A. Observe that for any system \((e) x' = Ax \),
we have \( x' = A e A_{-1} x_{-1} A_{-1} x_0 \). Hence \( A_{0} + p = A_{-1} \).

(1) \( x' = B^J x_0 \). (Take a few examples \( p = 1, 2, 3 \)).
Hence (2) \( x' = A_{0} A_{-1} B^J x_0 \) for \( r = 1, \ldots, p - 1 \).

(a) (2) Hence (0) is stable \( \Rightarrow (1) \) is stable, i.e., \( \rho(B) < 1 \),
which is equivalent to all eigenvalues of \( B \) in the unit disk \( |z| < 1 \). It is equivalent to \( \lim_{j \to \infty} B^j = 0 \). (3)

Suppose (3) holds. Then \( \lim_{j \to \infty} A_{0} A_{-1} \ldots A_{-1} B^j = 0 \)
for \( r = 1, \ldots, p - 1 \). Hence (0) is stable.

(b) Suppose that for each \( x_0 \), \( \lim_{j \to \infty} x_j = 0 \).

By choosing \( x_0 = 0 \), we deduce that (0) is stable convergent if \( \lim_{j \to \infty} A_{0} A_{-1} \ldots A_{-1} = C \).

By taking \( l = p^j \), we get \( A_{p^0} A_{p^{-1}} \ldots A_{1} = B^J \). (4)
Hence (4) \( \lim_{j \to \infty} B^J = C \). i.e., \( B \) is power convergent.

As from (3) + (4), we deduce that \( A_{1} = C \).

Use (2) to deduce that \( \lim_{j \to \infty} A_{p^j} A_{p^{j-1}} \ldots A_{1} = C \).

Implying \( C = A_{0} A_{-1} C A_{-1} C \), continuing in the same manner deduce that recurrent conditions for (0) to be convergent is \( j \to \infty \).
(c) (4) yields that \( (0) \) is power bounded. Observe that \( 2B^{k} \) is a bounded sequence.

Hence, \( \text{rec. & suff. conditions for power boundedness of } (0) \) is that \( B \) is power bounded.

\[ A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \] is \( 2^{2} - 4 \text{ z } + 4 = (2 - z)^{2} \)

Since \( A \) is not \( 2I \), \( A \) is not diagonalizable. Hence the minimal pol of \( A \) is \( (2 - z)^{2} \). \( A \) has three components

\[ Z_{1}, Z_{2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} f(A) = f(2) Z_{1} + f'(2) Z_{2} = f(2) Z_{1} + f'(2) Z_{2} \]

\[ Z_{1} = \gamma_{1}(A), \quad Z_{2} = \gamma_{2}(A), \]

\[ \gamma_{1}(z) = 1, \quad \gamma_{1}'(z) = 0, \quad \gamma_{2}(z) = 0, \quad \gamma_{2}'(z) = 1 \]

Recall \( \gamma_{j}(z) = a_{j} + b_{j}(z-2) \)

\[ \gamma_{1}(z) = 1 \Rightarrow a_{1} = 1, \quad \gamma_{1}'(z) = b_{1} = 0 \Rightarrow \gamma_{1} = 1 \]

\[ \gamma_{2}(z) = a_{2} = 0, \quad \gamma_{2}'(z) = b_{2} = 1 \Rightarrow \gamma_{2} = z - 2 \]

\[ Z_{1} = I_{2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Z_{2} = A - 2I_{2} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \]

\[ A^{100} = f(A) = z^{100} = 2^{100}, \quad f'(z) = 100 z^{99} \]

\[ f'(z) = 100 z^{99} \Rightarrow A^{100} = 2^{100} Z_{1} + 100 z^{99} Z_{2} = \]

\[ = 2^{99} \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} + \begin{bmatrix} -100 & 100 \\ -100 & 100 \end{bmatrix} = 2^{99} \begin{bmatrix} -98 & 100 \\ -100 & 102 \end{bmatrix} \]

\[ e^{At} = f(A), \quad f(z) = e^{zt}, \quad f'(z) = z e^{zt}, \quad f(z) = e^{zt}, \quad f'(z) = z e^{zt} \]

\[ e^{At} = e^{zt} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = e^{zt} \begin{bmatrix} 1 & 1 \\ -t & 1 + t \end{bmatrix}. \]
2. charpol. $\hat{A} = \begin{bmatrix} 0 & 2 & -1 \\ 0 & -1 & 1 \\ -2 & 2 & 2 \end{bmatrix}$ $\Rightarrow$ $z(z^2 - 2) = z^2(z - 1)$

\[ \text{dim ker } A = \text{rank } A \leq A \begin{bmatrix} 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + A \]

So $A$ is rank $A = 2$. Hence the dimension of the eigenspace corresponding to 2 is 1. Thus the JCF of $A$ has $2 \times 2$ block $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Thus minpol of $A$ = char pol of $A$ $\lambda_1 = 2$ multiplicity 2 $\lambda_1 = 1$ multiplicity 1.

\[ f(A) = f(0)I + f(0)A + f(1)A^2 \]

\[ Z_i = \gamma_i(A), \quad Z_{21} = \gamma_{21}(A) \]

\[ \gamma_{11} = (a_1 + b_1 z)(z - 1), \quad \gamma_{12} = (a_2 + b_2 z)(z - 1) \]

\[ \gamma_{21} = z^2 z \]

\[ \gamma_{11}(0) = 1, \quad \gamma_{11}'(0) = 0, \quad \gamma_{11}(1) = 0 \]

\[ a_1 = a_1 = -1, \quad b_1 = b_1 + a_1, \quad b_2 = a_2 = -1 \]

\[ \gamma_{12}(0) = 0, \quad a_2 = 0, \quad a_3 = 0 \]

\[ \gamma_{12} = z(1-z) = z - z^2 \]

\[ \gamma_{12}(1) = 0 \]

\[ \gamma_{21}(1) = 1 \Rightarrow a_3 = 1 \]

\[ \gamma_{21}(0) = \gamma_{21}'(0) = 0 \]

\[ \gamma_{21} = z^2 \]

\[ A^2 = \begin{bmatrix} 0 & 2 & -1 \\ 0 & -1 & 1 \\ -2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 & -1 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

\[ Z_{11} = I - A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \]

\[ Z_{21} = A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

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\[ Z_{21} = A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]
So use (1):

\[ A^{100} = A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{bmatrix} \]

\[ f(x) = x^{100} \]

\[ f(0) = 0, \quad f'(0) = 0, \quad f(1) = 1 \]

\[ e^{At} = z_{11} + t z_{12} + e^t z_{21} \]

\[ f(x) = e^{xt}, \quad f'(x) = te^{xt} \]

\[ f(0) = 1, \quad f'(0) = t, \quad f(1) = e^t \]

\[ A = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 0 & 5 & -6 & -1 \\ 0 & 3 & -4 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \]

\[ \det (2I - A) = (2 - 2)(2^2 - 2 - 2)(2 + 1) = (2 - 2)(2 - 2)(2 + 1) = (2 - 2)^2(2 + 1), \quad \xi_1 = 2, \quad \xi_2 = -1 \]

\[ \text{rank} (A - 2I) = \text{rank} \left( \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 3 & -6 & -1 \\ 0 & 3 & -6 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

So \( A \) has one linearly independent eigenvector corresponding to 2.

The Jordan blocks of \( A \) correspond to \( 2^{2} \), \[
\begin{bmatrix}
2 & 1 \\
0 & 2
\end{bmatrix}
\]

So \( (2^{2}) \rightarrow \text{nullity} + 4 \) divides \( \min p(x) \) of \( A \).

\[ \text{rank} (A + I) = 3 \]

\[ \left( \begin{bmatrix} 3 & 1 & -1 & 0 \\ 0 & 6 & -6 & -1 \\ 0 & 3 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \sim \left( \begin{bmatrix} 3 & 1 & -1 & 0 \\ 0 & 6 & -6 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \]

\[ \text{rank} (A + I) = 2 \quad \text{null} (A + I) = 4 - 2 = 2, \quad A \text{ has two} \]

linearly independent eigenvectors corresponding to \( \lambda = -1 \).

Hence, there are two Jordan blocks of order 1 corresponding to \( \lambda = -1 \). The JCF of \( A \) is
\[
\begin{bmatrix}
2 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}
\]
\[ \text{HNn pol of } A \text{ is } \psi(t) = (t-A)^2 \text{ (or) } \]

So \[ \psi(A) = \psi(t)\bigg|_{t=A} = \psi'(t) \bigg|_{t=A} = \psi''(t) \bigg|_{t=A} \]

\[ \psi_{11} \text{ satisfies } \psi_{11}(2) = 1, \psi_{11}'(2) = 0, \psi_{11}(-1) = 0 \]

\[ \psi_{12} \quad \psi_{12}(2) = 0, \psi_{12}'(2) = 1, \psi_{12}(-1) = 0 \]

\[ \psi_{21} \quad \psi_{21}(2) = \psi_{21}'(2) = 0, \psi_{21}(-1) = 1 \]

So \[ \psi_{i} = (a_i + b_i (t-2)) (t+1) \quad i=1,2 \]

\[ \psi_{21} = (t-2) \]

Similar computations yield \[ \psi_{12} = \frac{1}{3} (t+1) (2-t) \]

\[ \psi_{11} = \frac{1}{q} (3 - (t-2)) (t+1) = \frac{1}{q} (5+t) (2-t) \]

Now compute \[ Z_{11} = \psi_{11}(A) = \frac{1}{q} (5I_2 - 4A + A^2) \]

\[ Z_{21} = \frac{1}{q} (A - 2I)^2 = \frac{1}{q} (A^2 - 4A + 4I) \]

\[ Z_{22} = \frac{1}{3} (A + I) (A - 2I) = \frac{1}{3} (A^2 - A - I) \]

Now use (1) for \[ f(t) = e^{100t} e^{2t} \]

Note \( A \in \mathbb{R}^{n \times n} \) is stochastic if \( A \geq 0 \)

(i.e., all entries are nonnegative) and \( A^T = A \)

when \( A = (1, -1)^T \).

So \( A^k \geq 0 \) and \( A^k \cdot 1 = 1 \cdot 1 = 1 \)

Hence \( A^k \) is stochastic for \( k \geq 2 \).
2. Since $A^k$ is stochastic, each entry of $A^k$ is in $[0,1)$. Hence $A$ is power bounded.

3. $A^k = 1, 1\cdot 1, \ldots, 1$ eigenvalue of $A$ with the corresponding eigenvector $1$.

4. Since $A$ is power bounded, Theorem 4.5, p. 79, of Math. 425 notes part 3, shows that each Jordan block corresponding to eigenvalue $1$ is of order $1$.

5. Part 3 of Theorem 4.6 yields that $|\lambda| \leq 1$.

6. Part 2 of Theorem 4.6 yields that if $\lim_{k \to \infty} A^k = B$, then for each eigenvalue $\lambda$ of $A$, if $|\lambda| \neq 1$, we must have $|\lambda| < 1$. 