1. \( \text{span}(v_1, \ldots, v_n) = P_n \)

b. It is left to show that \( v_j, j = 1, \ldots, n \) is l.c. Suppose that
\[
p(t) = \sum_{i=0}^{\infty} a_i t^i = 0
\]
So \( a_i = 0 \) for \( i = 0 \) and \( a_i = 0 \) for \( i > 0 \)

Hence \( v_j \) \( j = 1, \ldots, n \) is l.c.

2. \[
\begin{bmatrix}
p_0 \\
p_1 \\
p_n
\end{bmatrix} = \begin{bmatrix}
a & 0 & 0 & \cdots & 0 \\
a_0 & a & 0 & \cdots & 0 \\
na_0 & na_0 & a & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\begin{bmatrix}
\vdots \\
\vdots \\
\vdots
\end{bmatrix} = \begin{bmatrix}
a_0 & a_1 & a_2 & \cdots & a_n
\end{bmatrix}
\]

A is lower triangular, \( a_i \neq 0 \) since \( \deg p_i = i \)

So \( \det A = a_0 a_1 \cdots a_n \neq 0 \)

A is invertible, hence \( p_0, \ldots, p_n \) is a basis too.

3. Take \( p_k(t) = x^{k-1} x^k \) for \( k = 0, \ldots, n \)

Then
\[
\begin{bmatrix}
p_0 \\
p_1 \\
p_n
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
\vdots \\
\vdots \\
\vdots
\end{bmatrix} = A
\]

A is upper triangular

\( \det A = 1 \)

Hence \( p_0, p_1, \ldots, p_n \) is a basis.

5. Put a matrix from the given columns
\[
A = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}
\rightarrow
B = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & -1 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

\( B \) has full rank, so column space of \( A \) is \( \mathbb{C}^4 (\mathbb{R}^4) \)

The first 4 columns form a basis. Probably any 4 of them form a basis. This needs \( \binom{4}{2} = \binom{4}{3} = 15 \) checks!
(1) \( Ax = b \) soluble. Means that
\[
\begin{align*}
x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + \cdots + x_n \mathbf{e}_n &= b \\
A = \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix} &\in \mathbb{F}^{n \times 1}, \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}
\end{align*}
\]
That is, \( b \) is a linear combination of columns of \( A \).

In particular, the vectors \( e_1, -e_2, b \) are linearly dependent in \( \mathbb{F}^{n+1} \). The augmented matrix
\[
B = \begin{bmatrix} A & b \end{bmatrix} \in \mathbb{F}^{(n+1) \times (n+1)}
\]
has a dependent column.

Equivalently,
\[
B \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ -1 \end{bmatrix} = 0 
\]
has a nontrivial solution.
Hence \( \det B = 0 \).

(2) Let \( A_n \) be the given matrix. Subtract row 2 from row 1 to get \( B_{1,n} \) with the first row
\[
(1, 0, \ldots, 0). \quad \text{Expand } \det B_{1,n} \text{ by the first row to obtain}
\]
\[
\det A_n = \det A_{n-1} = \cdots \det A_2 = 1.
\]
Hence \( \det A_n \) of dimension \( n \).

Exract HW: Assume by induction that
\[
V(z_1, z_2) = \prod_{i=1}^n (z_i - z_i) \quad \text{holds. (For } n=2 \text{ this is obvious.)}
\]
Consider \( V(z_1, z_n, z) \) expanded by the last row to see
\[
\sum_{i=1}^n \prod_{j \neq i} (z_j - z_i) z_i
\]
that it is polynomial of degree \( n \) and of the form
\[
V(z_1, z_n, z) = z^n + \text{lower terms}. \quad \text{Clearly } V(z_1, z_n, z) = 0
\]
for \( i=1, \ldots, n \) since \( V(z_1, z_n, z) \) has two identical rows.
Hence \( V(z_1, z_n) \) is a polynomial of degree \( n \) and of the form
\[
V(z_1, z_n) = V(z_1, z_n) (z - z_1) \cdots (z - z_n), \quad \text{which proves the claim.}