A. Observe that for any system (0) $\underline{x}_\ell = A_\ell \underline{x}_{\ell-1}$
we have $x_\ell = A_\ell A_{\ell-1} A_{\ell-2} \cdots A_1 x_0 \quad \ell=1,2,\ldots$
~~Hence~~ if $A_{\ell+p} = A_\ell$, it is easy to see that

(1) $\underline{x}_{pj} = B^j \underline{x}_0$ (Take a few examples $p=1,2,3$)

Hence (2) $\underline{x}_{pj+r} = A_r \cdots A_1 B^j x_0$ for $r=1,\ldots,p-1$

(a) (2) Hence (0) is stable $\Rightarrow$ (1) is stable, i.e. $\rho(B)<1$, which is equivalent to all eigenvalues of B in the unit disk $|z|<1$. It is equivalent to $\lim_{j\to\infty} B^j = 0$. (3)

Suppose (3) holds. Then $\lim_{j\to\infty} A_r A_{r-1} \cdots A_1 B^j = 0$ for $r=1,\ldots,p-1$. Hence (0) is stable.

(b) Suppose that for each $\underline{x}_0$ $\lim_{\ell\to\infty} x_\ell = \underline{y}(x_0)$

By chosing $\underline{x}_0 = \underline{e}_i = (\delta_{i1},\ldots,\delta_{in})^T$ we deduce that (0) is ~~stable~~ convergent iff $\lim_{\ell\to\infty} A_\ell A_{\ell-1}\cdots A_1 = C$. (3)

By taking $\ell = pj$ we get $A_{pj} A_{pj-1} \cdots A_1 = B^j$ (4)
Hence (4) $\Rightarrow \lim_{j\to\infty} B^j = C$ (5), i.e. B is power convergent.
Let $\ell = pj+1$. Then $A_{pj+1} A_{pj} \cdots A_1 = A_1 B^j$
~~As~~ From (3)+(4) we deduce that $A_1 C = C$
Use (2) to deduce that $\lim_{j\to\infty} A_{pj+2} A_{pj+1} \cdots A_1 = C$
~~implies $C = A_2 A_1 C = A_2 C$, Continuing in the same manner~~
deduce that nec. & suf. conditions for (0) to be convergent is I: condition (5), II $A_j C = C \quad j=1,2,\ldots,p$.

(c) (4) yields that (0) is power bdd $\Rightarrow$ B is power bounded. Observe if $\{B^j\}_{j=0}^{\infty}$ is bdd sequence then $\{B^j, A_1B^j, A_2A_1B^j, \ldots, A_{p-1}\cdots A_1B^j\}$ is a bounded sequence. Hence nec. & suf. conditions for power boundedness of (0) is that B is power bdd.

B. 1. Char pol $A=\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$ is $z^2-4z+4=(z-2)^2$

Since A is not $2I$, A is not diagonable. Hence the minimal pol of A is $(z-2)^2$. A has two components Z_1, Z_2 : (1) $f(A)=f(2)Z_1+\frac{f'(2)}{1!}Z_2 = f(2)Z_1+f'(2)Z_2$

$Z_1=\psi_1(A)$, $Z_2=\psi_2(A)$,

$\psi_1(2)=1$ $\psi_1'(2)=0$, $\psi_2(2)=0$ $\psi_2'(2)=1$

Recall $\psi_i(z) = a_i+b_i(z-2)$

$\psi_1(2)=1 \Rightarrow a_1=1$ $\psi_1'(2)=b_1=0 \Rightarrow \psi_1=1$.

$\psi_2(2)=a_2=0$ $\psi_2'(2)=b_2=1 \Rightarrow \psi_2=z-2$

$Z_1=I_2=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $Z_2=A-2I_2=\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$

$A^{100}=f(A)$, $f(z)=z^{100}$, $f(2)=2^{100}$, $f'(z)=100z^{99}$, $f'(2)=100\, 2^{99} \Rightarrow$

$$A^{100}=2^{100}Z_1+100\,2^{99}Z_2=$$

$$=2^{99}\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}+\begin{bmatrix} -100 & 100 \\ -100 & 100 \end{bmatrix}\right)=2^{99}\begin{bmatrix} -98 & 100 \\ -100 & 102 \end{bmatrix}$$

$e^{At}=f(A)$, $f(z)=e^{zt}$, $f'(z)=te^{zt}$, $f(2)=e^{2t}$, $f'(2)=te^{2t}$

$$e^{At}=e^{2t}\left[I_2+t\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\right)=e^{2t}\begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix}.$$

2. Charpol. of $A = \begin{bmatrix} 0 & 2 & -1 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{bmatrix} \Rightarrow z(z^2 - z) = z^2(z-1)$

RREF

dim ker A = 3 - rank A $\quad A \sim \begin{bmatrix} 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ of A.

~~So $Ax=$~~ rank $A = 2$ Hence the dimension of the eigenspace corresponding to 2 is 1 Thus the JCF of A has 2×2 block $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Thus min pol of A = char pol of A $\quad \psi(z) = z^2(z-1)$

$\lambda_1 = 0$ multiplicity 2 $\quad \lambda_1 = 1$ multip 1.

(1) $f(A) = f(0) Z_{11} + f'(0) Z_{12} + f(1) Z_{21}$

$Z_{11} = \psi_{11}(A), \quad Z_{12} = \psi_{12}(A), \quad Z_{21} = \psi_{21}(A)$

$\psi_{11} = (a_1 + b_1 z)(z-1), \quad \psi_{12} = (a_2 + b_2 z)(z-1)$

$\psi_{21} = z^2 a_3$

$\psi_{11}(0) = 1 \quad \psi_{11}'(0) = 0 \quad \psi_{11}(1) = 0$ ✓

" $-a_1 = $ $a_1 = -1$ $\quad 0 = \psi_{11}'(0) = b_1(0-1) + a_1 = -b_1 + a_1, \quad b_1 = a_1 = -1$

$\psi_{11} = (1-z)(1+z) = 1 - z^2$

$\psi_{12}(0) = 0 \quad -a_2 = 0 \quad a_2 = 0 \qquad \psi_{12}' = b_1(z-1) + b_1 z \big|_{z=0} = -b_1 = 1$ " $\psi_{12}'(0)$

$\psi_{12} = z(1-z) = z - z^2 \qquad \psi_{12}(1) = 0$

$\psi_{21}(1) = 1 \Rightarrow a_3 = 1 \qquad \psi_{21}(0) = \psi_{21}'(0) = 0 \qquad \psi_{21} = z^2$

$A^2 = \begin{bmatrix} 0 & 2 & -1 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 & -1 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{bmatrix}$

$Z_{11} = I - A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$ $\quad Z_{12} = A - A^2 = \begin{bmatrix} 0 & 2 & -1 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \end{bmatrix}$

$= \begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$Z_{21} = A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{bmatrix}$

Use (1):

So $A^{100} = A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{bmatrix}$

$f(z) = z^{100}$

$f(0) = 0 \quad f'(0) = 0 \quad f(1) = 1$

$e^{At} = Z_{11} + t Z_{12} + e^t Z_{21}$

$f(z) = e^{zt}, \; f'(z) = te^{zt}$

$f(0) = 1 \quad f'(0) = t, \; f(1) = e^t$

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3. $A = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 0 & 5 & -6 & -1 \\ 0 & 3 & -4 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

$\det[zI - A] = (z-2)(z^2 - z - 2)(z+1)$
$= (z-2)(z-2)(z+1)(z+1) =$
$(z-2)^2(z+1)^2 \quad z_1 = 2 \quad z_2 = -1$

$\text{rank}\,(A - 2I) = \text{rank} \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 3 & -6 & -1 \\ 0 & 3 & -6 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\| 3$

So A has one lin. ind eigenvalue corresponding to 2

The Jordan blocks of A corresponding to 2 is $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

So $(z-2)^2$ ~~is a root of~~ divides min pol. of A.

$\text{rank}\; A + I = \begin{bmatrix} 3 & 1 & -1 & 0 \\ 0 & 6 & -6 & -1 \\ 0 & 3 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 & -1 & 0 \\ 0 & 6 & -6 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\text{rank}(A+I) = 2 \quad \text{null}(A+I) = 4 - 2 = 2$, A has two lin. ind. eigenvectors corresponding to $\lambda = -1$

Hence there are two Jordan blocks of order 1 corresponding to $\lambda = -1$. J.C.F. of A is

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Min pol of ~~of~~ A is $\psi(z) = (z-2)^2(z+1)$

So

(1) $f(A) = f(2)Z_{11} + f'(2)Z_{12} + f(-1)Z_{21}$

ψ_{11} satisfies $\psi_{11}(2)=1$ $\psi_{11}'(2)=0$ $\psi_{11}(-1)=0$

ψ_{12} " $\psi_{12}(2)=0$ $\psi_{12}'(2)=1$ $\psi_{12}(-1)=0$

ψ_{21} " $\psi_{21}(2)=\psi_{21}'(2)=0$ $\psi_{21}(-1)=1$.

So $\psi_{1i} = (a_i + b_i(z-2))(z+1)$ $i=1,2$

$\psi_{21} = c(z-2)^2$

similar computations yield — to what I done in Problem 2

$\psi_{21} = \frac{1}{9}(z-2)^2$, $\psi_{12} = \frac{1}{3}(z+1)(z-2)$

$\psi_{11} = \frac{1}{9}(3-(z-2))(z+1) = \frac{1}{9}(5-z)(z+1)$

Now compute

$Z_{11} = \psi_{11}(A) = \frac{1}{9}(5I_2 - 4A + A^2)$

$Z_{21} = \frac{1}{9}(A-2I)^2 = \frac{1}{9}(A^2-4A+4I)$

$Z_{12} = \frac{1}{3}(A+I)(A-2I) = \frac{1}{3}(A^2-A-I)$

Now use (1) for $f(z)=z^{100}$, e^{zt}.

C. Note $A \in \mathbb{R}^{n\times n}$ is stochastic if $A \geq 0$ (i.e. all entries are nonnegative) and $A\mathbf{1} = \mathbf{1}$ where $\mathbf{1} = (1,\dots,1)^T$.

1. So $A^k \geq 0$ and $A^k\mathbf{1} = 1^k\mathbf{1} = \mathbf{1}$

Hence A^k is stochastic for $k=1,2,\dots$

6

2. Since A^k is stochastic, each entry of A^k is in $[0,1]$.
Hence A is power bounded.

3. $A\underline{1} = 1 \cdot \underline{1}$, i.e. 1 eigenvalue of A with the corresponding eigenvector $\underline{1}$.

4. Since A is power bounded Theorem 4.6 p' 79 of Math. 425 notes part 3. shows that each Jordan block corresponding to eigenvalue 1 is of order 1.

5. Part 3 of Theorem 4.6 yields that $|\lambda| \le 1$.

6. Part 2 of Theorem 4.6 yields that if $\lim_{k\to\infty} A^k = B$ then for each eigenvalue λ of A, $\lambda \ne 1$, we must have $|\lambda| < 1$.