_t is a finite set for any $t \geq 0$. The arguments of the proof of Lemma **??** yield that supp $\mu \subset \Lambda(\Gamma)$ for any $\mu \in \mathcal{M}_{\Gamma,A,p}$. □

Theorem 6.10 *Let Γ be a discrete Zariski dense subgroup of $\mathbf{Sp}(n,\mathbb{R})$. Then $\delta_p(\Gamma) > 0$ for any $p \in [1,\infty]$.*

Proof. The results of Tits [**?**] (see also the results on Schottky groups in [**?**]) imply that Γ contains a free subgroup Γ' on $k \geq 2$ generators such that Γ' is Zariski dense in $\mathbf{Sp}(n,\mathbb{R})$. Fix $p \in [1,\infty]$. Clearly, $\delta_p(\Gamma') \leq \delta_p(\Gamma)$. We use the results in [**?**] to show that $\delta_p(\Gamma') > 0$. From here until the end of the proof we refer by numbers to the displayed formulas, Theorems and Corollaries in [**?**]. Let $\gamma_1, ..., \gamma_k$ be a minimal set of generators of Γ'. Associate with these generators a subshift $\mathcal{S}$ of finite type on $2k$ letters $1, ..., 2k$. Here the letter $i \in [1,k]$ corresponds to the generator γ_i and the letter $j \in [k+1, 2k] \cap \mathbb{Z}$ corresponds the generator $\gamma_j := \gamma_{j-k}^{-1}$. $\mathcal{S}$ is the set of reduced infinite words

$$w = \gamma_{i_1}\gamma_{i_2}\ldots, \quad i_j \in [1,2k] \cap \mathbb{Z}, \ \ j = 1, ..., \quad |i_j - i_{j+1}| \neq k, \ \ j = 1, ... \tag{6.8}$$

$\mathcal{S}$ is a compact topological space respect to product topology. Let $\tau : \mathcal{S} \to \mathcal{S}$ be the shift map given by $\tau(w) = \gamma_{i_2}\gamma_{i_3}\ldots$. Let $w_m = \gamma_{1_1}\ldots\gamma_{i_m}$ be a reduced word of length m. Then $C(w_m) \subset \mathcal{S}$ is the set of all infinite words in $\mathcal{S}$ which start with w_m. Define the function $\phi_m : \mathcal{S} \to \mathbb{R}_+$ by assuming that ϕ_m is constant on each $C(w_m)$ and its value is equal to $d_p(I, w_m)$ which is denoted by $\phi_m(w_m)$. As $\mathbf{Sp}(n,\mathbb{R})$ acts as a subgroup of isometries with respect to the metric $d_p(\cdot,\cdot)$ on $\mathbf{Sp}(n,\mathbb{R})/\mathbf{K}_n$, we deduce that the family $\{\phi_m\}_1^\infty$ satisfies the conditions (0.1). Since Γ' is discrete the condition (0.2) holds. Hence the sequence $\{\phi_m\}_1^\infty$ defines a metric $d : \mathcal{S} \times \mathcal{S} \to \mathbb{R}^+$ given by (0.3). Let $\delta(\phi)$ be the Hausdorff dimension of $\mathcal{S}$ with respect to d. Observe next that $\kappa(\phi)$ defined in (1.12) of is equal to $\delta_p(\Gamma')$. Theorem 1.14 yields $\delta_p(\Gamma') \geq \delta(\phi)$. Let $\mathcal{E}$ be the set of ergodic measures on $\mathcal{S}$ with respect to τ. For $\nu \in \mathcal{E}$ one can define the ν-Hausdorff dimension of $\mathcal{E}$ denoted by $\delta(\nu,\phi)$. By the definition $\delta(\nu,\phi) \leq \delta(\phi)$. Theorem 2.4 and Corollary 2.6 yield $\delta(\nu,\phi) = \frac{h(\nu)}{\alpha(\nu)} \geq \frac{h(\nu)}{\alpha_1(\nu)}$. Here $h(\nu)$ is the entropy of ν and

$$\alpha_1(\nu) = \sum_{j=1}^{2k} d_p(I,\gamma_j)\nu(C(\gamma_j)) \leq \max_{1\leq j\leq 2k} d_p(I,\gamma_j).$$

Note that for any nontrivial $\gamma \in \Gamma'$ $d_p(I,\gamma) > 1$. Otherwise $\gamma \in \mathbf{K}_n$ and $< \gamma >$ is a discrete, hence a finite subgroup of $\mathbf{K}_n$. This contradicts the freeness of Γ'. Let ν_P be the equidistributed measure given by $\nu_P(C(w_m)) = \frac{1}{2k(2k-1)^{m-1}}$ for $m \in \mathbb{N}$. Then Corollary 2.10 yields

$$\delta_p(\Gamma') \geq \delta(\phi) \geq \delta(\nu_P,\phi) \geq \frac{\log(2k-1)}{\alpha_1(\nu_P)}, \ \alpha_1(\nu_P) = \frac{1}{2k}\sum_{j=1}^{2k} d_p(I,\gamma_j),$$

and the theorem follows. $\square$

In [**?**] we show that for a Kleinian Schottky group Γ we have equalities $\kappa(\phi) = \delta(\phi) = \sup_{\nu\in\mathcal{E}} \frac{h(\nu)}{\alpha(\nu)}$. It is an interesting problem if the above equalities hold for a generic subgroup $\Gamma \subset \mathbf{Sp}(n,\mathbb{R})$. Theorem **??** can be considered as a generalization of the result of Beardon [**?**] that the Hausdorff dimension of a nonelementary Kleinian group is positive (see [**?**, (4.1)]).

Corollary 6.11 *Let Γ be a generic subgroup of $\mathbf{Sp}(n,\mathbb{R})$. Then Γ is strongly p-regular for any $p \in [1,\infty]$.*

Proof. As Γ_t is a finite set $\delta_p(\Gamma_t) = 0$. Theorem **??** implies that $\delta_p(\Gamma) > 0$. $\square$

In what follows we restrict our attention to $p = 1$. Recall that Busemann 1-compactification of $\mathbf{Y}_n$ gives the compactification of $\mathbf{SH}_n$ as a bounded domain. In view of Corollary **??** we identify the compact strata of the Busemann 1-boundary of $\mathbf{Y}_n$ with the Shilov boundary of $\mathbf{SH}_n$. Let Γ be a discrete subgroup of $\mathbf{Sp}(n,\mathbb{R})$. By abuse of notation we view $\Lambda(\Gamma)$ as a closed subset of $\partial_1 \mathbf{Y}_n$. Use the definition of the Busemann functions and the standard arguments for Patterson-Sullivan measure as in [**?**] and [**?**] to obtain:

Theorem 6.12 *Let Γ be a discrete 1-regular subgroup of $\mathbf{Sp}(n,\mathbb{R})$. Choose a family of Patterson-Sullivan measures $\mu_X \in \mathcal{M}_{\Gamma,X,1}$ depending on a parameter $X \in \mathbf{Y}_n$. Then*
(a) $\gamma^* \mu_X = \mu_{\gamma^{-1}(X)}$ *for $\gamma \in \Gamma$;*
(b) $\frac{d\mu_X}{d\mu_{X_0}}(\xi) = -\delta_1(\Gamma) b_{\xi,1}(X)$ *for any $\xi \in \Lambda(\Gamma)$, where $b_{\xi,1}$ is computed for the reference point X_0.*

7 Modified Patterson-Sullivan measures

In this section we assume that $\Lambda(\Gamma) \neq \emptyset$. We suggest here another definition of the PS measures $\tilde{M}_{\Gamma,A,p}$ so that supp $\mu \subset \Lambda(\Gamma)$ for any $\mu \in \tilde{M}_{\Gamma,A,p}$. For any subgroup $G \subset \mathbf{Sp}(n,\mathbb{R})$ let $G^t := G \backslash \mathbf{Sp}(n,\mathbb{R})_t$. The assumption $\Lambda(\Gamma) \neq \emptyset$ yields that Γ^t is an infinite set for any $t \geq 0$. We now consider families $\mathcal{M}_{\Gamma^t,A,p}$. Clearly, supp $\mu \subset \mathrm{BCl}(\Gamma^t(B))$ for any $\mu \in \mathcal{M}_{\Gamma^t,A,p}$. Let $\tilde{\mathcal{M}}_{\Gamma,A,p}$ be the set of weak limits of measures in $\mathcal{M}_{\Gamma^t,A,p}$ as $t \to \infty$. Note that each $\mu \in \mathcal{M}_{\Gamma^t,B,p}$ is a probability measure on $\mathrm{BCl}(\mathbf{SH}_n)$. Hence $\tilde{\mathcal{M}}_{\Gamma,B,p}$ is a set of probability measures which is supported on $\Lambda(\Gamma)$. Thus $\tilde{\mathcal{M}}_{\Gamma,A,p}$ is a set of positive finite measures which is supported on $\Lambda(\Gamma)$.

Proposition 7.1 *Assume that Γ is strongly p-regular. If Γ is of p-divergence type then for each $t \geq 0$ $\mathcal{M}_{\Gamma^t,A,p} = \mathcal{M}_{\Gamma,A,p}$. In particular $\tilde{\mathcal{M}}_{\Gamma,A,p} = \mathcal{M}_{\Gamma,A,p}$. Assume that Γ is of p-convergence then for each $t \geq 0$ it is possible to choose $\mathcal{M}_{\Gamma^t,A,p}$ to be equal to $\mathcal{M}_{\Gamma,A,p}$. For these choices $\tilde{\mathcal{M}}_{\Gamma,A,p} = \mathcal{M}_{\Gamma,A,p}$.*

Proof. Assume first that Γ is of p-divergence type. Then Γ^t is of p-divergence type. The arguments of the proof of Lemma **??** imply the equality $\mathcal{M}_{\Gamma^t,A,p} = \mathcal{M}_{\Gamma,A,p}$. Assume that Γ is of convergence type. Fix the function $h_p : \mathbb{R}_+ \to \mathbb{R}_+$ such that the series $g^*_{s,p}(\Gamma, A, B)$ given by (**??**) diverges for $s = \delta_p(\Gamma)$. For $t \geq 0$ choose $h_{p,t} = h_p$. Then the series $g^*_{s,p}(\Gamma^t, A, B)$ diverges for $s = \delta_p(\Gamma)$ and converges for $s > \delta_p(\Gamma)$. For this choice of $h_{p,t}$ $\mathcal{M}_{\Gamma^t,A,p} = \mathcal{M}_{\Gamma,A,p}$.
$\square$

Let $\overline{\delta}_p(\Gamma) = \limsup_{t\to\infty} \delta_p(\Gamma^t)$ and $\underline{\delta}_p(\Gamma) = \liminf_{t\to\infty} \delta_p(\Gamma^t)$. We conjecture that for any Γ, $\Lambda(\Gamma) \neq \emptyset$ each $\mu_X \in \tilde{\mathcal{M}}_{\Gamma,X,p}$ is a β density:

$$\frac{d\mu_X}{d\mu_{X_0}}(\xi) = e^{-\beta b_{\xi,1}(X)}, \quad \xi \in \Lambda(\Gamma), \quad \beta \in [\underline{\delta}_1(\Gamma), \overline{\delta}_1(\Gamma)].$$

Theorem 7.2 *Let Γ be a discrete Zariski dense subgroup of $\mathbf{Sp}(n,\mathbb{R})$. Then $\underline{\delta}_p(\Gamma) > 0$ for any $p \in [1,\infty]$.*

Proof. We use the notations and the results of the proof of Theorem **??**. Consider the free Zariski dense group Γ' on $k > 1$ generators, the associated subshift of finite type $\mathcal{S}$ on $2k$ letters and an ergodic measure $\nu \in \mathcal{E}$. With each infinite reducible word $w \in \mathcal{S}$ of the form (**??**) we associate the matrix cocycle $\mathcal{A}(w,m) = w_m = \gamma_{i_1}\ldots\gamma_{i_m} \in \mathbf{Sp}(n,\mathbb{R})$. One can view $\mathcal{A}(w,m)$ as a random product of m matrices from the set $\{\gamma_1,...,\gamma_{2k}\}$ with respect to the stationary measure ν. The fundamental result of Oseledets [**?**] claims that

$$\lim_{m\to\infty} \frac{\log \sigma_i(w_m)}{m} = \lambda_i(w,\nu), \quad i = 1,...,2n, \tag{7.1}$$

for almost all $w \in \mathcal{S}$ with respect ν. Since $\Gamma' \subset \mathbf{Sp}(n,\mathbb{R})$ we deduce that $\lambda_i(w,\nu) = -\lambda_{2n+1-i}(w,\nu)$ for $i = 1,...,2n$. As ν is ergodic we obtain $\lambda_i(w,\nu) = \lambda_i(\nu),\ i = 1,...,2n$ are ν-Lyapunov exponents of Γ'. Since $\mathbf{Sp}(n,\mathbb{R})$ is a simple group, the fundamental result of Goldsheid-Margulis [**?**] claims that all the ν-Lyapunov exponents are simple:

$$\lambda_1(\nu) > \ldots > \lambda_n(\nu) > \lambda_{n+1}(\nu) > \ldots > \lambda_{2n}(\nu).$$

As $\lambda_{n+1}(\nu) = -\lambda_n(\nu)$ we deduce that $\lambda_n(\nu) > 0$. That is, for a.a. w with respect to ν $\sigma_n(w_m) \asymp e^{m\lambda_n(\nu)}$. The arguments of the proofs of Theorem 2.4 and Corollary 2.6 in [**?**] imply that $\delta_p((\Gamma')^t) \geq \delta(\phi) \geq \delta(\nu,\phi)$ for any $t \geq 0$. Hence $\underline{\delta}_p(\Gamma') \geq \delta(\phi) \geq \delta(\nu,\phi)$. Choose $\nu = \nu_p$ to deduce that $\underline{\delta}_p(\Gamma) \geq \underline{\delta}_p(\Gamma') \geq \delta(\nu_p,\phi) > 0$. □

8 Critical exponent

We now define the critical exponent for the action of Γ on the Shilov boundary $\partial_n\mathbf{SH}_n$ as it done for Kleinian groups [**?**, (1.1)]. As in Proposition **??** we identify ξ with a unit vector in $\mathbb{R}^{\binom{2n}{n}}$ in the one dimensional subspace $\wedge_n\Xi \subset \mathbb{R}^{\binom{2n}{n}}$. Note that ξ is determined up to a sign. Then

$$\mathrm{dist}(\xi,\eta) := \min(||\xi-\eta||_2, ||\xi+\eta||_2), \quad \xi,\eta \in \partial_n\mathbf{SH}_n.$$

Definition 8.1 *A discrete group $\Gamma \subset \mathbf{Sp}(n,\mathbb{R})$ is called a Siegel group, if $n > 1$, $\Lambda(\Gamma) \neq \emptyset$ and $\Lambda_d(\Gamma)$ is strictly contained in the Shilov boundary of $\mathbf{SH}_n$.*

Note that the Siegel modular group $\mathbf{Sp}(n,\mathbb{Z})$ is a lattice in $\mathbf{Sp}(n,\mathbb{R})$. It can be shown that $\Lambda(\mathbf{Sp}(n,\mathbb{Z}))$ is the Shilov boundary of $\mathbf{SH}_n$. Hence $\mathbf{Sp}(n,\mathbb{Z})$ is not a Siegel group. It is not difficult to show that if $\Gamma_i \in \mathbf{SL}(2,\mathbb{R})$ are Schottky Fuchsian groups for $i = 1,...,n$

then $\Gamma := \Gamma_1 \odot \ldots \odot \Gamma_n < \mathbf{Sp}(n, \mathbb{R})$ is a Siegel group. The critical exponent $\epsilon(\Gamma, \xi)$ of Siegel group Γ is defined as

$$\epsilon(\Gamma, \xi) := \inf \ \{s > 0 : \ \sum_{\gamma \in \Gamma} \text{dist}(\gamma(\xi), \Lambda_d(\Gamma))^s\}, \quad \xi \in \Omega(\Gamma). \tag{8.1}$$

It seems that $\epsilon(\Gamma, \xi)$ does not depend on $\xi \in \Omega(\Gamma)$. The interesting question is how $\epsilon(\Gamma, \xi)$ is related to $\delta_1(\Gamma)$.

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