

# MULTI-DIMENSIONAL CAPACITY

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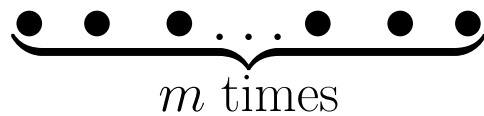
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## 1 INTR: UNRESTRICTED CAPACITY

$$\langle n \rangle := \{1, \dots, n\}$$

IS AN ALPHABET ON  $n$  LETTERS.



LINEAR STORAGE OF LENGTH  $m$ .

# STORAGE MESSAGES  $n^m$ .

(UNRESTRICTED) CAPACITY:

$$\log_2 n^m = \frac{\log_2 n^m}{m}.$$

## 2 INTR: (0 – 1) LIMITED CHANNEL

$$n = 2, \langle 2 \rangle = \{1, 2\} = \{1, 0\} \quad (2 \equiv 0).$$

NO TWO 1's ARE NEIGHBORS.

# OF  $m$ -MESSAGES IS  $u_m$ ;

$$u_{m+1} = u_m + u_{m-1}, \quad m = 1, 2, \dots$$

FIBONACCI SEQUENCE 2, 3, 5, 8, ...

$$\text{CAPACITY: } \log_2 \frac{1+\sqrt{5}}{2} = 0.694241914.$$

### 3 INTR: MULTI-DIMENSIONAL CAPACITY

$\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$  – integers

$\mathbb{Z}_+ = \{0, 1, \dots\}$  – nonnegative integers

$\mathbb{N} = \{1, 2, \dots\}$  – positive integers

$d$  – the dimension,  $1 < d \in \mathbb{N}$

$\mathbf{m} := (m_1, \dots, m_d) \in \mathbb{Z}^d$

$|\mathbf{m}| = |m_1| + \dots + |m_d|$

$|\mathbf{m}|_{pr} := |m_1| \times \dots \times |m_d|$

for  $\mathbf{m} \in \mathbb{N}^d$

$\langle \mathbf{m} \rangle := \langle m_1 \rangle \times \dots \times \langle m_d \rangle$

the storage box of dimension  $m_1, \dots, m_d$

with # of storage places  $|\mathbf{m}|_{pr}$

at all lattice points

$$\mathbf{i} = (i_1, i_2, \dots, i_d), \quad i_j \in \langle m_j \rangle, \quad j = 1, \dots, d.$$

$$\langle (4, 3) \rangle := \begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{array}$$

# messages in  $\langle \mathbf{m} \rangle$  is  $n^{|\langle \mathbf{m} \rangle|_{pr}}$ .

(UNRESTRICTED) CAPACITY

$$\log_2 n = \frac{\log_2 n^{|\mathbf{m}|_{pr}}}{|\mathbf{m}|_{pr}}.$$

**DEFINITION:**  $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^d$  neighbors  
 $\iff |\mathbf{i} - \mathbf{j}| = 1$ .

in  $\mathbb{Z}^d$  every point has  $2d$  neighbors.

in  $\langle (3, 4) \rangle - (2, 1)$  has 3 neighbors.

#### 4 INTR: $d$ -dimensional (0 – 1) LIMITED CHANNEL

NO TWO 1's ARE NEIGHBORS.

$d > 1$ : NO FORMULA for capacity.

$d = 2$ : capacity:

$$0.587891162 \pm 0.5 \cdot 10^{-9}.$$

$d = 3$ : capacity:  $0.524 \pm 0.003$ .

**REASON**: computer performance

$d=3$ : need spectral radius of  
 $6,000 \times 6,000$  - 0 – 1 matrix.

## 5 1-DIMENSIONAL CAPACITY

$$a = a_1 \dots a_m = (a_i)_1^m : \langle m \rangle \rightarrow \langle n \rangle$$

WORD OF LENGTH  $m$ .

$\langle n \rangle^{\langle m \rangle}$  all messages of length  $m$ .

$$\langle n \rangle^{\mathbb{Z}} = \{a = (\dots a_{-1} a_0 a_1 \dots = (a_i)_{i \in \mathbb{Z}} : \mathbb{Z} \rightarrow \langle n \rangle\}$$

$$\langle n \rangle^{\mathbb{N}} = \{a = (a_1 \dots = (a_i)_{i \in \mathbb{N}} : \mathbb{N} \rightarrow \langle n \rangle\}$$

## COMPACT METRIC SPACES

## METRICS ON $\langle n \rangle^{\mathbb{Z}}$ & $\langle n \rangle^{\mathbb{N}}$

Hamming metric on  $\langle n \rangle$ :

$$\begin{aligned}d_h(i, i) &= 0, & i \in \langle n \rangle, \\d_h(i, j) &= 1, & i \neq j \in \langle n \rangle.\end{aligned}$$

$$d(a, b) = \sum_{i \in \mathbb{N}} \frac{d_h(a_i, b_i)}{2^i},$$

$$a = (a_i)_{i \in \mathbb{N}}, b = (b_i)_{i \in \mathbb{N}} \in \langle n \rangle^{\mathbb{N}},$$

$$d(a, b) = \sum_{i \in \mathbb{Z}} \frac{d_h(a_i, b_i)}{2^{|i|}},$$

$$a = (a_i)_{i \in \mathbb{Z}}, b = (b_i)_{i \in \mathbb{Z}} \in \langle n \rangle^{\mathbb{Z}}.$$

sequence  $a^1, a^2, \dots$  converges

$\iff$  it converges coordinatewise.

## SHIFT MAP

$$\sigma : \langle n \rangle^{\mathbb{N}} \rightarrow \langle n \rangle^{\mathbb{N}},$$

$$a_1 a_2 \dots \mapsto a_2 a_3 \dots$$

$$\sigma : \langle n \rangle^{\mathbb{Z}} \rightarrow \langle n \rangle^{\mathbb{Z}},$$

$$\begin{array}{cccc} \dots & a_{-1} & a_0 & a_1 \dots \\ \dots & -1 & 0 & 1 \dots \end{array} \mapsto \begin{array}{cccc} \dots & a_0 & a_1 & a_2 \dots \\ \dots & -1 & 0 & 1 \dots \end{array}$$

$\mathcal{S} \subset \langle n \rangle^{\mathbb{N}}$  ( $\langle n \rangle^{\mathbb{Z}}$ ) is subshift if

- (a)  $\mathcal{S}$  is closed set
- (b)  $\sigma(\mathcal{S}) = \mathcal{S}$ .

allowable states independent of origin

## 6 SUBSHIFT OF FINITE TYPE

PROJECTION:  $\pi_m((a_i)) = (a_i)_{i=1}^m$ .

$\mathcal{S}$  is SOFT if its maximal subshift for  
 $\exists P \subset \langle n \rangle \langle r \rangle$  such that  $\pi_r(\mathcal{S}) \subset P$ .

Equivalently:  $a \in \mathcal{S} \iff$  any consecutive string of  $r$  letters in  $a$  is in  $P$ .

**EXAMPLE:**  $\Gamma \subset \langle n \rangle \times \langle n \rangle$ .

ALLOWABLE WORD OF LENGTH  $m$

A WALK OF LENGTH  $m$  ON  $\Gamma$

$$\Gamma^m = \{(a_i)_{i=1}^{m+1} \in \langle n \rangle^{m+1}: (a_i, a_{i+1}) \in \Gamma\}$$

$$\Gamma^{\mathbb{N}} = \{(a_i) \in \langle n \rangle^{\mathbb{N}}: (a_i, a_{i+1}) \in \Gamma\}$$

$$\Gamma^{\mathbb{Z}} = \{(a_i) \in \langle n \rangle^{\mathbb{Z}}: (a_i, a_{i+1}) \in \Gamma\}$$

(0 – 1) LIMITED CHANNEL:

$\Gamma$ :            •        •

any **SOFT** can be coded as a walk on

$$\Gamma \subset \langle N \rangle \times \langle N \rangle$$

$$N = \# \pi_{r-1}(P)$$

$$((a_i)_1^{r-1}, (b_i)_1^{r-1}) \in \Gamma \iff$$

$$b_1 = a_2, \dots, b_{r-2} = a_{r-1}$$

$$a_1 \quad a_2 \quad \dots \quad a_{r-1}$$

$$b_1 \quad \dots \quad b_{r-2} \quad b_{r-1}$$

$$\& a_1 a_2 \dots a_{r-1} b_{r-1}$$

is allowable word of length  $r$

## ASSUMPTION: $\mathcal{S}$ -SOFT

$(a_i)_{i \in \mathbb{N}}$  is  $m$ -periodic

$$a_{i+m} = a_i, \quad i \in \mathbb{N}.$$

$\delta_m$ -log#  $P$ -allow. words of length  $m$ .

$$\tilde{\delta}_m = \log \# \pi_m(\mathcal{S})$$

$\delta_{m,per}$ -log#  $m$ -periodic words in  $\mathcal{S}$ .

$$\delta_{m,per} \leq \tilde{\delta}_m \leq \delta_m.$$

# EXAMPLE

1



2



3



$\{t_i\}_{i \in \mathbb{N}} \subset \mathbb{R}$  IS SUBADDITIVE (SA):

$$t_{p+q} \leq t_p + t_q \quad \text{for all } p, q \in \mathbb{N}.$$

$$\{t_i\}_{i \in \mathbb{N}} \text{ SA} \Rightarrow \lim_{i \rightarrow \infty} \frac{t_i}{i} = \tau \leq \frac{t_p}{p}.$$

**CLAIM:**  $\{\delta_m\}, \{\tilde{\delta}_m\}$ - ARE SA.

$$h_{com} := \lim_{m \rightarrow \infty} \frac{\delta_m}{m} \quad \text{CAPACITY}$$

$$h := \lim_{m \rightarrow \infty} \frac{\tilde{\delta}_m}{m} \quad \text{ENTROPY}$$

$$h_{per} := \lim_{m \rightarrow \infty} \frac{\delta_{m,per}}{m} \quad \text{PERIODIC ENTROPY}$$

$$-\infty \leq h_{per} \leq h \leq h_{com}$$

## MAIN THM

for 1- dimensional SOFT  $\mathcal{S}$

$$h_{per} = h = h_{com} = \log \rho(\Gamma)$$

$\Gamma$  -GRAPH INDUCED BY SOFT.

**COR.**  $\mathcal{S}$  is decidable:

EITHER  $\mathcal{S} = \emptyset \iff :$

$\exists m$  with no allow. word of length  $m$

OR  $\mathcal{S}$  contains an  $m$  periodic state.

## 7 MATRICES AND GRAPHS

$$\Gamma \subset \langle n \rangle \times \langle n \rangle \Rightarrow A(\Gamma) = (a_{ij})_1^n :$$

$$a_{ij} \in \{0, 1\}, \quad a_{ij} = 1 \iff (i, j) \in \Gamma$$

$$A = (a_{ij})_1^n \geq 0 \Rightarrow \Gamma(A) \subset \langle n \rangle \times \langle n \rangle :$$

$$(i, j) \in \Gamma(A) \iff a_{ij} > 0.$$

$$A = (a_{ij})_1^n \in M_n(\mathbb{R}):$$

$\text{spec}(A)$  : set of eigenvalues of  $A$

$$\rho(A) := \max_{\lambda \in \text{spec}(A)} |\lambda|$$

## PERRON-FROBENIUS:

$$A = (a_{ij})_{\mathbf{1}}^{\mathbf{n}} \geq 0 \Rightarrow$$

- $\rho(A) \in \text{spec}(A)$ .
- IF  $\rho(A) \neq 0 \Rightarrow$   
 $\text{spec}(\rho(A)^{-1}A) \cap \{|z| = 1\}$ -roots of 1.
- $\exists x \geq 0 \ Ax = \rho(A)x$ .
- $\limsup_{m \rightarrow \infty} \frac{\log \text{tr } A^m}{m} = \log \rho(A)$ .
- $\lim_{m \rightarrow \infty} \frac{\log \mathbf{1}A^m\mathbf{1}^T}{m} = \log \rho(A)$ .

$A$  irreducible ( $\Gamma(A)$  connected):

- $\rho(A) > 0$ .
- all e.v. on  $|z| = \rho(A)$  are simple.
- $\text{spec}(\rho(A)^{-1}A) \cap \{|z| = 1\}$   
are  $k$ -roots of 1 and  $k|n$ .
- $x \gg 0$ .

$A$ -primitive iff  $A^m \gg 0$  iff  $A$ -irr. &  
 $\text{spec}(\rho(A)^{-1}A) \cap \{|z| = 1\} = \{1\}$

$$A = (a_{ij})_1^n, \quad a_{ij} \in \mathbb{Z}_+$$

- EITHER

$$\rho(A) = 0 \iff A^n = 0 \iff \Gamma(A) \text{ cycleless}$$

OR  $\rho(A) \geq 1.$

- $\rho(A)$  is an algebraic integer

$$\rho(A)^n + \sum_{i=1}^n d_i \rho(A)^{n-i} = 0, \quad d_1, \dots, d_n \in \mathbb{Z}.$$

## MINIMAX CHAR OF $\rho(A)$

- $A \geq 0$ ,  $x = (x_1, \dots, x_n)^T \gg 0$ .

$$\rho(A) \geq \min_{i \in \langle n \rangle} \frac{\sum_{j=1}^n a_{ij} x_j}{x_i}$$

$$\rho(A) \leq \max_{i \in \langle n \rangle} \frac{\sum_{j=1}^n a_{ij} x_j}{x_i}$$

$A$  primitive

$$x^{(m)} = \frac{Ax^{(m-1)}}{a_{m-1}}, \quad x^{(0)} = x$$

lower & upper bounds converge to  $\rho(A)$   
 $x^{(m)}$  converge to e.v.

$A$  irreducible  $\Rightarrow A + I$  primitive

$$A = A^T = (a_{ij})_1^n \geq 0$$

$$\text{tr } A^{2m} = \sum_{i=1}^n \lambda_i^{2m} \geq \rho(A)^{2m}.$$

$$\rho(A) = \max_{x \in \mathbb{R}^n \setminus \{0\}} \frac{x^T A x}{x^T x}.$$

$$\rho(A) \geq \frac{\sum_{i=1}^n (\sum_{j=1}^n a_{ij} x_j) x_i}{\sum_{i=1}^n x_i x_i} \geq \min_{i \in \langle n \rangle} \frac{\sum_{j=1}^n a_{ij} x_j}{x_i}.$$

$$\Gamma \text{ symmetric (undirected)} \iff A(\Gamma) = A(\Gamma)^T$$

## 8 PROOF MAIN THM for 1-dim SOFT

$$A = A(\Gamma) = (a_{ij})_1^n \quad A^m = (a_{ij}^{(m)})_1^n$$

$$a_{ij}^{(m)} = \sum_{i_1, \dots, i_{m-1} \in \langle n \rangle} a_{ii_1} a_{i_1 i_2} \dots a_{i_{m-1} j}$$

$$a_{ii_1} a_{i_1 i_2} \dots a_{i_{m-1} j} \in \{0, 1\}$$

equal to 1 iff  $\Gamma$  has path

$$i \rightarrow i_1 \rightarrow \dots \rightarrow i_{m-1} \rightarrow j$$

$$a_{ij}^{(m)} = \# \text{ paths length } m \text{ from } i \text{ to } j$$

$$\mathbf{1} A^m \mathbf{1}^T = \#\Gamma^m$$

$$\text{tr } A^m = \sum_{i=1}^n a_{ii}^{(m)} = \#\Gamma_{\text{per}}^m$$

$$h_{per} = \log \rho(\Gamma) \leq h \leq h_{com} = \log \rho(\Gamma)$$

## 9 MULTI-DIMENSIONAL CAPACITY

$2 \leq d$  dimension

$\mathbf{e}_j = (\delta_{1j}, \dots, \delta_{dj})$   $j \in \langle d \rangle$  basis

$\mathbf{m} = (m_1, \dots, m_d) \leq \mathbf{n} = (n_1, \dots, n_d) \iff$

$$m_i \leq n_i, \quad i = 1, \dots, d$$

$a : \langle \mathbf{m} \rangle \rightarrow \langle n \rangle$  is  $(a_{\mathbf{i}})_{\mathbf{i} \in \langle \mathbf{m} \rangle}$

$\langle n \rangle^{\langle \mathbf{m} \rangle}$  set of all maps  $a$ .

$\langle n \rangle^{\mathbb{N}^d}$  &  $\langle n \rangle^{\mathbb{Z}^d}$  all maps

from  $\mathbb{N}^d$  &  $\mathbb{Z}^d$  to  $\langle n \rangle$ .

shift  $\sigma_j : \mathbb{N}^d \rightarrow \mathbb{N}^d$  ( $\sigma_j : \mathbb{Z}^d \rightarrow \mathbb{Z}^d$ )

$$\sigma_j((a_{\mathbf{i}})) = (a_{\mathbf{i} + \mathbf{e}_j}), \quad j = 1, \dots, d.$$

$\sigma_1, \dots, \sigma_d$  commuting

$\sigma_j$  invertible on  $\mathbb{Z}^d$

$\langle n \rangle^{\mathbb{Z}^d}$  &  $\langle n \rangle^{\mathbb{N}^d}$  compact metric

$\mathcal{S} \subset \langle n \rangle^{\mathbb{N}^d} (\langle n \rangle^{\mathbb{Z}^d})$  subshift if

(a)  $\mathcal{S}$  is closed set

(b)  $\sigma_i(\mathcal{S}) = \mathcal{S}$  for  $i = 1, \dots, d$

allowable states independent of origin

$\pi_{\mathbf{m}}((a_{\mathbf{i}})) = (a_{\mathbf{i}})_{\mathbf{i} \in \langle \mathbf{m} \rangle}$  proj. on  $\langle \mathbf{m} \rangle$ .

$\mathcal{S}$  is SOFT if its maximal subshift for

$\exists P \subset \langle n \rangle^{\langle \mathbf{r} \rangle}$  such that  $\pi_{\mathbf{r}}(\mathcal{S}) \subset P$ .

Equivalently:  $a \in \mathcal{S} \iff$  any

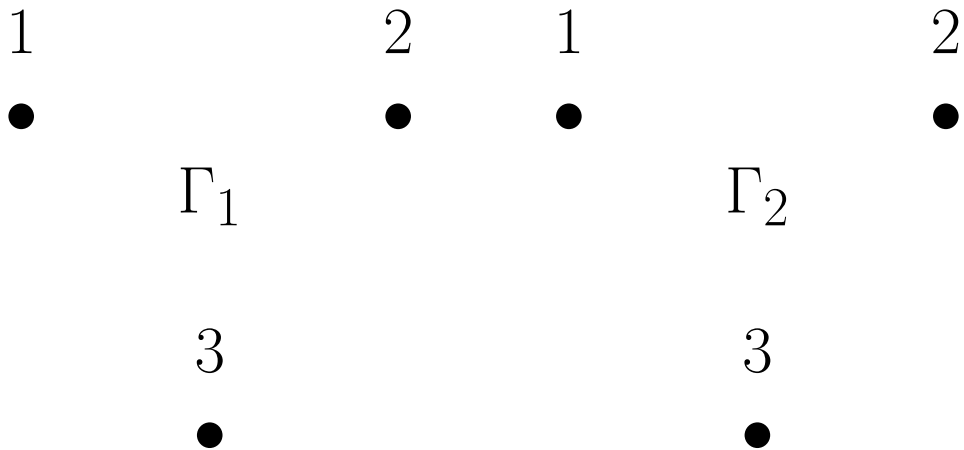
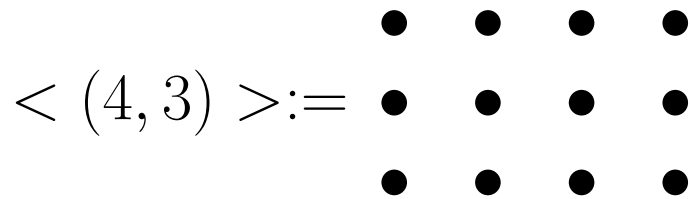
consecutive box of  $\mathbf{r}$  letters in  $a$  is in  $P$ .

**EXAMPLE:**  $\Gamma = (\Gamma_1, \dots, \Gamma_d)$

$(a_{\mathbf{i}})_{\mathbf{i} \in \langle \mathbf{m} \rangle}$  IS ALLOWABLE WORD

if any line through  $\mathbf{i} \in \langle \mathbf{m} \rangle$

in direction  $\mathbf{e}_k$  is in  $\Gamma_k^{m_k-1}$  for  $k \in \langle d \rangle$



$\Gamma^{\mathbf{m}}$  all allowable words in  $\langle \mathbf{m} + \mathbf{1} \rangle$

$\Gamma_{per}^{\mathbf{m}} \subset \Gamma^{\mathbf{m}}$  all  $\mathbf{m}$  periodic words

$$a_{\mathbf{i}+\mathbf{m}} = a_{\mathbf{i}}$$

(we view  $\Gamma^{\mathbf{m}}$  as subset of  $\Gamma^{\mathbf{m}-\mathbf{1}}$ )

$\Gamma^{\mathbb{N}^d}$  all allowable words in  $\mathbb{N}^d$

$\Gamma^{\mathbb{Z}^d}$  all allowable words in  $\mathbb{Z}^d$

**any SOFT** can be coded as  $\Gamma^{\mathbb{N}^d}$  ( $\Gamma^{\mathbb{Z}^d}$ )

$\exists \mathcal{S}$  **SOFT NOT DECIDABLE:**

$\exists \Gamma^{\mathbb{N}^2} \neq \emptyset$  with no periodic state

Berger 1966

$$\Gamma^{\mathbb{N}^d} = \emptyset \iff \exists \mathbf{m} \Gamma^{\mathbf{m}} = \emptyset$$

## ASSUMPTION: $\mathcal{S}$ -SOFT

$\delta_{\mathbf{m}}\text{-log}\#\ P\text{-allow. words of dim. } \mathbf{m}.$

$$\tilde{\delta}_{\mathbf{m}} = \log \#\pi_{\mathbf{m}}(\mathcal{S})$$

$\delta_{\mathbf{m},per}\text{-log}\#\ \mathbf{m}\text{-periodic words in } \mathcal{S}.$

$$\delta_{\mathbf{m},per} \leq \tilde{\delta}_{\mathbf{m}} \leq \delta_{\mathbf{m}}.$$

$\{\delta_{\mathbf{m}}\}, \{\tilde{\delta}_{\mathbf{m}}\}$ - are SA in each coordinate

(split box  $\langle \mathbf{m} \rangle$  to 2 boxes by  $x_k = i_k$ )

$$h_{com} := \lim_{\mathbf{m} \rightarrow \infty} \frac{\delta_{\mathbf{m}}}{|\mathbf{m}|_{pr}} \quad \text{CAPACITY}$$

$$h := \lim_{m \rightarrow \infty} \frac{\tilde{\delta}_{\mathbf{m}}}{|\mathbf{m}|_{pr}} \quad \text{ENTROPY}$$

$$h_{per} := \lim_{\mathbf{m} \rightarrow \infty} \frac{\delta_{\mathbf{m},per}}{|\mathbf{m}|_{per}} \quad \text{PERIODIC ENTROPY}$$

$$-\infty \leq h_{per} \leq h \leq h_{com} \leq \frac{\delta_{\mathbf{m}}}{|\mathbf{m}|_{pr}}$$

Berger:  $-\infty = h_{per} < 0 \leq h \leq h_{com}$

Friedland 97: for all  $\mathcal{S}$   $h = h_{com}$

10 UPPER ESTIMATES OF MDC

$$\tau \subset \langle d \rangle \ \& \ \mathbf{m} = (m_1, \dots, m_d) \in \mathbb{N}^d$$

$$\tau^c := \langle d \rangle \setminus \tau = \{q_1, q_2, \dots, q_p\}$$

$$1 \leq q_1 < \dots < q_p \leq d, \ p = d - \#\tau$$

$$\mathbf{m}^\tau = (m_{q_1}, m_{q_2}, \dots, m_{q_p})$$

$$\mathbf{m}^{\{i\}} = (m_1, \dots, m_{i-1}, m_{i+1}, \dots, m_d)$$

$$\mathbf{m} = (\mathbf{m}^{\{i\}}, m_i),$$

$$\Gamma^\tau = (\Gamma_{q_1}, \dots, \Gamma_{q_p})$$

$$\Gamma(k, \mathbf{m}^{\{k\}}) \subset (\Gamma^{\{k\}})^{\mathbf{m}^{\{k\}} - \mathbf{1}^{\{k\}}} \times (\Gamma^{\{k\}})^{\mathbf{m}^{\{k\}} - \mathbf{1}^{\{k\}}}$$

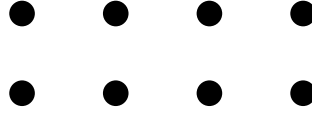
graph on vertices  $(\Gamma^{\{k\}})^{\mathbf{m}^{\{k\}} - \mathbf{1}^{\{k\}}}$ :

$$((b_{\mathbf{i}^{\{k\}}})_{\mathbf{i}^{\{k\}} \in \langle \mathbf{m}^{\{k\}} \rangle}, (c_{\mathbf{i}^{\{k\}}})_{\mathbf{i}^{\{k\}} \in \langle \mathbf{m}^{\{k\}} \rangle})$$

$$\text{in } \Gamma(k, \mathbf{m}^{\{k\}}) \iff (b_{\mathbf{i}^{\{k\}}}, c_{\mathbf{i}^{\{k\}}}) \in \Gamma_k$$

for each  $\mathbf{i}^{\{k\}} \in \langle \mathbf{m}^{\{k\}} \rangle$

**EXAMPLE:**  $\Gamma(2, 4)$



$$\Gamma_{per}(k, \mathbf{m}^{\{k\}}) \subset (\Gamma^{\{k\}})_{per}^{\mathbf{m}^{\{k\}}} \times (\Gamma^{\{k\}})_{per}^{\mathbf{m}^{\{k\}}}$$

graph on vertices  $(\Gamma^{\{k\}})_{per}^{\mathbf{m}^{\{k\}}}$ :

$$((b_{\mathbf{i}^{\{k\}}})_{\mathbf{i}^{\{k\}} \in \langle \mathbf{m}^{\{k\}} + \mathbf{1}^{\{k\}} \rangle}, (c_{\mathbf{i}^{\{k\}}})_{\mathbf{i}^{\{k\}} \in \langle \mathbf{m}^{\{k\}} + \mathbf{1}^{\{k\}} \rangle})$$

in  $\Gamma_{per}(k, \mathbf{m}^{\{k\}})$  iff  $(b_{\mathbf{i}^{\{k\}}}, c_{\mathbf{i}^{\{k\}}}) \in \Gamma_k$

for each  $\mathbf{i}^{\{k\}} \in \langle \mathbf{m}^{\{k\}} + \mathbf{1}^{\{k\}} \rangle$

$$\rho(\Gamma(k, \mathbf{m}^{\{k\}})) \ \& \ \rho(\Gamma_{per}(k, \mathbf{m}^{\{k\}}))$$

$$\text{spec. rad. } \Gamma(k, \mathbf{m}^{\{k\}}) \ \& \ \Gamma_{per}(k, \mathbf{m}^{\{k\}})$$

$$\Gamma(k, \mathbf{m}^{\{k\}})^{\mathbb{N}} \ \& \ \Gamma(k, \mathbf{m}^{\{k\}})^{\mathbb{Z}}$$

1-dim. SOFT  $\Gamma$  allow. conf.:

”strip” in direction  $k$  with

basis dimension  $\mathbf{m}^{\{k\}}$

$$\Gamma_{per}(k, \mathbf{m}^{\{k\}})^{\mathbb{N}} \ \& \ \Gamma_{per}(k, \mathbf{m}^{\{k\}})^{\mathbb{Z}}$$

1-dim. SOFT  $\Gamma$  allow. conf.:

”strip” in direction  $k$  with

$\mathbf{m}^{\{k\}}$ -periodic basis dim.  $\mathbf{m}^{\{k\}} + \mathbf{1}^{\{k\}}$

view  $\mathbf{m}^{\{k\}}$  periodic conf.

$$(b_{\mathbf{i}^{\{k\}}})_{\mathbf{i}^{\{k\}} \in \langle \mathbf{m}^{\{k\}} + \mathbf{1}^{\{k\}} \rangle} \in (\Gamma_{per}^{\{k\}})^{\mathbf{m}^{\{k\}}}$$

$$\text{as } (b_{\mathbf{i}^{\{k\}}})_{\mathbf{i}^{\{k\}} \in \langle \mathbf{m}^{\{k\}} \rangle} \in (\Gamma^{\{k\}})^{\mathbf{m}^{\{k\}} - \mathbf{1}^{\{k\}}}$$

periodic exten. to a configuration

$$(b_{\mathbf{i}^{\{k\}}})_{\mathbf{i}^{\{k\}} \in \langle \mathbf{m}^{\{k\}} + \mathbf{1}^{\{k\}} \rangle} \in (\Gamma^{\mathbf{k}})^{\mathbf{m}^{\{k\}}}.$$

$$(\Gamma^{\{k\}})^{\mathbf{m}^{\{k\}}}_{per} \ \& \ \Gamma_{per}(k, \mathbf{m}^{\{k\}}) \text{ subsets}$$

$$(\Gamma^{\{k\}})^{\mathbf{m}^{\{k\}} - \mathbf{1}^{\{k\}}} \ \& \ \Gamma(k, \mathbf{m}^{\{k\}})$$

statistical mechanics *transfer* matrix:

$$A(\Gamma(k, \mathbf{m}^{\{k\}}))$$

$$\lim_{m_k \rightarrow \infty} \frac{\delta_{\mathbf{m}}}{m_k} = \log \rho(\Gamma(k, \mathbf{m}^{\{k\}})) \geq |\mathbf{m}^{\{k\}}|_{per} h$$

$$\lim_{m_k \rightarrow \infty} \frac{\delta_{\mathbf{m},per}}{m_k} = \log \rho(\Gamma_{per}(k, \mathbf{m}^{\{k\}}))$$

$$\lim_{\mathbf{m}^{\{k\}} \rightarrow \infty} \frac{\log \rho(\Gamma(k, \mathbf{m}^{\{k\}}))}{|\mathbf{m}^{\{k\}}|_{per}} = h$$

$$\lim_{\mathbf{m}^{\{k\}} \rightarrow \infty} \frac{\log \rho(\Gamma_{per}(k, \mathbf{m}^{\{k\}}))}{|\mathbf{m}^{\{k\}}|_{per}} = h_{per}$$

$$\rho(\Gamma(k, \mathbf{m}^{\{k\}})) \geq \rho(\Gamma_{per}(k, \mathbf{m}^{\{k\}}))$$

11 LOWER ESTIMATES FOR  $d = 2$

**THM:**  $\mathcal{S} \subset \langle n \rangle^{\mathbb{N}^2}$  ( $\langle n \rangle^{\mathbb{Z}^2}$ )

$\Gamma = (\Gamma_1, \Gamma_2)$ -SOFT

$\Gamma_2$  **SYMMETRIC**

$$\frac{\log \rho(\Gamma(1, p + 2q + 1)) - \log \rho(\Gamma(1, 2q + 1))}{2m} \leq$$

$$h \leq \frac{\log \rho(\Gamma_{per}^p(1, 2m))}{2m} \left( \leq \frac{\log \rho(\Gamma(1, 2m))}{2m} \right)$$

for any  $m, p \geq 1$  and  $q \geq 0$ .

**$\mathcal{S}$  DECIDABLE:**

$$\mathcal{S} \neq \emptyset \iff \Gamma(1, 2) \text{ has cycle}$$

**AND COMPUTABLE**

$$h_{per} = h$$

## Proof.

$\Gamma_2$  sym.  $\Rightarrow \Gamma(2, i) \& \Gamma_{per}(2, i)$  sym.

eigenvalues of  $A(\Gamma(2, i))$  are real

$\rho(\Gamma(2, i))$  max. eig.val.  $A(\Gamma(2, i))$ .

$$\rho(\Gamma(2, i))^{2m} \leq \text{tr } A(\Gamma(2, i))^{2m} = \#\Gamma_{per}(1, 2m)^{i-1}$$

$$\frac{\log \rho(\Gamma(2, i))}{i} \leq \frac{1}{2m} \frac{\log \#\Gamma_{per}(1, 2m)^{i-1}}{i}.$$

$$i \rightarrow \infty \Rightarrow h \leq \frac{\log \rho(\Gamma_{per}(1, 2m))}{2m}$$

$$N = \#\Gamma_1^{i-1} \ \& \ 0 \neq x \in \mathbb{R}^N$$

$$\rho(\Gamma(2, i))^p \geq \frac{x^T A(\Gamma(2, i))^p x}{x^T x}.$$

$$x = A(\Gamma(2, i))^q \mathbf{1}^T.$$

$$\rho(\Gamma(2, i))^p \geq \frac{\mathbf{1}A(\Gamma(2, i))^{p+2q}\mathbf{1}^T}{\mathbf{1}A(\Gamma(2, i))^{2q}\mathbf{1}^T}$$

$$\#\Gamma^{(i-1, \ell-1)} = \#\Gamma(1, \ell)^{i-1}$$

$$\#\Gamma(2, i)^{\ell-1} = \mathbf{1}A(\Gamma(2, i))^{\ell-1}\mathbf{1}^T \Rightarrow$$

$$\frac{\log \rho(\Gamma(2, i))}{i} \geq$$

$$\frac{\log \#\Gamma(1, p + 2q + 1)^{i-1} - \log \#\Gamma(1, 2q + 1)^{i-1}}{pi}$$

$$i \rightarrow \infty \Rightarrow$$

$$h \geq \frac{\log \rho(\Gamma(1, p + 2q + 1)) - \log \rho(\Gamma(1, 2q + 1))}{p}$$

$$M = \#\Gamma_2$$

$$\Gamma(1, 2) \text{ cycleless} \Rightarrow \Gamma(1, 2)^M = \emptyset \Rightarrow \mathcal{S} = \emptyset$$

$$\begin{aligned} & (u_1, v_1) \rightarrow \dots \rightarrow (u_{L+1}, v_{L+1}) = (u_1, v_1) \in \\ & \Gamma_{per}(1, 2)^L \quad \& \Gamma_2 \text{ sym.} \Rightarrow \\ & (u_1, v_1, u_1), \dots, (u_{L+1}, v_{L+1}, u_{L+1}) \in \\ & \Gamma_{per}(1, 2)^L \Rightarrow \Gamma_{per}^{<L, 2>} \neq \emptyset \Rightarrow \mathcal{S} \neq \emptyset \end{aligned}$$

$$\begin{aligned} \limsup_{i \rightarrow \infty} \frac{\delta_{(i, 2m), per}}{i \ 2m} &= \frac{\log \rho(\Gamma_{per}(1, 2m))}{2m} \Rightarrow \\ h_{per} &= \limsup_{m \rightarrow \infty} \frac{\log \rho(\Gamma_{per}(1, 2m))}{2m} \geq h \geq h_{per} \\ \Rightarrow h_{per} &= h \end{aligned}$$

$$q = 0 \Rightarrow$$

$$\frac{\log \rho(\Gamma(1, p + 1))}{p} - \frac{\log(\Gamma(1, 1))}{p} \leq h$$

$$\leq \frac{\log \rho(\Gamma(1, p + 1))}{p + 1}, \quad p \geq 1 \Rightarrow$$

$$\frac{\log \rho(\Gamma(1, p + 1))}{p + 1} -$$

$$\left( \frac{\log \rho(\Gamma(1, p + 1))}{p} - \frac{\log \rho(\Gamma(1, 1))}{p} \right) =$$

$$- \frac{\log \rho(\Gamma(1, p + 1))}{p(p + 1)} + \frac{\log \rho(\Gamma_1)}{p + 1}$$

$$\leq \frac{\log \rho(\Gamma_1) - h}{p} \leq \frac{\log \rho(\Gamma_1)}{p} \Rightarrow$$

COMPUTABILITY

## Historical Remarks

Friedland 97:  $q = 0$  and  $h_{per} = h$

Calkin-Wilf 98: any  $q$  and  
improved u.b. for  $h$  in special  $\mathbb{Z}^2$  SOFT

Sharper l.b. for  $h$  for  $p = 1$  and  $q \nearrow$ :

Weeks-Blahut 98

Forschhammer-Justenes 99

Nagy-Zeger 00

12  $d \geq 3$

**THM:**  $\mathcal{S} \subset \langle n \rangle^{\mathbb{N}^d}$  ( $\langle n \rangle^{\mathbb{Z}^d}$ )

$\Gamma = (\Gamma_1, \dots, \Gamma_d)$ -SOFT

$\Gamma_j$  **SYMMETRIC** for  $j \in \tau \subset \langle d \rangle$

$i \in \langle d \rangle \setminus \tau$ ,  $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{N}^d$ ,  
 $2|m_j, j \in \tau$

$$h \leq \frac{\log \rho(\Gamma_{\tau, per}(i, \mathbf{m}^{\{i\}}))}{|\mathbf{m}^{\{i\}}|_{pr}}$$

## LOWER BOUNDS

$\tilde{\mathcal{S}} \subset \langle N \rangle^{\mathbb{N}^{d-1}}$  ( $\langle N \rangle^{\mathbb{Z}^{d-1}}$ ) SOFT:

$$\Gamma\{i\},q = (\Gamma_1\{i\},q, \dots, \Gamma_{i-1}\{i\},q, \Gamma_{i+1}\{i\},q, \dots, \Gamma_d\{i\},q)$$

Inf. conf. in dir.  $j \neq i$ , width  $q$  in dir.  $i$

$h\{i\},q$  entropy of  $\tilde{\mathcal{S}}$

$$h\{i\} = h\{i\},1 \quad (\Gamma\{i\},1 = \Gamma\{i\})$$

$j \in \langle d \rangle \setminus \{i\} \Rightarrow$

$$\lim_{m_i \rightarrow \infty} \frac{\log \rho(\Gamma(j, \mathbf{m}^{\{j\}}))}{m_i} = \log \rho_{\{i\}}(j, \mathbf{m}^{(i,j)})$$

THM:  $\Gamma_j$  sym. for  $j \neq i$

$$\log \rho(\Gamma(i, (\mathbf{m}^{\{i,j\}}, p + 2q + 1))) \leq$$

$$\log \rho(\Gamma(i, (\mathbf{m}^{\{i,j\}}, 2q + 1))) + p \log \rho_{\{i\}}(j, \mathbf{m}^{\{i,j\}})$$

$$\frac{h_{\{j\}, p+2q+1} - h_{\{j\}, 2q+1}}{p} \leq h \leq \frac{h_{\{j\}, p+2q+1}}{p + 2q + 1}$$

$$p \geq 1, q \geq 0$$

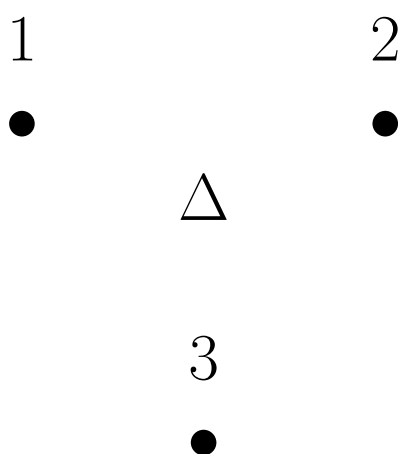
If  $\Gamma_j$  sym. for each  $j \in \langle d \rangle \setminus \{i\}$

$h = h_{per}$  **COMPUTABLE**

13 EXAMPLES

**Ex.1**-residual entropy of square ice:

$\mathbb{Z}^2$ -SOFT given  $\Gamma = (\Delta, \Delta)$  ( $\Delta = K_3$ )



3 col. of  $\mathbb{Z}^2$  s.t. no same colors adjac.

$$\text{Lieb 67} \Rightarrow h_{per} = \frac{3}{2} \log \frac{4}{3} = 0.43152\dots$$

Brascamp-Kunz-Wu 73  $h_{per} = h_{com}$

$$h = \log \text{ of } u = \left(\frac{4}{3}\right)^{\frac{3}{2}} \text{ alg. num.}$$

$$27u^2 = 64 \text{ } u \neq \text{ alg. int.}$$

$$\rho(\Delta) = \rho(\Gamma(1, 1)) = \rho(\Gamma_{per}(1, 3)) = 2,$$

$$\rho(\Gamma(1, 2)) = \rho(\Gamma_{per}(1, 2)) = 3,$$

$$\rho(\Gamma(1, 3)) = 4.561552, \quad \rho(\Gamma_{per}(1, 4)) = 6.372281$$

$$p = 1, q = 0 \Rightarrow h \geq \log \frac{3}{2} = 0.405465108.$$

Pauling 1935 estimate  $h \approx \log \frac{3}{2}$

$$p = 2, q = 0 \Rightarrow h \geq 0.4122579570.$$

$$\frac{\log \rho(\Gamma(1, p))}{p} \geq h \text{ for } p = 1, 2, 3 \Rightarrow$$

$$0.693147 > 0.549306 > 0.505887 \geq h$$

$$\frac{\log \rho(\Gamma_{per}(1, 2m))}{2m} \geq h \text{ for } m = 1, 2 \Rightarrow$$

$$0.549306 > 0.462989 \geq h$$

Physicists use asymptotic expansions

Nagle 66  $h = 0.432 \pm 0.001$

**Ex2:**  $d$ - dimen.  $(0, 1)$  r.l.l.c.

$$\Gamma_1 = \dots = \Gamma_d = \Delta \subset \langle 2 \rangle \times \langle 2 \rangle:$$

$$\Delta : \quad \bullet \quad \bullet$$

$$\bar{h}_d = \frac{h_d}{\log 2} \text{ entropy (capacity) base 2}$$

$$\bar{h}_1 = \log_2 \rho(\Delta) = \log_2 \frac{1 + \sqrt{5}}{2} = 0.694241914$$

$$0.587891161775 \leq \bar{h}_2 \leq 0.587891161868$$

Calkin-Wilf 98, Weeks-Blahut 98, Forschhammer-Justesen 99, Nagy-Zeger 00

$$0.5225017411838 \leq \bar{h}_3 \leq 0.526880847825.$$

Nagy-Zeger 00

need spec. rad. of  $A$  with 40 mill. elem.

## NEAREST NEIGHBOR $\mathbb{Z}^d$ -SOFT:

For  $\Delta \subset \langle n \rangle \times \langle n \rangle$

$\mathcal{S}_d(\Delta)$  given by  $\Gamma_1 = \dots = \Gamma_d = \Delta$

$h_d(\Delta)$  -entropy of  $\mathcal{S}_d(\Delta)$ .

$$h_1(\Delta) \geq h_2(\Delta) \dots \geq h_d(\Delta)$$

Find  $h(\Delta) = \lim_{d \rightarrow \infty} h_d(\Delta)$

**Ex3:**  $d$ -dimensional dimers

Dimer:  $(\mathbf{i}, \mathbf{j}), \mathbf{i} \neq \mathbf{j} \in \mathbb{Z}^d$  s.t.  $|\mathbf{i} - \mathbf{j}| = 1$ .

any partition of  $\mathbb{Z}^d$  to dimers (1-factor)

$\mathcal{S}_d - \Gamma(d) = (\Gamma_{1,d}, \dots, \Gamma_{d,d})$  - SOFT

$\Gamma_{i,d} \subset \langle 2d \rangle \times \langle 2d \rangle$ :

$(i, i+d) \in \Gamma_i$  &  $(i, j) \notin \Gamma_i$  for  $j \neq i+d$

For  $k \neq i$   $(k, i+d) \notin \Gamma_i$ .

For  $k \neq i, j \neq i+d$   $(k, j) \in \Gamma_i$ .

$\Gamma_{i,d}$  is not symmetric

$$0 = h_1 \leq h_2 \leq \dots \leq h_d \leq \dots$$

Fisher, Kasteleyn and Temperley 61

$$h_2 = \frac{1}{\pi} \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)^2} = 0.29156090\dots$$

no formula is for  $h_3$ .

Schrijver 98 (l.b. for permanents)

$$h_d \geq \frac{1}{2}((2d - 1) \log(2d - 1) - (2d - 2) \log(2d))$$

$$\Rightarrow h_3 > 0.440075842$$

Ciucu 98 (without permanents)

$$h_3 \leq 0.463107.$$

Heuristical results:

$$\text{Nagle 66 } h_3 = 0.44645 \pm 5 \cdot 10^{-5}$$

$$\text{Beichl-Sullivan 99 } h_3 = 0.4466 \pm 6 \cdot 10^{-4}$$

Classical results on permanents

$$h_d = \frac{1}{2} \log(2d) - \frac{1}{2} + O\left(\frac{\log d}{d}\right).$$

14 PROBABILISTIC ASPECTS OF MDC

$$C((a_{\mathbf{i}})_{\mathbf{i} \in \langle \mathbf{m} \rangle}) := \{(b_{\mathbf{i}})_{\mathbf{i} \in \mathbb{N}^d} \in \mathbb{N}^d \\ b_{\mathbf{i}} = a_{\mathbf{i}} \text{ for } \mathbf{i} \in \langle \mathbf{m} \rangle\}$$

$\mathcal{B} \subset 2^{\mathbb{N}^d}$   $\sigma$ -algebra gener. by cylinders  
 $\Pi$  probability measures on the  $\mathbb{N}^d$ .

$\mu \in \Pi$  is  $\sigma$ -invariant  $\iff$

$$\mu(T) = \mu(\sigma_i^{-1}(T)), \quad \forall T \in \mathcal{B} \& i \in \langle d \rangle$$

$\Pi \supset \Pi_i$  -  $\sigma$ -invariant measures

$\mu \in \Pi$  ergodic:  $\mu \in \Pi_i \&$

$$\sigma_i^{-1}(T) = T \quad \forall i \in \langle d \rangle \implies \mu(T) = 0, 1$$

$\Pi_i \supset \Pi_e$ -all ergodic measures.

$\Pi_e$  set of extreme points in  $\Pi_i$

Kolmogorov-Sinai entropy of  $\mu \in \Pi_i$ :

$$h(\mu) = \lim_{\mathbf{m} \rightarrow \infty} \frac{1}{|\mathbf{m}|_{pr}} \sum_{(a_{\mathbf{i}})_{\mathbf{i} \in \langle n \rangle} \in \langle \mathbf{m} \rangle} -\mu(C((a_{\mathbf{i}})_{\mathbf{i} \in \langle \mathbf{m} \rangle})) \log \mu(C((a_{\mathbf{i}})_{\mathbf{i} \in \langle \mathbf{m} \rangle}))$$

Shannon's probabilistic capacity

$$\mathcal{S} \subset \langle n \rangle^{\mathbb{N}^d}\text{-SOFT}$$

$$\Pi_i(\mathcal{S}) \supset \Pi_e(\mathcal{S}) \text{ all } \mu, \mu(\mathcal{S}) = 1.$$

**MAXIMAL CHARAC.**

$$h(\mathcal{S}) = \sup_{\mu \in \Pi_i(\mathcal{S})} h(\mu) = \sup_{\mu \in \Pi_e(\mathcal{S})} h(\mu)$$

**MAXIMUM ACHIEVED**

TRANSPARENCIES OF TALK

BASED ON THE PAPER:

S. Friedland

”MULTI-DIMENSIONAL CAPACITY,

PRESSURE AND

HAUSDORFF DIMENSION”

[www.math.uic.edu/~friedlan\(d\)](http://www.math.uic.edu/~friedlan(d))