The number of singular vector tuples and approximation of symmetric tensors

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Notations

Indices: $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d, \quad [m] := \{1, \ldots, m\}$,

$\mathbb{F} = \mathbb{C}, \mathbb{R}$, $J = \{j_1, \ldots, j_k\} \subset [d]$

Tensors: $\otimes_{i=1}^d \mathbb{F}^{m_i} = \mathbb{F}^{m_1 \times \ldots \times m_d} = \mathbb{F}^{\mathbf{m}}$

Contraction of $\mathcal{T} = [t_{i_1,\ldots,i_d}] \in \mathbb{F}^{\mathbf{m}}$ with $\mathcal{X} = [x_{i_{j_1},\ldots,i_{j_k}}] \in \otimes_{j_p \in J} \mathbb{F}^{m_{j_p}}$:

$\mathcal{T} \times \mathcal{X} = \sum_{i_{j_p} \in [m_{j_p}], j_p \in J} t_{i_1,\ldots,i_d} x_{i_{j_1},\ldots,i_{j_k}} \in \otimes_{l \in [d] \setminus J} \mathbb{F}^{m_l}$

Example $\mathcal{T} \times (\mathbf{x}_1 \otimes \ldots \otimes \mathbf{x}_{k-1} \otimes \mathbf{x}_{k+1} \otimes \ldots \otimes \mathbf{x}_d) =$
$\sum_{i_j \in [m_j], j \in [d] \setminus \{k\}} t_{i_1,\ldots,i_d} \prod_{j \in [d] \setminus \{k\}} x_{i_j,j}$

is a vector in $\mathbb{F}^{m_k}$

$\|\mathcal{T}\| = \sqrt{\mathcal{T} \times \mathcal{T}}$ - Hilbert-Schmidt norm of $\mathcal{T} \in \mathbb{R}^{\mathbf{m}}$

Singular values and vectors for tensors

Introduced by Lek-Heng Lim 2005

$\mathcal{T} \times (\mathbf{x}_1 \otimes \ldots \otimes \mathbf{x}_{k-1} \otimes \mathbf{x}_{k+1} \otimes \ldots \otimes \mathbf{x}_d) = \lambda \mathbf{x}_k, \ \|\mathbf{x}_k\| = 1, \ k \in [d]$ (1)

critical points of d-linear form $\mathcal{T} \times \otimes_{j\in[d]}\mathbf{x}_j$ restricted to $\mathrm{S}(\mathbf{m})$ where

$\mathrm{S}(\mathbf{m}) = S^{m_1-1} \times \ldots \times S^{m_d-1}, \mathrm{S}^{m-1} := \{\mathbf{x} \in \mathbb{R}^m, \ \|\mathbf{x}\| = 1\}$

$C(\mathbf{m}) := \mathbb{R}^{m_1} \times \ldots \mathbb{R}^{m_d}$ variety of rank one tensors (+zero tensor)

Claim: Singular tuples of $\mathcal{T}$ are the critical points of $\mathrm{dist}(\mathcal{T}, C(\mathbf{m}))$.

$\min_{t\in\mathbb{R}} \|\mathcal{T} - t \otimes_{j\in[d]} \mathbf{x}_j\|_2 = \|\mathcal{T} - \mathrm{Proj}_{\mathrm{span}(\otimes_{j\in[d]}\mathbf{x}_j)}(\mathcal{T})\|_2$

$\|\mathrm{Proj}_{\mathrm{span}(\otimes_{j\in[d]}\mathbf{x}_j)^\perp(\mathcal{T})}\|_2$

$\|\mathcal{T}\|_2^2 = \|\mathrm{Proj}_{\mathrm{span}(\otimes_{j\in[d]}\mathbf{x}_j)}(\mathcal{T})\|_2^2 + \|\mathrm{Proj}_{\mathrm{span}(\otimes_{j\in[d]}\mathbf{x}_j)^\perp(\mathcal{T})}\|_2^2$

$\|\mathrm{Proj}_{\mathrm{span}(\otimes_{j\in[d]}\mathbf{x}_j)}(\mathcal{T})\|_2 = |\mathcal{T} \times \otimes_{j\in[d]}\mathbf{x}_j|$ for $(\mathbf{x}_1, \ldots, \mathbf{x}_d) \in \mathrm{S}(\mathbf{m})$

$\mathrm{dist}(\mathcal{T}, C(\mathbf{m}))^2 = |\mathcal{T}|_2^2 - \max_{(\mathbf{x}_1,\ldots,\mathbf{x}_m)\in\mathrm{S}(\mathbf{m})}(\mathcal{T} \times \otimes_{j\in[d]}\mathbf{x}_j)^2$

Number of singular tuples of a generic tensor

Problem: Is the number of singular values and singular vectors is finite for a generic $\mathcal{T} \in \mathbb{R}^{\mathbf{m}}$ an if yes what is the number?

More precisely $(\mathbf{x}_1, \ldots, \mathbf{x}_d) \in \mathbb{P}(\mathbb{R}^{m_1}) \times \ldots \mathbb{P}(\mathbb{R}^{m_d})$ is a singular tuple if

$\mathcal{T} \times (\mathbf{x}_1 \otimes \ldots \otimes \mathbf{x}_{k-1} \otimes \mathbf{x}_{k+1} \otimes \ldots \otimes \mathbf{x}_d) = \lambda_j \mathbf{x}_k, \ k \in [d]$ (2)

We will consider complex sigular tuples $(\mathbf{x}_1, \ldots, \mathbf{x}_d)$

in Segre variety $\Sigma(\mathbf{m}, \mathbb{C}) := \mathbb{P}(\mathbb{C}^{m_1}) \times \ldots \times \mathbb{P}(\mathbb{C}^{m_d})$

For real tensors and real singular tuples (2) reduces to (1) with $\pm\lambda$

Number of complex singular tuples

Number of complex singular tuples is $c(\mathbf{m})$,

the coefficient of $t^{m_1-1}\ldots t^{m_d-1}$ in

$f(\mathbf{t},\mathbf{m}) := \prod_{i=1}^{d} \frac{\hat{t}_i^{m_i}-t_i^{m_i}}{\hat{t}_i-t_i}$, where $\hat{t}_i = t_1+\ldots+t_{i-1}+t_{i+1}+\ldots+t_d$

For $d=2$, $c(m_1,m_2)=\min(m_1,m_2)$ as expected:

$$\frac{\hat{t}_1^{m_1}-t_1^{m_1}}{\hat{t}_1-t_1}\frac{\hat{t}_2^{m_2}-t_2^{m_2}}{\hat{t}_2-t_2} = \frac{t_2^{m_1}-t_1^{m_1}}{t_2-t_1}\frac{t_1^{m_2}-t_2^{m_2}}{t_1-t_2} = (\sum_{i=1}^{m_1} t_1^{m_1-i}t_2^{i-1})(\sum_{j=1}^{m_2} t_2^{m_2-j}t_1^{j-1})$$

Stabilization: $c(m,n,p)=c(m,,n,p(m,n))$ for

$p\ge p(m,n)\ge n\ge m\ge 2$ where $p(m,n)=m+n-1$

$p(m,n)=m+n-1$ boundary format case for hyperdeterminants

For any $m_1,\ldots,m_d\ge 1$ $c(\mathbf{m})$ stabilizes for

$p(m_1,\ldots,m_d)=m_1+m_2+\ldots+m_{d-1}-(d-2)$

Note this is valid also for $d=2$

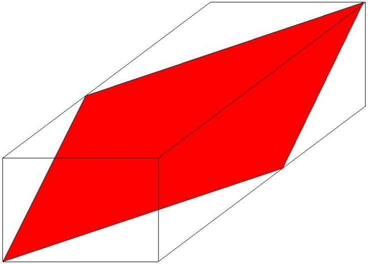
Some values of $c(m, n, p)$ I

d_1, d_2, d_3	$c(d_1, d_2, d_3)$	
$2, 2, 2$	6	
$2, 2, n$	8	$n \geq 3$
$2, 3, 3$	15	
$2, 3, 4$	18	$n \geq 4$
$2, 4, 4$	28	
$2, 4, n$	32	$n \geq 5$
$2, 5, 5$	45	
$2, 5, n$	50	$n \geq 6$
$2, m, m+1$	$2m^2$	
$3, 3, 3$	37	
$3, 3, 4$	55	
$3, 3, n$	61	$n \geq 5$
$3, 4, 4$	104	
$3, 4, 5$	138	
$3, 4, n$	148	$n \geq 6$
$3, 5, 5$	225	

Some values of $c(m, n, p)$ II

d_1, d_2, d_3	$c(d_1, d_2, d_3)$	
$3, 5, 6$	280	
$3, 5, n$	295	$n \geq 7$
$3, m, m+2$	$\frac{8}{3}m^3 - 2m^2 + \frac{7}{3}m$	
$4, 4, 4$	240	
$4, 4, 5$	380	
$4, 4, 6$	460	
$4, 4, n$	480	$n \geq 7$
$4, 5, 5$	725	
$4, 5, 6$	1030	
$4, 5, 7$	1185	
$4, 5, n$	1220	$n \geq 8$
$5, 5, 5$	1621	
$5, 5, 6$	2671	
$5, 5, 7$	3461	
$5, 5, 8$	3811	
$5, 5, n$	3881	$n \geq 9$

Stabilization



An outline for computation of $c(\mathbf{m})$

We construct a natural vector vector bundle $E(\mathbf{m})$ on Segre variety

$\Sigma(\mathbf{m}) := \mathbb{P}(\mathbb{C}^{m_1}) \times \ldots \times \mathbb{P}(\mathbb{C}^{m_d})$

At each factor $\mathbb{P}(\mathbb{C}^{m_i})$ associate v.b. E_i, dual to quotient of

tautological bundle at $[\mathbf{x}_i] \in \mathbb{P}(\mathbb{C}^{m_i})$ v.b. $E_i|_{[\mathbf{x}_i]} = (\mathbb{C}^{m_i}/\mathrm{span}(\mathbf{x}_i))'$

Then $E(\mathbf{m})|_{([\mathbf{x}_1],\ldots,[\mathbf{x}_d])} = \oplus_{i=1}^d E_i([\mathbf{x}_i])$

Each $\mathcal{T} \in \mathbb{C}^{\mathbf{m}}$ induces a section in $E(\mathbf{m})$

For each $([\mathbf{x}_1],\ldots,[\mathbf{x}_d]) \in \Sigma(\mathbf{m})$ the vector in $E(\mathbf{m})$ is

$\mathbf{u}(\mathbf{x}_1,\ldots,\mathbf{x}_d) := \oplus_{i=1}^d \mathcal{T} \times \otimes_{j\in[d]\setminus\{i\}} \mathbf{x}_j \in E(\mathbf{m})|_{([\mathbf{x}_1],\ldots,[\mathbf{x}_d])}$

$(\mathbf{x}_1,\ldots,\mathbf{x}_d)$ is a singular tuple of $\mathcal{T}$ iff $\mathbf{u}(\mathbf{x}_1,\ldots,\mathbf{x}_d) = \mathbf{0}$

Bertini's type theorem yields: section of generic $\mathcal{T}$ has finite number

of zeros. This number is the top Chern class of $E(\mathbf{m})$

Approximation of symmetric tensors: rank one at most

$m^{\times d} := (\underbrace{m, \ldots, m}_{d})$, $\mathbb{R}^n = \mathbb{R}^{m^{\times d}}$, $\mathrm{S}(d, \mathbb{R}^n)$ symmetric tensors

C_k - tensors of border rank at most k

Thm There exists a semi-algebraic set $Q \subset \mathrm{S}(d, \mathbb{R}^m), \dim Q < \binom{m+d-1}{d}$

for $\mathcal{T} \in \mathrm{S}(d, \mathbb{R}^n) \setminus Q$ best rank 1-approximation unique, and symmetric

Prf. 1. Banach 1939, Chen-He-Li-Zhang 2012, Friedland 2013:

best rank 1-approxim. of symmetric tensor can be chosen symmetric

2. Friedland-Ottaviani: $f := \mathrm{dist}(\cdot, C_1)|\mathrm{S}(d, \mathbb{R}^m)$. If f differentiable at $\mathcal{T}$

then best rank 1-approximation unique up to permutation of factors in

$\mathcal{X} = \mathbf{x}_1 \otimes \ldots \otimes \mathbf{x}_d$. Use 1. to deduce $\mathcal{X}$ symmetric

3. Friedland-Stawiska: the set Q of symmetric tensor with

not unique best rank approximation is semi-algebraic

Approximation of symmetric tensors: b. rank k at most

$N(m,d)=\frac{1}{2}(\binom{m+d-3}{d-2}+2m-2)$ for $d\ge 3$, $\quad (N(m,3)=\frac{3m-2}{2})$

Thm For $d\ge 3, 2\le k\le N(m,d)$ the semi-algebraic set of all symmetric tensors for which best border rank k approximation is unique, (denoted as $P_k\subset\mathbb{R}^n$), has dimension $\binom{m+d-1}{d}$.

Use Kruskal's theorem to show that a symmetric tensor of the form

$\mathcal{T}=\sum_{i=1}^k\otimes^d\mathbf{u}_i$, k-as above

has rank k if any $\min(m,k)$ vectors from $\mathbf{u}_1,\ldots,\mathbf{u}_k$ and $\min(k,\binom{m+d-3}{d-2})$ vectors from $\otimes^{d-2}\mathbf{u}_1,\ldots,\otimes^{d-2}\mathbf{u}_k$ linearly independent

Problem : Is $\dim(\mathbb{R}^n\setminus P_k)<\binom{m+d-1}{d}$?

Weaker problem: Is the best border rank k-approximation to a symmetric tensor can be chosen symmetric?

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