[1] §23 Pr. 9

(a) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$AA^T = I_2 \Rightarrow \quad a^2+b^2=1 \quad ac+bd=0 \quad c^2+d^2=1$

$\det A = 1 \therefore ad-bc=1$

$a=\cos\theta \quad b=\sin\theta, \qquad c=-\sin\varphi, \quad d=\cos\varphi$

$0 = ac+bd = -\cos\theta\sin\varphi + \sin\theta\cos\varphi = \sin(\theta-\varphi) = 0$

possibility 1 $\quad \theta-\varphi=0 \Rightarrow \quad \theta=\varphi$

possibility 2 $\quad \theta-\varphi=\pi \qquad$ ~~$\theta=\varphi+\pi$~~ $\varphi=\theta-\pi$

Possibility 1 $\quad A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad (1) \qquad \det A = 1$

Possibility 2 $\quad A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$

$\det A = -1$

So A must have form (1).

b. $e^B = I_n + B + \dfrac{B^2}{2!} + \dfrac{B^3}{3!} + \cdots$

$B = \begin{bmatrix} 0 & \theta \\ -\theta & 0 \end{bmatrix} \qquad B^2 = -\begin{bmatrix} \theta^2 & 0 \\ 0 & \theta^2 \end{bmatrix}$

$B^{2k} = (-1)^k \begin{bmatrix} \theta^{2k} & 0 \\ 0 & \theta^{2k} \end{bmatrix} \qquad B^{2k-1} = (-1)^{k-1} \begin{bmatrix} 0 & \theta^{2k-1} \\ -\theta^{2k-1} & 0 \end{bmatrix}$

$I_2 + \dfrac{B^2}{2!} + \dfrac{B^4}{4!} + \cdots = \begin{bmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{bmatrix}$

$\dfrac{B}{1!} + \dfrac{B^3}{3!} + \cdots = \begin{bmatrix} 0 & \sin\theta \\ -\sin\theta & 0 \end{bmatrix} \Rightarrow e^B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

(c) observe 1) $B = TAT^{-1}$ $\quad B^k = TA^kT^{-1}$

So for any polynomial $p(z) = a_0 + a_1 z + \cdots + a_m z^m$

$$p(B) = a_0 I + a_1 B + a_2 B^2 + \cdots + a_m B^m = T\,p(A)\,T^{-1}$$

~~Hence~~ $e^B = Te^AT^{-1}$

2) $A = \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_m \end{bmatrix}$, Then $p(A) = \begin{bmatrix} p(A_1) & & 0 \\ & \ddots & \\ 0 & & p(A_m) \end{bmatrix}$.

~~Hence~~ $e^A = \begin{bmatrix} e^{A_1} & & & 0 \\ & e^{A_2} & & \\ 0 & & \ddots & e^{A_m} \end{bmatrix}$

Recall that if B is skew symmetric ~~and~~ then

$$B = Q \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & B_1 & & \\ & & & \ddots & \\ & & & & B_p \end{bmatrix} Q^T \quad \text{where } B_p = \begin{bmatrix} 0 & b_p \\ -b_p & 0 \end{bmatrix}$$

(Proposition 2.22)

$Q^T = Q^{-1}$

So $e^B = Q \begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & e^{B_1} & & \\ & & & \ddots & \\ 0 & & & & e^{B_p} \end{bmatrix} Q^T$ (2)

~~By Proposition 2.21 any special orthogonal matrix~~ is of form (2).

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10 Any skew hermitian $= A^* = -A$ is $A = iB$ where $B^* = B$ - hermitian.

~~So~~ $B = U\Delta U^*$ $(U^* = U^{-1})$ $\Delta = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{bmatrix}$

λ_j are real. $A = U\, i\Delta\, U^*$

(3)

$$e^{A} = U e^{i\Delta} U^{*} \overset{U^{-1}}{=} U \begin{bmatrix} e^{i\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\lambda_n} \end{bmatrix} U^{*}$$

No $e^{i\lambda} = \zeta$, $\lambda \in \mathbb{R}$, is the form of any complex number ζ satisfying $|\zeta| = 1$.

T ~~is~~ unitary is normal with eigenvalues ζ, $|\zeta| = 1$

So any $T = U \begin{bmatrix} \zeta_1 & & 0 \\ & \ddots & \\ 0 & & \zeta_n \end{bmatrix} U^{*}$ $|\zeta_j| = 1$

$$= U \begin{bmatrix} e^{i\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\lambda_n} \end{bmatrix} U^{*} = e^{A}.$$

12. $B = \begin{bmatrix} b_{11} & & * \\ & \ddots & \\ 0 & & b_{nn} \end{bmatrix}$ $B^{*}B = BB^{*}$

$B^{*} = \begin{bmatrix} \bar{b}_{11} & & 0 \\ \bar{b}_{12} & \ddots & \\ \vdots & & \bar{b}_{n} \end{bmatrix}$ The diagonal entry of $(1,1)$ $(B^{*}B)_{11}$ is $|b_{11}|^2$

The diagonal entry of $(1,1)$ of BB^{*} is $|b_{11}|^2 + |b_{12}|^2 + \cdots + |b_{1n}|^2$

So $b_{12} = \cdots b_{1n} = 0$. Continue now to the entry $(2,2)$ of BB^{*} and $B^{*}B$ to deduce that $b_{23} = \cdots b_{2n} = 0$

Continuing in this manner we deduce that B is diagonal. Hence B is normal

4a $A = \begin{bmatrix} 1-i & 2 \\ 2 & 3 \end{bmatrix}$ not hermitian since a_{11} not real

$A^* = \begin{bmatrix} 1+i & 2 \\ 2 & 3 \end{bmatrix}$ $AA^* = \begin{bmatrix} 1^2+1^2+2^2 & 2(1-i)+6 \\ & \end{bmatrix}$

$A^*A = \begin{bmatrix} 1^2+1^2+2^2 & 2(1+i)+6 \\ x & x \end{bmatrix}$

NOT NORMAL

b) Hermitian, hence normal

c) Not hermitian since $a_{12} \neq a_{21}$
Orthogonal, hence normal $Q^*Q = Q^TQ = I = QQ^* = QQ^T$

(d) $A = \begin{bmatrix} \frac{1}{\sqrt{2}} i & +\frac{1}{\sqrt{2}} \\ +\frac{1}{\sqrt{2}} i & \frac{1}{\sqrt{2}} i \end{bmatrix}$ not HERMITIAN SINCE diagonal entries are not real

$A^* = \begin{bmatrix} -\frac{1}{\sqrt{2}} i & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & +\frac{1}{\sqrt{2}} i \end{bmatrix}$ $AA^* = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$

$A^*A = \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$ not normal

(e) Not hermitian $a_{12} \neq \overline{a_{21}}$ $(AA^*)_{12} = -2-i$

$A = \begin{bmatrix} 0 & i & 1 \\ i & 0 & -2+i \\ -1 & 2+i & 0 \end{bmatrix}$ $A^* = \begin{bmatrix} 0 & -i & -1 \\ -i & 0 & 2-i \\ 1 & -2i & 0 \end{bmatrix} =$ $(A^*A)_{12} = -2-i$