① MCS-421 – FALL 2010, SOLUTIONS TO HW3 – 9.16.10 – chapter 3 – Brualdi

1. Use same notation as in Application 4, p'71

$1 \le a_1 < a_2 < \dots < a_{77} \le 132$, $k = 1, 2, \dots, 21$

$1+k \le a_1 < a_2 < \dots < a_{77} \le 123+k$

Each of 154 numbers $a_1, \dots, a_{77}, a_1+k, \dots, a_{77}+k$ is between 1 and $132+k \le 132+21 = 153$. By pigeon hole principle there must be two same number. Note for $i \ne j$ $a_i \ne a_j$ and $a_i+k \ne a_j+k$. So the two equal members are $a_i = a_j + k$, i.e. $a_i - a_j = k \ge 1$

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2. Choose 100 integers from $1, 2, \dots, 200$ and call this set S. Each member in S is of the form $2^k \times a$, a - odd. Call S_{odd} all odd member a obtained from each $b \in S$. We have 100 such numbers. If two are equal then $b_1 = 2^{k_1} \times a$, $b_2 = 2^{k_2} \times a$. Assuming $k_1 < k_2$ we get $b_1 | b_2$, otherwise $b_2 | b_1$.

We are left with the case $S_{odd} = \{1, 3, 5, \dots, 199\}$.

Case I Suppose S contains one of the members $1, 3, 5, 7, 9, 11, 13, 15\}$

If 1 is there it divides all other members in S. Done.

If $3 \in S$ then 3 must divide each of the member $\{27, 54, 108\}$ (whose odd part is 27) Done. Similarly for 3 and 15 choose $\{45, 90, 180\}$. For 7 choose $\{49, 98, 196\}$ for 11 choose $\{33, 66, 132\}$ for 13 choose $\{39, 78, 156\}$.

Case II S does not contain any number in case I, contains one the numbers $\{6, 10, 12, 14\}$. For 6 consider $\{27, 54, 108\}$ and $\{81, 162\}$. If $27 \in S$ it divides 81 and 62. Done. Otherwise if 54 or 108 in S then 6 divides them. For 10 consider $\{25, 50, 100\}$ and $\{125\}$, For ~~12~~ 12 consider $\{36, 72, 144\}$ and ~~$\{162\}$~~ $\{27, 54, 108\}$ ~~If 18 chosen~~ and $\{81, 162\}$ if 162 chosen then if 27, or 54 chosen. Done. So otherwise chose 108 then $12 | 108$. Chose 81 from $\{81, 162\}$, Then if 27 chosen ✓, So choose 54 from $\{27, 54, 108\}$ If 18 chosen from $\{18, 36, 72, 144\}$ then $18 | 54$. Other number ~~from~~ $\{36, 72, 144\}$ divisible by 12 ✓. Similarly for 14 - use $\{21, 42, 84, 168\}$ and $\{63, 126\}$

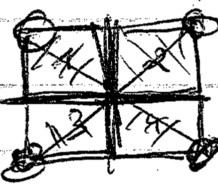
Case 3 S does not contain $\{1,3,5,6,7,9,10,11,12,13,14,15\}$.
So S contains one of $\{2,4,8\}$.
S must contain an integer of the form 3×2^k but not 3, 6, 12. So $k \geq 3$. Hence 3×2^k is divisible by 2, 4, 8.

Problem 4 Consider the intervals $[1,2]\ [3,4] \ldots, [2n-1, 2n]$.
By pigeon hole principle two integers must lie in one of the intervals (Since any integer ~~belong~~ from 1 to $2n$ belongs to one of the intervals).
Hence their difference is at most 1.

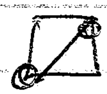
Problem 5 Apply the same argument for intervals
$[1,3], [4,6], \ldots, [3n-3, 3n]$

Problem 8 Consider the long division process applied to m/n. At each stage, we write the highest factor of n in $\alpha \times 10$ where α is the remainder of the previous division by n. So $0 \leq \alpha < n$. (If at certain stage $\alpha = 0$, i.e. $3.4567000000\ldots0$ this OK). Otherwise $1 \leq \alpha < n$ and by the pigeon hole principle we must repeat α after n steps at most (e.g. $1,2,3,4,\ldots,n-1, 1,2,3$). The period is at most n.

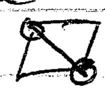
Problem 18 :



~~The only way~~ Divide the square to 4 1×1 squares.
If you have 5 points. Two of them must be in same unit square. The distance between them is $\sqrt{2}$ at most



or



(3)

22. Start with the case $r(3,3,3)$ and recall that $r(3,3)=6$

Assume that $p = 3\cdot(r(3,3)-1)+2 = 17$

Choose one vertex $v \in K_{17}$. It has 16 edges. They are colored 3 colors. So there is one color for which at least 6 edges of v are colored in this color say $c1$. Consider the complete subgraph of K_{17} induced by v and its six neighbors $v_1, \dots, v_6$ such that (v, v_i) are colored in $c1$.

If one of edges of (v_i, v_j) $1 \le i < j \le 6$ is colored by $c1$ we have a ~~et~~ triangle colored by $c1$ out of vertices v, v_i, v_j.

So assume that each edge (v_i, v_j) is colored by $c2$ or $c3$. ~~Thus~~ The subgraph base on $v_1, \dots, v_6$ is K_6 colored with 2 colors. Since $6 = r(3,3)$ it must contain K_3 with one color. $\square$

Same proof works for $R_k(3) = r(\underbrace{3,\dots,3}_{k})$, $k \ge 4$.

Claim $R_k(3) \le (k+1)!$

For $k=2$ $(2+1)! = 3! = 6$ ✓

$R_{k+1} \le (k+1)\left((k+1)! - 1\right) + 2 = (k+1)(k+1)! - (k+1-2) <$ $k \ge 2$

$(k+1)(k+1)! < (k+2)!$ $\square$

27. Divide the set of all subsets of $\{1,\dots,n\}$ to 2^{n-1} pairs $\{A_1, A_1^c\}$ $\dots$ $\{A_{2^{n-1}}, A_{2^{n-1}}^c\}$.

If we have $B_1, \dots, B_{2^{n-1}+1}$ distinct subsets of $\{1,\dots,n\}$ at least we have a pair B_i, B_j which belong to $\{A_k, A_k^c\}$.

So $B_i \cap B_j = \phi$ — contradiction.