# MULTI-DIMENSIONAL CAPACITY

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$$< n >:= \{1, ..., n\}$$

IS AN ALPHABET ON $n$ LETTERS.

$$\underbrace{\bullet\ \bullet\ \bullet \ldots \bullet\ \bullet\ \bullet}_{m \text{ times}}$$

LINEAR STORAGE OF LENGTH $m$.

# STORAGE MESSAGES $n^m$.

(UNRESTRICTED) CAPACITY:

$$\log_2 n \quad = \frac{\log_2 n^m}{m}.$$

## 2 INTR: $(0-1)$ LIMITED CHANNEL

$$n = 2, < 2 >= \{1, 2\} = \{1, 0\}\ (2 \equiv 0).$$

NO TWO $1's$ ARE NEIGHBORS.

# OF $m$-MESSAGES IS $u_m$;

$$u_{m+1} = u_m + u_{m-1},\ m = 1, 2, ...$$

FIBONACCI SEQUENCE $2, 3, 5, 8, \ldots$

CAPACITY: $\log_2 \frac{1+\sqrt{5}}{2} = 0.694241914.$

$\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$ – integers

$\mathbb{Z}_+ = \{0, 1, \ldots\}$ – nonnegative integers

$\mathbb{N} = \{1, 2, ...\}$ – positive integers

$d$ – the dimension, $\quad 1 < d \in \mathbb{N}$

$\mathbf{m} := (m_1, ..., m_d) \in \mathbb{Z}^d$

$|\mathbf{m}| = |m_1| + ... + |m_d|$

$|\mathbf{m}|_{pr} := |m_1| \times \ldots \times |m_d|$

for $\mathbf{m} \in \mathbb{N}^d$

$< \mathbf{m} > := < m_1 > \times \ldots \times < m_d >$

the storage box of dimension $m_1, ..., m_d$

with # of storage places $|\mathbf{m}|_{pr}$

at all lattice points

$$\mathbf{i} = (i_1, i_2, .., i_d),\ i_j \in < m_j >, j = 1, ..., d.$$

$$< (4,3) > := \begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{array}$$

# messages in $< \mathbf{m} >$ is $n^{|<\mathbf{m}>|_{pr}}$.

(UNRESTRICTED) CAPACITY

$$\log_2 n \quad = \frac{\log_2 n^{|\mathbf{m}|_{pr}}}{|\mathbf{m}|_{pr}}.$$

**DEFINITION**: $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^d$ neighbors $\iff |\mathbf{i} - \mathbf{j}| = 1$.

in $\mathbb{Z}^d$ every point has $2d$ neighbors.
in $< (3,4) >$ - $(2,1)$ has 3 neighbors.

NO TWO $1's$ ARE NEIGHBORS.

$d > 1$: NO FORMULA for capacity.

$d = 2$: capacity:

$$0.587891162 \pm 0.5 \cdot 10^{-9}.$$

$d = 3$: capacity: $0.524 \pm 0.003$.

**REASON**: computer performance

d=3: need spectral radius of
$6,000 \times 6,000$ - $0 - 1$ matrix.

$$a = a_1 ... a_m = (a_i)_1^m :< m >\to< n >$$

WORD OF LENGTH $m$.

$< n >^{<m>}$ all meassages of length $m$.

$$< n >^{\mathbb{Z}} = \{a = (\dots a_{-1} a_0 a_1 \dots = (a_i)_{i\in\mathbb{Z}} : \mathbb{Z} \to < n >\}$$

$$< n >^{\mathbb{N}} = \{a = (a_1 \dots = (a_i)_{i\in\mathbb{N}} : \mathbb{N} \to < n >\}$$

COMPACT METRIC SPACES

# METRICS ON $< n >^{\mathbb{Z}}$ & $< n >^{\mathbb{N}}$

Hamming metric on $< n >$:

$$d_h(i,i) = 0, \quad i \in < n >,$$
$$d_h(i,j) = 1, \quad i \neq j \in < n > .$$

$$d(a,b) = \sum_{i \in \mathbb{N}} \frac{d_h(a_i, b_i)}{2^i},$$
$$a = (a_i)_{i \in \mathbb{N}}, b = (b_i)_{i \in \mathbb{N}} \in < n >^{\mathbb{N}},$$
$$d(a,b) = \sum_{i \in \mathbb{Z}} \frac{d_h(a_i, b_i)}{2^{|i|}},$$
$$a = (a_i)_{i \in \mathbb{Z}}, b = (b_i)_{i \in \mathbb{Z}} \in < n >^{\mathbb{Z}} .$$

sequence $a^1, a^2, \ldots$ converges
$\iff$ it converges coordinatewise.

## SHIFT MAP

$\sigma :< n >^{\mathbb{N}} \rightarrow < n >^{\mathbb{N}},$

$a_1 a_2 \ldots \mapsto a_2 a_3 \ldots$

$\sigma :< n >^{\mathbb{Z}} \rightarrow < n >^{\mathbb{Z}},$

$$\begin{matrix} \ldots & a_{-1} & a_0 & a_1 \ldots \\ \ldots & -1 & 0 & 1 \ldots \end{matrix} \mapsto \begin{matrix} \ldots & a_0 & a_1 & a_2 \ldots \\ \ldots & -1 & 0 & 1 \ldots \end{matrix}$$

$\mathcal{S} \subset < n >^{\mathbb{N}} \ (< n >^{\mathbb{Z}})$ is subshift if

(a) $\mathcal{S}$ is closed set

(b) $\sigma(\mathcal{S}) = \mathcal{S}$.

allowable states independent of origin

## 6 SUBSHIFT OF FINITE TYPE

PROJECTION: $\pi_m((a_i)) = (a_i)_{i=1}^{m}$.

$\mathcal{S}$ is SOFT if its maximal subshift for $\exists P \subset < n >^{<r>}$ such that $\pi_r(\mathcal{S}) \subset P$.

Equivalently: $a \in \mathcal{S} \iff$ any consecutive string of $r$ letters in $a$ is in $P$.

**EXAMPLE**: $\Gamma \subset < n > \times < n >$.

ALLOWABLE WORD OF LENGTH $m$

A WALK OF LENGTH $m$ ON $\Gamma$

$$\Gamma^m = \{(a_i)_1^{m+1} \in < n >^{m+1}\colon\ (a_i, a_{i+1}) \in \Gamma\}$$

$$\Gamma^{\mathbb{N}} = \{(a_i) \in < n >^{\mathbb{N}}\colon\ (a_i, a_{i+1}) \in \Gamma\}$$

$$\Gamma^{\mathbb{Z}} = \{(a_i) \in < n >^{\mathbb{Z}}\colon\ (a_i, a_{i+1}) \in \Gamma\}$$

$(0-1)$ LIMITED CHANNEL:

$\Gamma$:

**any SOFT** can be coded as a walk on

$$\Gamma \subset < N > \times < N >$$

$$N = \#\pi_{r-1}(P)$$

$$((a_i)_1^{r-1}, (b_i)_1^{r-1}) \in \Gamma \iff$$
$$b_1 = a_2, \ldots, b_{r-2} = a_{r-1}$$

$$\begin{array}{ccccc} a_1 & a_2 & \ldots & a_{r-1} & \\ & b_1 & \ldots & b_{r-2} & b_{r-1} \end{array}$$

& $a_1$ $a_2$ ... $a_{r-1}$ $b_{r-1}$

is allowable word of length $r$

**ASSUMPTION**: $\mathcal{S}$ -SOFT

$(a_i)_{i\in\mathbb{N}}$ is $m$-periodic

$$a_{i+m} = a_i,\ i \in \mathbb{N}.$$

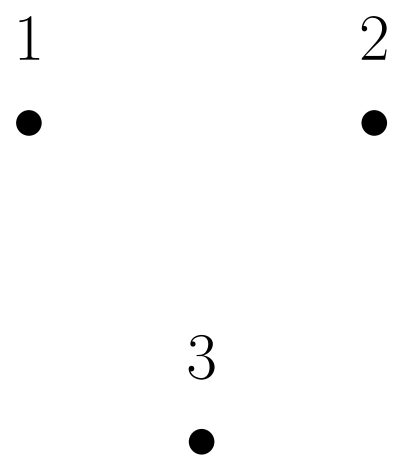
$\delta_m$-log# $P$-allow. words of legth $m$.

$$\tilde{\delta}_m = \log \#\pi_m(\mathcal{S})$$

$\delta_{m,per}$-log# $m$-periodic words in $\mathcal{S}$.

$$\delta_{m,per} \leq \tilde{\delta}_m \leq \delta_m.$$

## EXAMPLE



$\{t_i\}_{i \in \mathbb{N}} \subset \mathbb{R}$ IS SUBADDITIVE (SA):

$$t_{p+q} \leq t_p + t_q \quad \text{for all } p, q \in \mathbb{N}.$$

$$\{t_i\}_{i \in \mathbb{N}} \text{ SA} \Rightarrow \lim_{i \to \infty} \frac{t_i}{i} = \tau \leq \frac{t_p}{p}.$$

**CLAIM**: $\{\delta_m\}, \{\tilde{\delta}_m\}$- ARE SA.

$$h_{com} := \lim_{m \to \infty} \frac{\delta_m}{m} \quad \text{CAPACITY}$$

$$h := \lim_{m \to \infty} \frac{\tilde{\delta}_m}{m} \quad \text{ENTROPY}$$

$$h_{per} := \lim_{m \to \infty} \frac{\delta_{m,per}}{m} \quad \text{PERIODIC ENTROPY}$$

$$-\infty \leq h_{per} \leq h \leq h_{com}$$

**MAIN THM**
for 1- dimensional SOFT $\mathcal{S}$

$$h_{per} = h = h_{com} = \log \rho(\Gamma)$$

$\Gamma$ -GRAPH INDUCED BY SOFT.

**COR**. $\mathcal{S}$ is decidable:

EITHER $\mathcal{S} = \emptyset \iff$ :
$\exists m$ with no allow. word of length $m$

OR $\mathcal{S}$ contains an $m$ periodic state.

## 7 MATRICES AND GRAPHS

$$\Gamma \subset < n > \times < n > \Rightarrow A(\Gamma) = (a_{ij})_1^n \quad :$$

$$a_{ij} \in \{0, 1\}, \quad a_{ij} = 1 \iff (i, j) \in \Gamma$$

$$A = (a_{ij})_1^n \geq 0 \Rightarrow \Gamma(A) \subset < n > \times < n > \quad :$$

$$(i, j) \in \Gamma(A) \iff a_{ij} > 0.$$

$$A = (a_{ij})_1^n \in \mathrm{M}_n(\mathbb{R}):$$

$$\mathrm{spec}(A) : \quad \text{set of eigenvalues of } A$$

$$\rho(A) := \max_{\lambda \in \operatorname{spec}(A)} |\lambda|$$

**PERRON-FROBENIUS**:

$A = (a_{ij})_1^n) \geq 0 \Rightarrow$

- $\rho(A) \in \mathrm{spec}(A)$.
- IF $\rho(A) \neq 0 \Rightarrow$

$\mathrm{spec}(\rho(A)^{-1}A) \cap \{|z| = 1\}$-roots of 1.

- $\exists x \geq 0 \ Ax = \rho(A)x$.
- $\limsup_{m\to\infty} \frac{\log \mathrm{tr}\, A^m}{m} = \log \rho(A)$.
- $\lim_{m\to\infty} \frac{\log \mathbf{1} A^m \mathbf{1}^{\mathrm{T}}}{m} = \log \rho(A)$.

$A$ irreducible ($\Gamma(A)$ connected):

- $\rho(A) > 0$.
- all e.v. on $|z| = \rho(A)$ are simple.
- $\mathrm{spec}(\rho(A)^{-1}A) \cap \{|z| = 1\}$ are $k$-roots of 1 and $k|n$.
- $x >> 0$.

$A$-primitive iff $A^m >> 0$ iff $A$-irr. & $\mathrm{spec}(\rho(A)^{-1}A) \cap \{|z| = 1\} = \{1\}$

$$A = (a_{ij})_1^n,\ a_{ij} \in \mathbb{Z}_+$$

- EITHER

$$\rho(A) = 0 \iff A^n = 0 \iff \Gamma(A) \text{ cycleless}$$

OR $\quad \rho(A) \geq 1.$

- $\rho(A)$ is an algebraic integer

$$\rho(A)^n + \sum_{i=1}^{n} d_i \rho(A)^{n-i} = 0, \quad d_1, ..., d_n \in \mathbb{Z}.$$

## MINIMAX CHAR OF $\rho(A)$

- $A \geq 0$, $x = (x_1, ..., x_n)^{\mathrm{T}} >> 0$.

$$\rho(A) \geq \min_{i \in <n>} \frac{\sum_{j=1}^{n} a_{ij} x_j}{x_i}$$

$$\rho(A) \leq \max_{i \in <n>} \frac{\sum_{j=1}^{n} a_{ij} x_j}{x_i}$$

$A$ primitive

$$x^{(m)} = \frac{A x^{(m-1)}}{a_{m-1}}, \; x^{(0)} = x$$

lower& upper bounds converge to $\rho(A)$
$x^{(m)}$ converge to e.v.

$A$ irreducible $\Rightarrow$ $A + I$ primitive

$$A = A^{\mathrm{T}} = (a_{ij})_1^n \geq 0$$

$$\operatorname{tr} A^{2m} = \sum_{i=1}^{n} \lambda_i^{2m} \geq \rho(A)^{2m}.$$

$$\rho(A) = \max_{x \in \mathbb{R}^n \setminus \{0\}} \frac{x^{\mathrm{T}} A x}{x^{\mathrm{T}} x}.$$

$$\rho(A) \geq \frac{\sum_{i=1}^{n} (\sum_{j=1}^{n} a_{ij} x_j) x_i}{\sum_{i=1}^{n} x_i x_i} \geq$$

$$\min_{i \in <n>} \frac{\sum_{j=1}^{n} a_{ij} x_j}{x_i}.$$

$\Gamma$ symmetric (undirected) $\iff$
$A(\Gamma) = A(\Gamma)^{\mathrm{T}}$

## 8 PROOF MAIN THM for 1-dim SOFT

$$A = A(\Gamma) = (a_{ij})_1^n \quad A^m = (a_{ij}^{(m)})_1^n$$

$$a_{ij}^{(m)} = \sum_{i_1,\ldots,i_{m-1} \in <n>} a_{ii_1} a_{i_1 i_2} \ldots a_{i_{m-1} j}$$

$$a_{ii_1} a_{i_1 i_2} \ldots a_{i_{m-1} j} \in \{0, 1\}$$

equal to 1 iff $\Gamma$ has path

$$i \rightarrow i_1 \rightarrow .. \rightarrow i_{m-1} \rightarrow j$$

$a_{ij}^{(m)}$ # paths length $m$ from $i$ to $j$

$$\mathbf{1} A^m \mathbf{1}^{\mathrm{T}} = \#\Gamma^m$$

$$\operatorname{tr} A^m = \sum_{i=1}^{n} a_{ii}^{(m)} = \#\Gamma_{per}^m$$

$$h_{per} = \log \rho(\Gamma) \leq h \leq h_{com} = \log \rho(\Gamma)$$

## 9 MULTI-DIMENSIONAL CAPACITY

$2 \leq d$ dimension

$\mathbf{e}_j = (\delta_{1j}, ..., \delta_{dj})$ $j \in < d >$ basis

$\mathbf{m} = (m_1, ..., m_d) \leq \mathbf{n} = (n_1, ..., n_d) \iff$

$$m_i \leq n_i, \; i = 1, ..., d$$

$a :< \mathbf{m} > \to < n >$ is $(a_{\mathbf{i}})_{\mathbf{i} \in < \mathbf{m} >}$

$< n >^{< \mathbf{m} >}$ set of all maps $a$.

$< n >^{\mathbb{N}^d}$ & $< n >^{\mathbb{Z}^d}$ all maps from $\mathbb{N}^d$ & $\mathbb{Z}^d$ to $< n >$.

shift $\sigma_j : \mathbb{N}^d \to \mathbb{N}^d$ $(\sigma_j : \mathbb{Z}^d \to \mathbb{Z}^d)$

$$\sigma_j((a_{\mathbf{i}})) = (a_{\mathbf{i}+\mathbf{e}_j}), \quad j = 1, ..., d.$$

$\sigma_1, ..., \sigma_d$ commuting

$\sigma_j$ invertible on $\mathbb{Z}^d$

$< n >^{\mathbb{Z}^d}$ & $< n >^{\mathbb{N}^d}$ compact metric

$\mathcal{S} \subset < n >^{\mathbb{N}^d}$ $(< n >^{\mathbb{Z}^d})$ subshift if

(a) $\mathcal{S}$ is closed set

(b) $\sigma_i(\mathcal{S}) = \mathcal{S}$ for $i = 1, ..., d$

allowable states independent of origin

$\pi_{\mathbf{m}}((a_{\mathbf{i}})) = (a_{\mathbf{i}})_{\mathbf{i} \in < \mathbf{m} >}$ proj. on $< \mathbf{m} >$.

$\mathcal{S}$ is SOFT if its maximal subshift for

$\exists P \subset < n >^{< \mathbf{r} >}$ such that $\pi_{\mathbf{r}}(\mathcal{S}) \subset P$.

Equivalently: $a \in \mathcal{S} \iff$ any

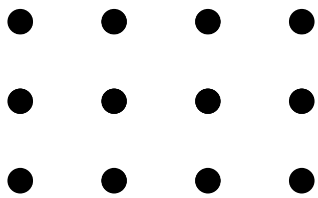
consecutive box of $\mathbf{r}$ letters in $a$ is in $P$.

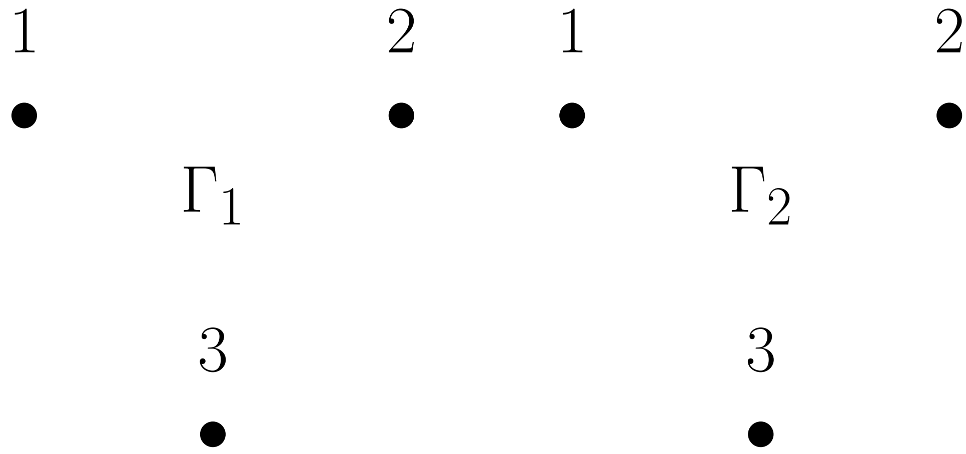
**EXAMPLE**: $\Gamma = (\Gamma_1, ..., \Gamma_d)$

$(a_{\mathbf{i}})_{\mathbf{i} \in <\mathbf{m}>}$ IS ALLOWABLE WORD

if any line through $\mathbf{i} \in < \mathbf{m} >$

in direction $\mathbf{e}_k$ is in $\Gamma_k^{m_k - 1}$ for $k \in < d >$

$< (4, 3) > :=$ 



$\Gamma^{\mathbf{m}}$ all allowable words in $< \mathbf{m} + \mathbf{1} >$

$\Gamma^{\mathbf{m}}_{per} \subset \Gamma^{\mathbf{m}}$ all $\mathbf{m}$ periodic words

$$a_{\mathbf{i}+\mathbf{m}} = a_{\mathbf{i}}$$

(we view $\Gamma^{\mathbf{m}}$ as subset of $\Gamma^{\mathbf{m}-\mathbf{1}}$)

$\Gamma^{\mathbb{N}^d}$ all allowable words in $\mathbb{N}^d$

$\Gamma^{\mathbb{Z}^d}$ all allowable words in $\mathbb{Z}^d$

**any SOFT** can be coded as $\Gamma^{\mathbb{N}^d}$ $(\Gamma^{\mathbb{Z}^d})$

$\exists$ $\mathcal{S}$ SOFT **NOT DECIDABLE**:

$\exists \Gamma^{\mathbb{N}^2} \neq \emptyset$ with no periodic state

Berger 1966

$$\Gamma^{\mathbb{N}^d} = \emptyset \iff \exists \mathbf{m}\, \Gamma^{\mathbf{m}} = \emptyset$$

**ASSUMPTION**: $\mathcal{S}$ -SOFT

$\delta_{\mathbf{m}}$-log# $P$-allow. words of dim. $\mathbf{m}$.

$$\tilde{\delta}_{\mathbf{m}} = \log \#\pi_{\mathbf{m}}(\mathcal{S})$$

$\delta_{\mathbf{m},per}$-log# $\mathbf{m}$-periodic words in $\mathcal{S}$.

$$\delta_{\mathbf{m},per} \le \tilde{\delta}_m \le \delta_m.$$

$\{\delta_{\mathbf{m}}\}, \{\tilde{\delta}_{\mathbf{m}}\}$- are SA in each coordinate

(split box $< \mathbf{m} >$ to 2 boxes by $x_k = i_k$)

$$h_{com} := \lim_{\mathbf{m}\to\infty} \frac{\delta_{\mathbf{m}}}{|\mathbf{m}|_{pr}} \quad \text{CAPACITY}$$

$$h := \lim_{m\to\infty} \frac{\tilde{\delta}_{\mathbf{m}}}{|\mathbf{m}|_{pr}} \quad \text{ENTROPY}$$

$$h_{per} := \lim_{\mathbf{m}\to\infty} \frac{\delta_{\mathbf{m},per}}{|\mathbf{m}|_{per}} \quad \text{PERIODIC ENTROPY}$$

$$-\infty \le h_{per} \le h \le h_{com} \le \frac{\delta_{\mathbf{m}}}{|\mathbf{m}|_{pr}}$$

Berger:$-\infty = h_{per} < 0 \le h \le h_{com}$

Friedland 97: for all $\mathcal{S}$ $h = h_{com}$

## 10 UPPER ESTIMATES OF MDC

$$\tau \subset < d > \; \& \; \mathbf{m} = (m_1, ..., m_d) \in \mathbb{N}^d$$

$$\tau^c := < d > \setminus \tau = \{q_1, q_2, ..., q_p\}$$

$$1 \leq q_1 < ... < q_p \leq d, \; p = d - \#\tau$$

$$\mathbf{m}^\tau = (m_{q_1}, m_{q_2}, ..., m_{q_p})$$

$$\mathbf{m}^{\{i\}} = (m_1, ..., m_{i-1}, m_{i+1}, ..., m_d)$$

$$\mathbf{m} = (\mathbf{m}^{\{i\}}, m_i),$$

$$\Gamma^\tau = (\Gamma_{q_1}, ..., \Gamma_{q_p})$$

$$\Gamma(k, \mathbf{m}^{\{k\}}) \subset (\Gamma^{\{k\}})^{\mathbf{m}^{\{k\}} - \mathbf{1}^{\{k\}}} \times (\Gamma^{\{k\}})^{\mathbf{m}^{\{k\}} - \mathbf{1}^{\{k}}$$

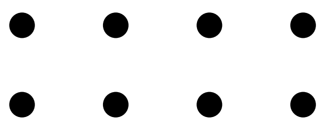
graph on vertices $(\Gamma^{\{k\}})^{\mathbf{m}^{\{k\}} - \mathbf{1}^{\{k\}}}$:

$$((b_{\mathbf{i}^{\{k\}}})_{\mathbf{i}^{\{k\}} \in < \mathbf{m}^{\{k\}} >}, (c_{\mathbf{i}^{\{k\}}})_{\mathbf{i}^{\{k\}} \in < \mathbf{m}^{\{k\}} >})$$

in $\Gamma(k, \mathbf{m}^{\{k\}}) \iff (b_{\mathbf{i}^{\{k\}}}, c_{\mathbf{i}^{\{k\}}}) \in \Gamma_k$

for each $\mathbf{i}^{\{k\}} \in < \mathbf{m}^{\{k\}} >$

**EXAMPLE:** $\Gamma(2,4)$



$$\Gamma_{per}(k,\mathbf{m}^{\{k\}}) \subset (\Gamma^{\{k\}})^{\mathbf{m}^{\{k\}}}_{per} \times (\Gamma^{\{k\}})^{\mathbf{m}^{\{k\}}}_{per}$$

graph on vertices $(\Gamma^{\{k\}})^{\mathbf{m}^{\{k\}}}_{per}$:

$((b_{\mathbf{i}^{\{k\}}})_{\mathbf{i}^{\{k\}} \in <\mathbf{m}^{\{k\}}+\mathbf{1}^{\{k\}}>}, (c_{\mathbf{i}^{\{k\}}})_{\mathbf{i}^{\{k\}} \in <\mathbf{m}^{\{k\}}+\mathbf{1}^{\{k\}}$

in $\Gamma_{per}(k,\mathbf{m}^{\{k\}})$ iff $(b_{\mathbf{i}^{\{k\}}}, c_{\mathbf{i}^{\{k\}}}) \in \Gamma_k$

for each $\mathbf{i}^{\{k\}} \in <\mathbf{m}^{\{k\}} + \mathbf{1}^{\{k\}}>$

$\rho(\Gamma(k,\mathbf{m}^{\{k\}}))$ & $\rho(\Gamma_{per}(k,\mathbf{m}^{\{k\}}))$

spec. rad. $\Gamma(k,\mathbf{m}^{\{k\}})$ & $\Gamma_{per}(k,\mathbf{m}^{\{k\}})$

$\Gamma(k, \mathbf{m}^{\{k\}})^{\mathbb{N}}$ & $\Gamma(k, \mathbf{m}^{\{k\}})^{\mathbb{Z}}$

1-dim. SOFT $\Gamma$ allow. conf.:

"strip" in direction $k$ with

basis dimension $\mathbf{m}^{\{k\}}$

$\Gamma_{per}(k, \mathbf{m}^{\{k\}})^{\mathbb{N}}$ & $\Gamma_{per}(k, \mathbf{m}^{\{k\}})^{\mathbb{Z}}$

1-dim. SOFT $\Gamma$ allow. conf.:

"strip" in direction $k$ with

$\mathbf{m}^{\{k\}}$-periodic basis dim. $\mathbf{m}^{\{k\}}+\mathbf{1}^{\{k\}}$

view $\mathbf{m}^{\{k\}}$ periodic conf.

$(b_{\mathbf{i}^{\{k\}}})_{\mathbf{i}^{\{k\}} \in <\mathbf{m}^{\{k\}}+\mathbf{1}^{\{k\}}>} \in (\Gamma_{per}^{\{k\}})^{\mathbf{m}^{\{k\}}}$

as $(b_{\mathbf{i}^{\{k\}}})_{\mathbf{i}^{\{k\}} \in <\mathbf{m}^{\{k\}}>} \in (\Gamma^{\{k\}})^{\mathbf{m}^{\{k\}}-\mathbf{1}^{\{k\}}}$

periodic exten. to a configuration

$(b_{\mathbf{i}^{\{k\}}})_{\mathbf{i}^{\{k\}} \in <\mathbf{m}^{\{k\}}+\mathbf{1}^{\{k\}}>} \in (\Gamma^{\mathbf{k}\}})^{\mathbf{m}^{\{k\}}}.$

$(\Gamma^{\{k\}})_{per}^{\mathbf{m}^{\{k\}}}$ & $\Gamma_{per}(k, \mathbf{m}^{\{k\}})$ subsets

$(\Gamma^{\{k\}})^{\mathbf{m}^{\{k\}}-\mathbf{1}^{\{k\}}}$ & $\Gamma(k, \mathbf{m}^{\{k\}})$

statistical mechanics *transfer* matrix:

$A(\Gamma(k, \mathbf{m}^{\{k\}}))$

$$\lim_{m_k \to \infty} \frac{\delta_{\mathbf{m}}}{m_k} = \log \rho(\Gamma(k, \mathbf{m}^{\{k\}})) \geq |\mathbf{m}^{\{k\}}|_{pr} h$$

$$\lim_{m_k \to \infty} \frac{\delta_{\mathbf{m},per}}{m_k} = \log \rho(\Gamma_{per}(k, \mathbf{m}^{\{k\}}))$$

$$\lim_{\mathbf{m}^{\{k\}} \to \infty} \frac{\log \rho(\Gamma(k, \mathbf{m}^{\{k\}}))}{|\mathbf{m}^{\{k\}}|_{per}} = h$$

$$\lim_{\mathbf{m}^{\{k\}} \to \infty} \frac{\log \rho(\Gamma_{per}(k, \mathbf{m}^{\{k\}}))}{|\mathbf{m}^{\{k\}}|_{per}} = h_{per}$$

$$\rho(\Gamma(k, \mathbf{m}^{\{k\}})) \geq \rho(\Gamma_{per}(k, \mathbf{m}^{\{k\}}))$$

**THM**: $\mathcal{S} \subset < n >^{\mathbb{N}^2} \ (< n >^{\mathbb{Z}^2})$

$\Gamma = (\Gamma_1, \Gamma_2)$-SOFT

$\Gamma_2$ **SYMMETRIC**

$$\frac{\log \rho(\Gamma(1, p + 2q + 1) - \log \rho(\Gamma(1, 2q + 1))}{p} \leq$$

$$h \leq \frac{\log \rho(\Gamma_{per}(1, 2m))}{2m} (\leq \frac{\log \rho(\Gamma(1, 2m)}{2m})$$

for any $m, p \geq 1$ and $q \geq 0$.

$\mathcal{S}$ **DECIDABLE**:

$$\mathcal{S} \neq \emptyset \iff \Gamma(1, 2) \text{ has cycle}$$

**AND COMPUTABLE**

$$h_{per} = h$$

**Proof.**

$\Gamma_2$ sym. $\Rightarrow \Gamma(2,i) \& \Gamma_{per}(2,i)$ sym.

eigenvalues of $A(\Gamma(2,i))$ are real

$\rho(\Gamma(2,i))$ max. eig.val. $A(\Gamma(2,i))$.

$$\rho(\Gamma(2,i))^{2m} \leq \operatorname{tr} A(\Gamma(2,i))^{2m} = \#\Gamma_{per}(1,2m)^{i-}$$

$$\frac{\log \rho(\Gamma(2,i))}{i} \leq \frac{1}{2m} \frac{\log \#\Gamma_{per}(1,2m)^{i-1}}{i}.$$

$$i \to \infty \Rightarrow h \leq \frac{\log \rho(\Gamma_{per}(1,2m))}{2m}$$

$N = \#\Gamma_1^{i-1}$ & $0 \neq x \in \mathbb{R}^N$

$$\rho(\Gamma(2,i))^p \geq \frac{x^{\mathrm{T}} A(\Gamma(2,i))^p x}{x^{\mathrm{T}} x}.$$

$$x = A(\Gamma(2,i))^q \mathbf{1}^{\mathrm{T}}.$$

$$\rho(\Gamma(2,i))^p \geq \frac{\mathbf{1}A(\Gamma(2,i))^{p+2q}\mathbf{1}^{\mathrm{T}}}{\mathbf{1}A(\Gamma(2,i))^{2q}\mathbf{1}^{\mathrm{T}}}$$

$$\#\Gamma^{(i-1,\ell-1)} = \#\Gamma(1,\ell)^{i-1}$$

$$\#\Gamma(2,i)^{\ell-1} = \mathbf{1}A(\Gamma(2,i))^{\ell-1}\mathbf{1}^{\mathrm{T}} \Rightarrow$$

$$\frac{\log\rho(\Gamma(2,i))}{i} \geq$$

$$\frac{\log\#\Gamma(1,p+2q+1)^{i-1} - \log\#\Gamma(1,2q+1)^{i-}}{pi}$$

$$i \to \infty \Rightarrow$$

$$h \geq \frac{\log\rho(\Gamma(1,p+2q+1) - \log\rho(\Gamma(1,2q+1))}{p}$$

$$M = \#\Gamma_2$$

$$\Gamma(1,2) \text{ cycleless} \Rightarrow \Gamma(1,2)^M = \emptyset \Rightarrow \mathcal{S} = \emptyset$$

$$(u_1, v_1) \rightarrow ... \rightarrow (u_{L+1}, v_{L+1}) = (u_1, v_1) \in$$
$$\Gamma_{per}(1,2)^L \quad \& \Gamma_2 \text{ sym.} \Rightarrow$$
$$(u_1, v_1, u_1), ..., (u_{L+1}, v_{L+1}, u_{L+1}) \in$$
$$\Gamma_{per}(1,2)^L \Rightarrow \Gamma_{per}^{<L,2>} \neq \emptyset \Rightarrow \mathcal{S} \neq \emptyset$$

$$\limsup_{i\rightarrow\infty} \frac{\delta_{(i,2m),per}}{i\ 2m} = \frac{\log \rho(\Gamma_{per}(1,2m))}{2m} \Rightarrow$$
$$h_{per} = \limsup_{m\rightarrow\infty} \frac{\log \rho(\Gamma_{per}(1,2m))}{2m} \geq h \geq h_{per}$$
$$\Rightarrow h_{per} = h$$

$$q = 0 \Rightarrow$$

$$\frac{\log\rho(\Gamma(1,p+1))}{p} - \frac{\log(\Gamma(1,1))}{p} \leq h$$

$$\leq \frac{\log\rho(\Gamma(1,p+1))}{p+1}, \quad p \geq 1 \Rightarrow$$

$$\frac{\log\rho(\Gamma(1,p+1))}{p+1} -$$

$$(\frac{\log\rho(\Gamma(1,p+1))}{p} - \frac{\log\rho(\Gamma(1,1))}{p}) =$$

$$-\frac{\log\rho(\Gamma(1,p+1))}{p(p+1)} + \frac{\log\rho(\Gamma_1)}{p+1}$$

$$\leq \frac{\log\rho(\Gamma_1) - h}{p} \leq \frac{\log\rho(\Gamma_1)}{p} \Rightarrow$$

COMPUTABILITY

## Historical Remarks

Friedland 97: $q = 0$ and $h_{per} = h$

Calkin-Wilf 98: any $q$ and
improved u.b. for $h$ in special $\mathbb{Z}^2$ SOFT

Sharper l.b. for $h$ for $p = 1$ and $q \nearrow$:

Weeks-Blahut 98

Forschhammer-Justenes 99

Nagy-Zeger 00

**12** $d \geq 3$

**THM**: $\mathcal{S} \subset < n >^{\mathbb{N}^d}$ $(< n >^{\mathbb{Z}^d})$

$\Gamma = (\Gamma_1, ..., \Gamma_d)$-SOFT

$\Gamma_j$ **SYMMETRIC** for $j \in \tau \subset < d >$

$i \in < d > \setminus \tau$, $\mathbf{m} = (m_1, ..., m_d) \in \mathbb{N}^d$, $2|m_j, j \in \tau$

$$h \leq \frac{\log \rho(\Gamma_{\tau,per}(i, \mathbf{m}^{\{i\}}))}{|\mathbf{m}^{\{i\}}|_{pr}}$$

# LOWER BOUNDS

$\tilde{\mathcal{S}} \subset < N >^{\mathbb{N}^{d-1}} \;\; (< N >^{\mathbb{Z}^{d-1}})$ SOFT:

$\Gamma^{\{i\},q} = (\Gamma_1^{\{i\},q}, ..., \Gamma_{i-1}^{\{i\},q}, \Gamma_{i+1}^{\{i\},q}, ..., \Gamma_d^{\{i\},q})$

Inf. conf. in dir. $j \neq i$, width $q$ in dir. $i$

$h^{\{i\},q}$ entropy of $\tilde{\mathcal{S}}$

$h^{\{i\}} = h^{\{i\},1} \; (\Gamma^{\{i\},1} = \Gamma^{\{i\}})$

$j \in < d > \setminus \{i\} \Rightarrow$

$$\lim_{m_i \to \infty} \frac{\log \rho(\Gamma(j, \mathbf{m}^{\{j\}})}{m_i} = \log \rho_{\{i\}}(j, \mathbf{m}^{(i,j\}})$$

THM: $\Gamma_j$ sym. for $j \neq i$

$$\log \rho(\Gamma(i, (\mathbf{m}^{\{i,j\}}, p + 2q + 1))) \leq$$

$$\log \rho(\Gamma(i, (\mathbf{m}^{\{i,j\}}, 2q + 1)) + p \log \rho_{\{i\}}(j, \mathbf{m}^{\{i,j}$$

$$\frac{h^{\{j\},p+2q+1} - h^{\{j\},2q+1}}{p} \leq h \leq \frac{h^{\{j\},p+2q+1}}{p + 2q + 1}$$

$$p \geq 1, \ q \geq 0$$

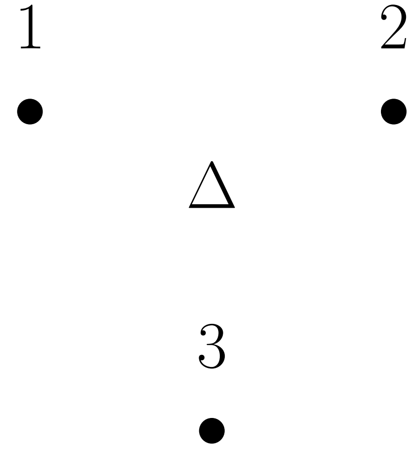
If $\Gamma_j$ sym. for each $j \in < d > \setminus \{i\}$

$h = h_{per}$ **COMPUTABLE**

## 13 EXAMPLES

**Ex.1**-residual entropy of square ice:

$\mathbb{Z}^2$-SOFT given $\Gamma = (\Delta, \Delta)$ $(\Delta = K_3)$




3 col. of $\mathbb{Z}^2$ s.t. no same colors adjac.

Lieb 67 $\Rightarrow h_{per} = \frac{3}{2} \log \frac{4}{3} = 0.43152...$

Brascamp-Kunz-Wu 73 $h_{per} = h_{com}$

$h = \log$ of $u = (\frac{4}{3})^{\frac{3}{2}}$ alg. num.:
$27u^2 = 64$ $u \neq$ alg. int.

$$\rho(\Delta) = \rho(\Gamma(1,1)) = \rho(\Gamma_{per}(1,3)) = 2,$$
$$\rho(\Gamma(1,2)) = \rho(\Gamma_{per}(1,2)) = 3,$$
$$\rho(\Gamma(1,3)) = 4.561552,\ \rho(\Gamma_{per}(1,4)) = 6.372281$$

$$p = 1,\ q = 0 \Rightarrow h \geq \log \frac{3}{2} = 0.405465108.$$

Pauling 1935 estimate $h \approx \log \frac{3}{2}$

$$p = 2,\ q = 0 \Rightarrow h \geq 0.4122579570.$$

$$\frac{\log \rho(\Gamma(1,p))}{p} \geq h \text{ for } p = 1,2,3 \Rightarrow$$

$$0.693147 > 0.549306 > 0.505887 \geq h$$

$$\frac{\log \rho(\Gamma_{per}(1,2m))}{2m} \geq h \text{ for } m = 1,2 \Rightarrow$$
$$0.549306 > 0.462989 \geq h$$

Physicists use asymptotic expansions

Nagle 66 $h = 0.432 \pm 0.001$

**Ex2:** $d-$ dimen. $(0,1)$ r.l.l.c.

$\Gamma_1 = ... = \Gamma_d = \Delta \subset < 2 > \times < 2 >$:

$\Delta$ : • •

$\overline{h}_d = \frac{h_d}{\log 2}$ entropy (capacity) base 2

$$\overline{h}_1 = \log_2 \rho(\Delta) = \log_2 \frac{1+\sqrt{5}}{2} = 0.694241914$$

$0.587891161775 \leq \overline{h}_2 \leq 0.587891161868$

Calkin-Wilf 98, Weeks-Blahut 98, Forschhammer-Justesen 99, Nagy-Zeger 00

$0.5225017411838 \leq \overline{h}_3 \leq 0.526880847825.$

Nagy-Zeger 00

need spec. rad. of $A$ with 40 mill. elem.

**NEARST NEIGHBOR** $\mathbb{Z}^d$-SOFT:

For $\Delta \subset < n > \times < n >$

$\mathcal{S}_d(\Delta)$ given by $\Gamma_1 = ... = \Gamma_d = \Delta$

$h_d(\Delta)$ -entropy of $\mathcal{S}_d(\Delta)$.

$$h_1(\Delta) \geq h_2(\Delta) ... \geq h_d(\Delta)$$

Find $h(\Delta) = \lim_{d\to\infty} h_d(\Delta)$

**Ex3:** $d$-dimensional dimers

Dimer: $(\mathbf{i},\mathbf{j})$, $\mathbf{i} \neq \mathbf{j} \in \mathbb{Z}^d$ s.t. $|\mathbf{i}-\mathbf{j}| = 1$.

any partition of $\mathbb{Z}^d$ to dimers (1-factor)

$\mathcal{S}_d$ -$\Gamma(d) = (\Gamma_{1,d}, ..., \Gamma_{d,d})$ - SOFT

$\Gamma_{i,d} \subset < 2d > \times < 2d >$:

$(i, i+d) \in \Gamma_i$ & $(i,j) \notin \Gamma_i$ for $j \neq i+d$

For $k \neq i$ $(k, i+d) \notin \Gamma_i$.

For $k \neq i,\ j \neq i+d$ $(k,j) \in \Gamma_i$.

$\Gamma_{i,d}$ is not symmetric

$$0 = h_1 \leq h_2 \leq ... \leq h_d \leq ...$$

Fisher, Kasteleyn and Tempreley 61

$$h_2 = \frac{1}{\pi} \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)^2} = 0.29156090...$$

no formula is for $h_3$.

Schrijver 98 (l.b. for permanents)

$$h_d \geq \frac{1}{2}((2d-1)\log(2d-1) - (2d-2)\log(2d)$$

$$\Rightarrow h_3 > 0.440075842$$

Ciucu 98 (without permanents)

$$h_3 \leq 0.463107.$$

Heuristical results:

Nagle 66 $h_3 = 0.44645 \pm 5 \cdot 10^{-5}$

Beichl-Sullivan 99 $h_3 = 0.4466 \pm 6 \cdot 10^{-4}$

Classical results on permanents

$$h_d = \frac{1}{2}\log(2d) - \frac{1}{2} + O(\frac{\log d}{d}).$$

$$C((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>}) := \{(b_{\mathbf{i}})_{\mathbf{i}\in\mathbb{N}^d} \in \mathbb{N}^d$$
$$b_{\mathbf{i}} = a_{\mathbf{i}} \text{ for } \mathbf{i} \in< \mathbf{m} >\}$$

$\mathcal{B} \subset 2^{\mathbb{N}^d}$ $\sigma$-algebra gener. by cylinders
$\Pi$ probability measures on the $\mathbb{N}^d$.
$\mu \in \Pi$ is $\sigma$-invariant $\iff$

$\mu(T) = \mu(\sigma_i^{-1}(T)), \ \forall T \in \mathcal{B} \& i \in< d >$

$\Pi \supset \Pi_i$ - $\sigma$-invariant measures
$\mu \in \Pi$ ergodic: $\mu \in \Pi_i$ &

$\sigma_i^{-1}(T) = T \ \forall i \in< d >\Rightarrow \mu(T) = 0, 1$

$\Pi_i \supset \Pi_e$-all ergodic measures.
$\Pi_e$ set of extreme points in $\Pi_i$

Kolmogorov-Sinai entropy of $\mu \in \Pi_i$:

$$h(\mu) = \lim_{\mathbf{m}\to\infty} \frac{1}{|\mathbf{m}|_{pr}} \sum_{(a_{\mathbf{i}})\in<n>^{<\mathbf{m}>}}$$

$$-\mu(C((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>}))\log\mu(C((a_{\mathbf{i}})_{\mathbf{i}\in<\mathbf{m}>}))$$

Shannon's probabilistic capacity

$\mathcal{S} \subset < n >^{\mathbb{N}^d}$-SOFT

$\Pi_i(\mathcal{S}) \supset \Pi_i(\mathcal{S})$ all $\mu$, $\mu(\mathcal{S}) = 1$.

**MAXIMAL CHARAC.**

$$h(\mathcal{S}) = \sup_{\mu\in\Pi_i(\mathcal{S})} h(\mu) = \sup_{\mu\in\Pi_e(\mathcal{S})} h(\mu)$$

**MAXIMUM ACHIEVED**

TRANSPARENCIES OF TALK

BASED ON THE PAPER:

S. Friedland

"MULTI-DIMENSIONAL CAPACITY,

PRESSURE AND

HAUSDORFF DIMENSION"

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