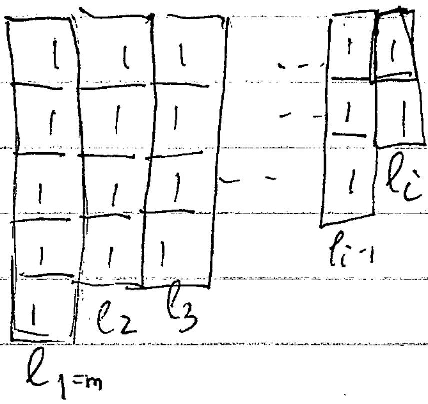
SOLUTIONS TO HW11 - MATH 425 - S12 - FRIEDLAND

p'69 of MATH 425 NOTES

1. Let A be nilpotent, and assume that $l_1 \ge l_2 \ge \ldots l_i \ge 1$ are the invariants of A, as in Herstein's book page 246



w_1 number of 1 in the first row

w_2 " " 1 in the second row

w_m numbers of 1 in the last row

Claim $\operatorname{rank} T^{j} = \sum_{j=1}^{i} \max(l_j - j, 0)$

But the formula given in Problem 1 is false!

2. Let $\psi(z) = (z-\lambda_1)^{m_1} \cdots (z-\lambda_k)^{m_k}$ $m_j \ge 1$, $j=1,\ldots,k$ where k is the number of distinct eigenvalues of A. Then m_j is the size of the largest Jordan block corresponding to λ_j. Hence A is diagonable iff $m_j = 1$, for $j=1,\ldots,k$. See Quiz 10.

3. Note that if A and B are similar matrices then $\operatorname{rank} A^p = \operatorname{rank} B^p$. Hence it is enough to prove the theorem if $A = \operatorname{diag}(J_1, \ldots, J_p)$ where J_j is a Jordan block.

Clearly $\operatorname{rank}(A-\lambda I)^p = \sum_{j=1} \operatorname{rank}(J_j - \lambda I_j)^p$

Now if λ is not eigenvalue of J_j then $\operatorname{rank}(J_j - \lambda I_j)^p = \operatorname{rank} I_j$

If λ is an eigenvalue of J_j then

$$\text{rank } (J_j - \lambda I_j)^p = \max(\text{rank } I_j - p, 0).$$

Combine these results to prove (a)

(b) follows from (a), since the ~~number of total~~ sum of sizes of Jordan block corresponding to λ_j is h_j.

4b $\begin{bmatrix} F & 0 \\ 0 & G \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow Fx_1 = 0, \; Fx_2 = 0$

Hence $\dim \ker \begin{bmatrix} F & 0 \\ 0 & G \end{bmatrix} = \dim \ker F + \dim \ker G$.

Additional problems

1. $J_4(2) \oplus J_3(5)$

2. $J_2(2) \oplus J_2(2) \oplus J_3(5)$, or $J_2(2) \oplus J_1(2) \oplus J_1(2) \oplus J_3(5)$

3. Diagonal matrix $\begin{bmatrix} 2 & & & & & & \\ & 2 & & & & 0 & \\ & & 2 & & & & \\ & & & 2 & & & \\ & 0 & & & 5 & & \\ & & & & & 5 & \\ & & & & & & 5 \end{bmatrix}$

4. $J_2(0) \oplus J_2(0) \oplus J_1(0) \oplus J_3(1)$, $J_2(0) \oplus J_2(0) \oplus J_3(1) \oplus J_1(1)$,

$J_2(0) \oplus J_1(0) \oplus J_1(0) \oplus J_1(1) \oplus J_3(1)$, $J_2(0) \oplus J_1(0) \oplus J_1(0) \oplus J_3(1) \oplus J_1(1)$

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$J_2(0) \oplus J_1(0) \oplus J_3(1) \oplus J_1(1) \oplus J_1(1)$, $J_2(0) \oplus J_3(1) \begin{cases} \oplus\, 3J_1(1) \\ \oplus\, J_2(1) \oplus J_1(1) \\ \oplus\, J_3(1) \end{cases}$

So we have to have $J_2(0) \oplus J_3(1)$

+ some Jordan blocks corresponding to $\lambda = 0$ of order 2 at most, and some Jordan blocks corresponding to $\lambda = 1$ of order 3 at most.

Problem 3 (a) Note that rank $(A-\lambda I)^p$ = rank $(A^T-\lambda I)^p$
As $\det(zI-A) = \det(zI-A^T)$ it follows that
A and A^T have the same eigenvalues.
So $s_i(A,\lambda_j) = s_i(A^T,\lambda_j)$ as defined in Problem 3
on page 69.
Hence A and A^T have the same number of
Jordan blocks of length i corresponding to λ_j.
I.e. A & A^T have the same Jordan canonical form.
I.e. $PAP^{-1}=J$ $QA^TQ^{-1}=J \Rightarrow A, A^T$ are similar.
(b) If det does vanish identically on some
subspace of $\mathbb{F}^{n\times n}$ then it can vanish accidentally
(~~when the subspace~~ $\mathbb{F} \neq \mathbb{F}_1$) then it vanish identically on
this subspace over the extension field $\mathbb{F}_1$.
(The subspace is defined by some ~~equations~~ linear
conditions over $\mathbb{F}$).

Problem 4 Look at the solution of Quiz 10.

Problem 5 rank $11^T = 1$, since it has one lin. ind. row.
null$(11^T)=n-1$, i.e. $A\underline{x}_i=\underline{0}$ for $n-1$ lin. ind. vectors
$i=1,\dots,n-1$. $(\underline{1}\underline{1}^T)\underline{1} = n\underline{1}$ So $x_n=\underline{1}$ is an eigenvector
of $A=11^T$ corresponding to n. If char F does not divide n
Then $n\neq 0$ in $\mathbb{F}$. Hence A has n lin. ind. vectors, so
A is similar to $\operatorname{diag}(n,0,\dots,0)$.
On the other hand if char $F | n$ the $\operatorname{diag}(n,0,\dots,0) = 0$
matrix. Clearly $A\neq 0$. So A and $\operatorname{diag}(n,0,\dots,0)$ are
not ~~line~~ similar.