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1. a. $\operatorname{span}(1, t, \dots, t^n) = P_n$.

b. It is left to show that $1, \dots, t^n$ l.i. Suppose that

$$p(t) = \sum_{i=0}^{n} a_i t^i \equiv 0 \qquad \text{so } 0 = p^{(j)}(0) = j!\, a_j \Rightarrow a_j = 0,\ j = 0, \dots, n$$

Hence $1, t, \dots, t^n$ l.i.

2. $$\begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_n \end{bmatrix} = \underbrace{\begin{bmatrix} a_{00} & 0 & \cdots & 0 \\ a_{10} & a_{11} & & 0 \\ \vdots & & \ddots & \\ a_{n0} & \cdots & & a_{nn} \end{bmatrix}}_{A} \begin{bmatrix} 1 \\ \vdots \\ t^n \end{bmatrix} ; \quad A = \begin{bmatrix} a_{00} & 0 & \cdots & 0 \\ a_{10} & a_{11} & & \\ \vdots & & \ddots & \end{bmatrix}$$

A is lower triangular, $a_{ii} \neq 0$ since $\deg p_i = i$

So $\det A = a_{00} a_{11} \cdots a_{nn} \neq 0$

A invertible. Hence $p_0, \dots, p_n$ is a basis too.

3. Take $p_k(t) = x^n + x^k$ for $k = 0, 1, \dots, n$

Then

$$\begin{bmatrix} p_0 \\ \vdots \\ p_n \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 0 & \cdots & 0\ 1 \\ & & \ddots & & \\ 0 & & & & 1 \end{bmatrix}}_{A} \begin{bmatrix} 1 \\ \vdots \\ t^n \end{bmatrix}, \quad A \text{ upper triangular}, \ \det A = 1$$

Hence $p_0, \dots, p_n$ basis.

5. put a matrix from the given columns

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \to \quad \text{Bring it to REF} \quad B = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 1 & -2 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

B = has full rank, So column space of A is $\mathbb{C}^4$ ($\mathbb{R}^4$)

The first 4 columns form a basis. Probably any 4 of them form a basis. This needs $\binom{6}{4} = \binom{6}{2} = \underline{\underline{15}}$ checks!

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(2)

(1) $A\underline{x} = \underline{b}$ solvable. Means that

$x_1\underline{c}_1 + x_2\underline{c}_2 + \cdots + x_n\underline{c}_n = \underline{b}$

where $A = [\underline{c}_1\ \underline{c}_2 \cdots \underline{c}_n]$, $\underline{c}_i \in \mathbb{F}^{n+1}$, $\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

That is $\underline{b}$ is a linear combination of columns of A.
In particular the vectors $\underline{c}_1, \ldots, \underline{c}_n, \underline{b}$ are linearly dependent in $\mathbb{F}^{n+1}$. The augmented matrix
$B = [A \,\vdots\, b] \in \mathbb{F}^{(n+1)\times(n+1)}$ has l. dependent column space. Equivalently

$$B \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ -1 \end{bmatrix} = \underline{0}$$ has a nontrivial solution.
Hence $\det B = 0$.

(2) Let A_n be the given matrix of dimension n. Subtract row 2 from row 1 to get $B_{1,n}$ with the first row $(1, 0, \ldots, 0)$. Expand $\det B_{1,n}$ by the first row to obtain

$$\det A_n = \det A_{n-1} = \cdots \det A_2 = 1 \ldots$$

Hence

Extra HW. Assume by induction that
$V(z_1, \ldots, z_n) = \prod_{1 \le i < j \le n} (z_j - z_i)$ holds. (For $n=2$ this is obvious).
Consider $V(z_1, \ldots, z_n, z)$ expand by the last row to see that is polynomial of degree n ~~where~~ of the form $V(z_1, \ldots, z_n)\, z^n +$ lower terms. Clearly $V(z_1, \ldots, z_n, z_i) = 0$ for $i = 1, \ldots, n$ since $V(z_1, \ldots, z_n, z_i)$ has two identical rows.
Hence $V(z_1, \ldots, z_n) = V(z_1, \ldots, z_n)(z - z_1) \cdots (z - z_n)$, which proves the claim.