① S. FRIEDLAND, May 6, 2013 - MCS 421 - Combinatorics - Final.

1a $\binom{50}{26}=\binom{50}{24}$, $\binom{50}{30}=\binom{50}{20}$, $\binom{50}{45}=\binom{50}{5}$

(5pts) Hence $\binom{50}{45}=\binom{50}{5}<\binom{50}{10}<\binom{50}{30}=\binom{50}{20}<\binom{50}{26}=\binom{50}{24}<\binom{50}{30}$

(b) Yes. $\binom{n-1}{n-k}=\binom{n-1}{k-1}$, $\binom{n-1}{n-k-1}=\binom{n-1}{k}$

(10pts) So $\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}$ - Combinatorial proof: $\binom{n}{k}$ all subsets of cardinality k. If n is not is a subsuset of cardinality k, then all choices of such sets is $\binom{n-1}{k}$. If n is in this set then we need to choose other k-1 elements from n-1 elements: $\binom{n-1}{k-1}$. QED.

(10pts) c. $\frac{10!}{3!4!2!1!}(1)^3(-1)^4(2)^2(-3) = -\frac{10!}{3!4!2!1!}\cdot 12 = -151,200$

(5pts) d. NO. what is true is ∃ three that either know each other or ~~the~~ there are three strangers.

(10pts) e. Let a_i be the number of hours of TV from day 1 to day i). So $1\le a_1 < a_2 < \dots < a_{49} \le 77$

Now $a_{i+1}\ge a_i+1$ $11\cdot 7=77$.

Consider ~~$1\le a_1<a_2<\dots<a_{44}<20$~~ $49\times 2=98$ numbers

$1\le a_1, a_2, \dots, a_{49}, 20+a_1, \dots, 20+a_{49}\le 97$

By pigeon hole principle ~~two must~~ at least 2 numbers must be the same, since total number of numbers is

No $a_i<a_j$ for $i<j$ and $20+a_i<20+a_j$ for $i<j$. Hence the two equal numbers are of the form $a_i+20=a_j$ so $a_j-a_i=20$. QED

Problem 2. total number of solutions is $\binom{s+r-1}{s} = \binom{s+r-1}{s}$. (2)

(5pts) (a) Look at number of $\underbrace{1\,1 \ldots 1}_{s}$ which add to s.

Take $r-1$ $*$: $\underbrace{x\,x \ldots x}_{r-1}$ put them together with

$$\underbrace{1\,1}_{x_1}\ x\ \underbrace{1\,1\,1}_{x_2}\ x \ldots *\ \underbrace{1\,1\,1}_{x_r}$$

So $\binom{s+r-1}{s}$ is # of all permutation of $s+r-1$ elements $-$ $(s+r-1)!$ divided by $s!\,(r-1)!$ since we have s identical 1's and $r-1$ identical $*$

$$\frac{(s+r-1)!}{s!\,(r-1)!} = \binom{s+r-1}{s}.$$

(b) let $y_1 = x_1 - 4$, $y_2 = x_2 \le 5$, $y_3 = x_3$, $y_4 = x_4$

So $y_1 + y_2 + y_3 + y_4 = 16$

let P_2: $x_2 \ge 6$, P_3: $x_3 \ge 8$

Use inclusion-exclusion $|S| = \binom{16+3}{3} = \binom{19}{3}$

P_2 $x_2 \ge 6$ $z_2 = x_2 - 6$ $y_1 + z_2 + y_3 + y_4 = 10$

(10pts) $|A_2| = \binom{10+3}{3} = \binom{13}{3}$

$|A_3| =$ $z_3 = x_3 - 8$ $y_1 + y_2 + z_3 + y_4 = 8$

$\binom{8+3}{3} = \binom{11}{3}$

$A_2 \cap A_3$: $y_1 + z_2 + z_3 + y_4 = 2$

$|A_2 \cap A_3| = \binom{2+3}{3} = \binom{5}{3}$

So the answer

$$\binom{19}{3} - \binom{13}{3} - \binom{11}{3} + \binom{5}{3} = 528$$

(c) P_i course i is not attended - inclusion exclusion

$|S| = 5^{20}$ since each student can take one of 5 courses.

② If course i is not taken then all possible courses to choose is 4^i, of one can choose $\binom{5}{1}$ such courses

If 2 courses ~~Ai~~ are not chosen $A_i \cap A_j$ then

$$|A_i \cap A_j| = 3^{20} \text{ and } \binom{5}{2} \text{ such choices of courses}$$

(10p) So number of

$$5^{20} - \binom{5}{1}4^{20} + \binom{5}{2}3^{20} - \binom{5}{3}2^{20} + \binom{5}{4}\cdot 1^{20}$$

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3. ~~Problem~~ (10pt)

$$(1-4x)^{1/2} = \sum_{m=0}^{\infty} (-4)^m \binom{\frac{1}{2}}{m} x^m$$

$$\binom{\frac{1}{2}}{m} = \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-(m-1)\right)}{m!} = \frac{1}{2^m}\frac{(-1)(-3)\cdots(-(2m-3))}{m!}$$

The coefficient of x^{m-1} for $m \geq 1$ is

(10p)

$$\frac{2^{m-1}\,1\cdot 3\cdots(2(m-1)-1)}{m!} = \frac{2^{m-1}\,1\cdot 3\cdots(2(m-1)-1)}{(m!)^2(m-1)!} = \frac{1}{m}\binom{2(m-1)}{m-1}$$

QED.

Going above the diagonal #1 ↑ $-1 \rightarrow$ C_{10}

~~C10~~ Going below the diagonal $+1 \rightarrow$ $-1 \uparrow$ C_{10}

(-10p) $2C_{10}$

Problem 4 (a) A Steiner triple is a block desing where

(10pts) $k=3$, i.e. the number of elements in each block is 3. Recall that $r = \frac{\lambda(v-1)}{k-1}$

as $k=3$ $r = \frac{\lambda(v-1)}{2}$

Next recall that $bk = vr \Rightarrow 3b = vr = v\frac{\lambda(v-1)}{2}$

$$b = \frac{\lambda(v-1)v}{6}$$

(b) In a difference set of order k there are $k(k-1)$ nonzero difference and $v-1$ divides $k(k-1)$

(10) $= 3\cdot 2 = 6$. $v-1 \geq 3$. So $v-1=3$ or $v-1=6$

$v=4$ $B=\{0,1,2\}$ it is a difference set $\{\{1,2\}\ \{1,3\},\ \{2,3\}\}$

So the triples are $\{0,1,2\}$, ~~$\{0,1,$~~ $\{1,2,3\}$, $\{0,2,3\}$, $\{0,1,3\}$

(10) $v=7$. $B_1 = B = \{0,1,3\}$ Block

$B_2 = \{1,2,4\}$, $B_3 = \{2,3,5\}$, $B_4 = \{3,4,6\}$, $B_5 = \{4,5,0\}$

$B_6 = \{5,6,1\}$ $B_7 = \{6,0,2\}$

5. (a) a Latin square is a $n \times n$ matrix with entries $\{0,1,\ldots,n-1\}$ such that in ~~each~~ each row or column

(5p) is a permutation of $\{0,1,\ldots,n-1\}$.

(b) (5p)

$$
\begin{array}{cc}
 & \text{col } 0 \\
\text{row } 0 & \begin{bmatrix} 0 & 1 & \cdots & n-1 \\ 1 & 2 & \cdots\ n-1 & 0 \\ \vdots & & & \\ n-1 & 0 & 1\ \cdots & n-2 \end{bmatrix}
\end{array}
$$

i.e. $a_{ij} = i+j \mod n$
$i,j = 0,\ldots,n-1$

(c). ~~An~~ ~~orthogo~~ Two orthogonal lattices $A=[a_{ij}]$, $B=[b_{ij}]$ are Latin squares such that when we consider the pairs (a_{ij}, b_{ij}) $(i,j) \in \mathbb{Z}_n$ each pair (p,q), $p,q \in \mathbb{Z}_n$ appears exactly one.

(5p)

(d) Let $a_{ij} = i+j$ $b_{ij} = i-j$

(10p) suppose that the pair $(i+j, i-j) = (p+q, p-q)$

so $i+j = p+q$
$i-j = p-q$

So $2i = 2p$ (add two equations)
but $(2,n)=1$ so $2(i-p)$ divisible by n.

n is odd so $n \mid c-p$

Hence $c \equiv p \in \mathbb{Z}_n$.

As $2j = (i+j)-(i-j) = (p+q)-(p-q) \Rightarrow$

$2(j-q) \equiv 0 \Rightarrow j \equiv q \in \mathbb{Z}_n$.

Hence we can not have the same pair appear twice. But we have exactly n^2 pairs (c,j). Hence A, B is a pair of orthogonal latin squares.

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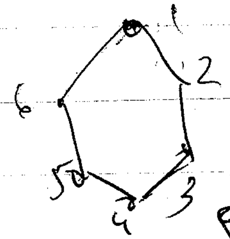
6 (15)



id — z_1^6

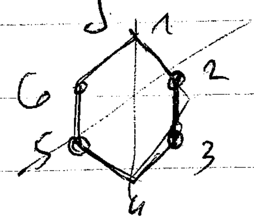
$\rho = (123456) \leftrightarrow z_6$

$\rho^2 = (135)(246) \leftrightarrow z_3^2$

$\rho^3 = (14)(25)(36)$ $\quad z_2^3$

$\rho^4 = \rho^6 \rho^{-2} = (153)(264)$ $\quad z_3^2$

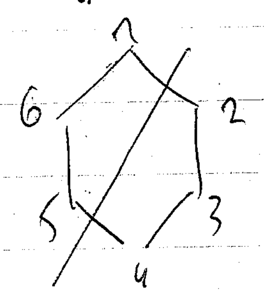
$\rho^5 = (165432)$ $\quad z_6$



$\tau_{14} = (1)(4)(26)(35)$ $\quad z_1^2 z_2^2 \times 3$

$\tau_{25} = (2)(5)(13)(46)$

$\omega_{12,45} = (12)(36)(45)$ $\quad z_2^3 \times 3$



$$P_{D_6} = \frac{1}{12}\left(z_1^6 + 2z_6 + 2z_3^2 + 3z_1^2 z_2^2 + 4z_2^3\right)$$

(5) # of nonequivalent colors for $k=3$

$$\frac{1}{12}\left(3^6 + 2\cdot 3 + 2\cdot 3^2 + 3\cdot 3^4 + 4\cdot 3^3\right) = 92$$

coefficient of

c(10) $r^2 b^2 g^2$

$$\frac{1}{12}\Big(\underset{A}{(r+b+g)^6} + 2\underset{B}{(r^6+b^6+g^6)} + \cancel{2}\underset{C}{(r^3+b^3+g^3)^2} + 3\underbrace{(r+b+g)^2(r^2+b^2+g^2)^2}_{D} + \cancel{4}\underbrace{(r^2+b^2+g^2)^3}_{F} \Big)$$

mistake

$$A: \frac{6!}{2!2!2!} = \frac{24 \times 5 \times 6}{8} = 90$$

$B = 0$

$C = 0$

$$D: 3\,(r^2+b^2+g^2)^2\,(r^2+b^2+g^2) = 3 \cdot (r^2+b^2+g^2)^3 \to 3 \times \frac{3!}{1!1!1!} = 18$$

$$F \Rightarrow \cancel{4} \times \frac{3!}{1!1!1!} = \cancel{18}\,24$$

$$\frac{1}{12}(90 + 18 + \cancel{24}) = \frac{132}{12} = \underline{\underline{11}} ✓$$