① SOLUTIONS HW 7, MCS-421, FALL 2010, S. FRIEDLAND

CHAPTER 7.

16. NUMBER OF CHOICES OF BAGS WITH n objects s.t.

FIRST OBJECT - 2 at most, SECOND OBJECT - EVEN # & 6 at most

THIRD OBJECT — EVEN NUMBERS, FOURTH OBJECT — AT LEAST ONE

OBJECT.

17. $(1+x^2+x^4+\cdots)(1+x+x^2)(1+x^3+x^6+\cdots)(1+x) =$

AP. OR. BAN. PE.

$$= \frac{1}{\cancel{1-x^2}} \cdot \frac{\cancel{1-x^3}}{1-x} \cdot \frac{1}{\cancel{1-x^3}} \cdot \frac{\cancel{1-x^2}}{1-x}$$

$$= \frac{1}{(1-x)^2} = (1-x)^{-2} = \sum_{n=0}^{\infty} (-1)^n \binom{-2}{n} x^n$$

$$1 - \binom{-2}{1}x + \binom{-2}{2}x^2 - \cdots$$

$$\binom{-2}{k} = \frac{(-2)(-2-1)\cdots(-2-k+1)}{k!} = (-1)^k \frac{2\cdot 3 \cdots (k+1)}{k!}$$

$$= (-1)^k (k+1)$$

$$1 + 2x + 3x^2 + \cdots + (n+1)x^n + \cdots$$

So $\boxed{h_n = (n+1)}$

18. Look at example p'220. (No need of this problem)

$$2e_1 \leftrightarrow 1+x^2+x^4+\cdots = \frac{1}{1-x^2}$$

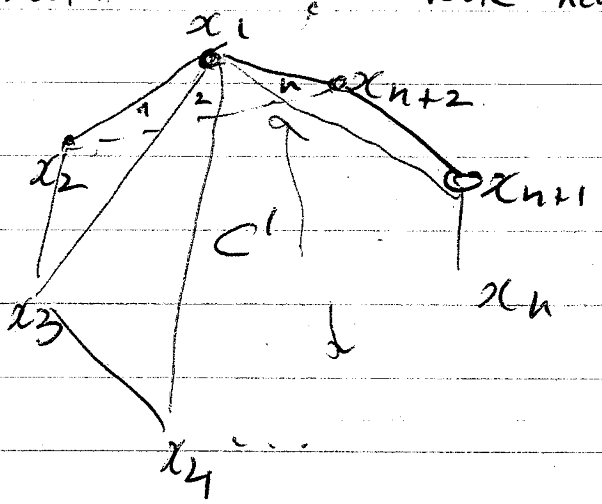
$$5e_2 \leftrightarrow 1+x^5+x^{10}+\cdots = \frac{1}{1-x^5}$$

$$e_3 \leftrightarrow 1+x+x^2+\cdots = \frac{1}{1-x}$$

$$7e_4 \leftrightarrow 1+x^7+x^{14}+\cdots = \frac{1}{1-x^7}$$

Generating function is $g(x) = \frac{1}{(1-x^2)} \frac{1}{(1-x^5)} \frac{1}{(1-x)} \frac{1}{(1-x^7)}$

21. Let C be a convex $(n+2)$-gon with $x_1, \dots, x_{n+2}$. Remove x_1 to obtain C'. Put back x_1. Create new n-regions



Also the diagonals from x_1 cut out new regions

$\#\ R$ in C'. So $h_n = h_{n-1} + n + R$, ~~where~~

Claim: $R = \binom{n+1}{3}$. Map bijectively each region in R to a set of three vertices in C'. Let the diagonal $x_1 x_i$ first encounter the region in the diagonal $x_j x_k$. So this region corresponds to $\{i,j,k\} \to \binom{n+1}{3}$.

So $h_n = h_{n-1} + n + \binom{n+1}{3}$

$\cancel{g(x) = \sum}$

Note $h_1 = 1$

$= h_0 + 1 + \binom{3}{3} = h_0 + 1 + 0 \Rightarrow h_0 = 0$

$$g(x) = \sum_{n=1}^{\infty} h_n x^n = x\left(\sum_{n=1}^{\infty} h_{n-1} x^{n-1}\right) + \sum_{n=1}^{\infty} n x^n + \sum_{n=1}^{\infty} \binom{n+1}{3} x^n$$

$$g(x) = x\, g(x) + \frac{x}{(1-x)^2} + x^2 \sum_{n-2=0}^{\infty} \binom{(n-2)+3}{3} x^{n-2}$$

$$(1-x) g(x) = \frac{x}{(1-x)^2} + \frac{x^2}{(1-x)^4} \Rightarrow g(x) = \frac{x}{(1-x)^3} + \frac{x^2}{(1-x)^5}$$

$$h_n = \binom{n+2}{4} + \binom{n+1}{2} \checkmark$$

3) 26: exponential generating function for even

$$1+\frac{x^2}{2!}+\frac{x^4}{4!}\cdots = \frac{e^x+e^{-x}}{2}$$ — red & green

regular exponential generating function

$$1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots = e^x$$

So the generating function is $\left(\frac{e^x+e^{-x}}{2}\right)^2\left(e^x\right)^2$

$$= \frac{e^{2x}+2+e^{-2x}}{4}\, e^{2x} = \frac{1}{4}\left[e^{4x}+2e^{2x}+1\right]$$

The coefficient of $\frac{x^n}{n!}$ is $\underbrace{4^{n-1}+2^{n-1}}_{\text{for } n\geq 1}$

~~for $n\geq 1$~~

for $n=0$ $\quad \frac{1}{4}(1+2+1)=\underline{\underline{1}}$

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33. $h_n = 3 h_{n-1} + 9h_{n-2} - 9h_{n-3}$ $\quad (n\geq 3)$

with initial values $h_0=0$, $h_1=1$, $h_2=2$

$q^3 - q^2 - 9q + 9 = 0$ $\quad q=1$ solution

$(q-1)(q^2 - 9) = 0$ $\quad q_1=1,\ q_2=3,\ q_3=-3$

So $h_n = A + B3^n + C(-3)^n$. The initial conditions

$A+B+C=0$, $A+3B-3C=1$, $A+9B+9C=2$. We obtain

$A=-\frac{1}{4}$, $B=\frac{1}{3}$, $C=-\frac{1}{12}$.

34. The characteristic equation is $x^2-8x+16=0$ $\quad x_{1,2}=4$

general solution $h_n=(A+Bn)4^n$

The initial values give $A=-1$, $B=1$.

42. The particular solution $n4^n$. A general solution for homogeneous $A4^n$. So $(A+n)4^n$. Initial value $A=3$

47. A particular solution $C+Dn$. Substitute to get $C=13$, $D=3$. General solution to homogeneous is $(A+Bn)2^n$. So $h_n=(A+Bn)2^n+13+3n$.

Initial values yield $A=-12$, $B=5$