1. Problem 1 p'41.

(a) Recall that if $T: V \to V$ and $\underline{u}_1, \dots, \underline{u}_n$ a basis of V then T is represented in this basis by a matrix $A = [a_{pq}]_{p,q=1}^{n}$ as follows (1) $T\underline{u}_p = \sum_{q=1}^{n} a_{q,p} \underline{u}_q$. In a different basis $\underline{v}_1, \dots, \underline{v}_n$ of V T is represented by $B \in \mathbb{F}^{n\times n}$ where $B = PAP^{-1}$ where P is the transition matrix from one basis to another. Since $\det(zI - B) = \det(zI - A)$ the characteristic polynomial of T is just $\det(zI - A)$, where A is a representation matrix in any basis of V.

As $\det(zI - A) = (z - \lambda_1) \cdots (z - \lambda_n)$, where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of T counted with their multiplicities.

Recall that $\sum_{i=1}^{n} \lambda_i$ is the trace of T and is given as the trace of the representation matrix A, $tr(A) = \sum_{p=1}^{n} a_{pp}$.

Now if $\underline{u}_1, \dots, \underline{u}_n$ orthonormal then from (1) it follows that $\langle T\underline{u}_p, \underline{u}_p \rangle = \langle \sum_{q=1}^{n} a_{qp} \underline{u}_q, \underline{u}_p \rangle$

$= \sum_{q=1}^{n} a_{qp} \langle \underline{u}_q, \underline{u}_p \rangle = a_{pp}$.

Hence $tr\, T = \sum_{p=1}^{n} \langle T\underline{u}_p, \underline{u}_p \rangle = \sum_{p=1}^{n} a_{pp}$

(b) In the o.n. basis S, T are represented by hermitian matrices A, B. So $tr(ST) = tr(AB) = tr(AB^*)$

$= \sum_{p,q=1} a_{pq} \bar{b}_{qp}$

So $\overline{tr(AB)} = \sum_{p,q} \overline{a_{pq} \bar{b}_{qp}} = \sum a_{qp} \bar{b}_{pq} = tr(AB)$ real

For any two ~~matrix~~ square matrices, A, B not hermitian

$tr(AB) = \sum_{p,q} a_{pq} b_{qp} = \sum_{p,q} b_{qp} a_{pq} = tr(BA)$.

Problems 3, 4, 9 - p'45

3. Let $P \in U(m)$, $Q \in U(n)$, $A \in \mathbb{C}^{m\times n}$, $A_1 = PAQ$

$A_1A_1^* = (PAQ)(PAQ)^* = PAQQ^*A^*P^* = PAA^*P^*$

So AA^* and $A_1A_1^*$ are unitarily similar.

Have the same eigenvalues $\lambda_j(AA^*) = \sigma_j^2(A) = \lambda_j(A_1A_1^*) = \sigma_j^2(A_1)$ $j = 1, \dots, n$. As singular values are nonnegative we deduce that $\sigma_j(A) = \sigma_j(A_1^*)$ for all $i = 1, 2, \dots$

4. If $A \in GL(n, \mathbb{C})$ rank $A = n$. So $\sigma_1(A) \geq \dots \geq \sigma_n(A) > 0$.

$A = U\Sigma V^*$ $\cancel{\Sigma =}$ [$(A^{-1})(A^{-1})^* = \cancel{(AA^*)^{-1}}\,(A^*A)^{-1}$

The eigenvalues of $(A^*A)^{-1}$ are inverses ~~of~~ of the eigenvalues of A^*A. ~~So~~ Since the eigenvalues arranged in a decreasing order

$$\lambda_1((A^*A)^{-1}) = \frac{1}{\lambda_n(A^*A)} \geq \lambda_2((A^*A)^{-1}) = \frac{1}{\lambda_{n-1}(A^*A)}$$

Hence $\sigma_1^2(A^{-1}) = \frac{1}{\sigma_n^2(A)} \Rightarrow \sigma_1(A^{-1}) = \frac{1}{\sigma_n(A)}$ -

missing assumption $U_1 \perp U_2$!!, $V_1 \perp V_2$!!

5. $U = U_1 \oplus U_2$. Choose an orthonormal basis in U which consists of the orthonormal basis of U_1 and U_2. ~~Same~~ Similar for V. Then in this basis T is represented by $A = \mathrm{diag}(A_1, A_2)$

$$\begin{array}{c|cc} & n_1 & n_2 \\ \hline m_1 & \boxed{A_1} & 0 \\ m_2 & 0 & \boxed{A_2} \end{array}$$

So $AA^* = \begin{bmatrix} A_1A_1^* & 0 \\ 0 & A_2A_2^* \end{bmatrix}$

The set of nonzero (positive) eigenvalues of AA^* is a union of ~~non~~ positive eigenvalues of $A_1A_1^*$, $A_2A_2^*$ and the rest of the problem follows straightforward.

9. $T \in L(V)$ normal, T is represented in any orthonormal basis by a normal matrix A. A normal iff

$$A = U \Delta U^* = \Delta = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

So $$AA^* = U\Delta U^* U \Delta^* U^* = U \Delta \bar{\Delta} U^* = U \begin{bmatrix} |\lambda_1|^2 & & 0 \\ & \ddots & \\ 0 & & |\lambda_n|^2 \end{bmatrix}$$

if $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_n|$, then

$\sigma_i(A) = \lambda_i$, $i = 1, \dots, n$.

Leon – p' 380–382, 1–6, 8

1. (1) $A = U \Sigma V^T$ (2) $A^T = V \Sigma^T U^T$ $\quad \Sigma^T = \begin{bmatrix} \sigma_1 & & & 0 \\ & \ddots & \sigma_r & \\ 0 & & & \ddots \end{bmatrix}$

As U, V orthogonal we see that the decomposition of (2) is the singular value decomposition with the singular values the same as of A.

2. (a) $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ $\quad AA^T = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$ charpol. $z^2 - 10z = 0$

$\lambda_1 = 10 \quad \lambda_2 = 0 \qquad \sigma_1 = \sqrt{10} \qquad \sigma_2 = 0$

$10I - AA^T = \begin{bmatrix} 8 & -4 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad u_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$x_2 \perp u_1 \qquad x_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \qquad u_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$U = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \qquad A^T u_1 = \sigma_1 v_1 \qquad v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \sqrt{5} & 0 \\ 0 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & -2 \\ 1 & 2 \end{bmatrix} \quad A^T = \begin{bmatrix} 2 & 1 \\ -2 & 2 \end{bmatrix} \quad AA^T = \begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix}$

Char pol. $z^2 - 13z + 36 = (z-4)(z-9)$

$\lambda_1 = 9, \ \sigma_1 = 3, \qquad \lambda_2 = 4, \qquad \sigma_2 = 2$

$AA^T - 9I = \begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 0 \qquad u_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad u_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$A^T u_2 = \frac{1}{2}\begin{bmatrix}2 & 1\\ -2 & 2\end{bmatrix}\frac{1}{\sqrt5}\begin{bmatrix}1\\ 2\end{bmatrix} = \frac{1}{2\sqrt5}\begin{bmatrix}4\\ 2\end{bmatrix} = \frac{1}{\sqrt5}\begin{bmatrix}2\\ 1\end{bmatrix} = \underline{v}_2$$

$$A^T \underline{u}_1 = \frac{1}{3}\begin{bmatrix}2 & 1\\ -2 & 2\end{bmatrix}\frac{1}{\sqrt5}\begin{bmatrix}2\\ -1\end{bmatrix} = \frac{1}{3\sqrt5}\begin{bmatrix}3\\ -6\end{bmatrix} = \frac{1}{\sqrt5}\begin{bmatrix}1\\ -2\end{bmatrix} = \underline{v}_1$$

$$U = \frac{1}{\sqrt5}\begin{bmatrix}2 & 1\\ -1 & 2\end{bmatrix} \quad V = \frac{1}{\sqrt5}\begin{bmatrix}1 & 2\\ -2 & 1\end{bmatrix} \quad \Sigma = \begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix}$$

(c) Since $A \in \mathbb{R}^{3\times2}$ it is more convenient to consider

$$A^TA = \begin{bmatrix}1 & 3 & 0\\ 3 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 3\\ 3 & 1\\ 0 & 0\end{bmatrix} = \begin{bmatrix}10 & 6\\ 6 & 10\end{bmatrix}$$

$\overset{4}{}$ $V\Sigma^T\Sigma V^T$

$$z^2 - 20z + 64 = 0 \qquad (z-4)(z-16) = 0$$

$\lambda_1 = 16 \quad \sigma_1 = 4 \qquad \lambda_2 = 4 \quad \sigma_2 = 2$

$\lambda = 16 \qquad \begin{bmatrix}-6 & 6\\ 6 & -6\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix} = 0 \qquad v_1 = \frac{1}{\sqrt2}\begin{bmatrix}1\\ 1\end{bmatrix} \qquad v_2 = \frac{1}{\sqrt2}\begin{bmatrix}1\\ -1\end{bmatrix}$

$V = \frac{1}{\sqrt2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \qquad \underline{u}_1 = \frac{1}{\sqrt2}\begin{bmatrix}1\\ 1\\ 0\end{bmatrix} \qquad u_2 = \frac{1}{\sqrt2}\begin{bmatrix}-1\\ 1\\ 0\end{bmatrix} \qquad u_3 = \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}$

as $u_3 \perp u_1$ & $\underline{u}_2$. So $u_3 \in$ null space of $\begin{bmatrix}1 & -1\\ 1 & 1\\ 0 & 0\end{bmatrix}^T$

$$\Sigma = \begin{bmatrix}4 & 0\\ 0 & 2\\ 0 & 0\end{bmatrix}$$

(d) As $A = \begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 1\\ 0 & 1 & 2\end{bmatrix}$ $\quad A^T = A,\ AA^T = A^2 = \begin{bmatrix}4 & 0 & 0\\ 0 & 5 & 4\\ 0 & 4 & 5\end{bmatrix}$

Char pol AA^T $\quad (z-4)(z^2 - 10z + 9) = (z-4)(z-9)(z-1)$

$\lambda_1 = 9 \quad \lambda_2 = 4 \quad \lambda_3 = 1$

$\sigma_1 = \cancel{\sqrt{9}}\ 3 \quad \sigma_2 = 2 \quad \sigma_3 = \cancel{\sqrt{1}}\ 1$

The eigenvectors of A^2 are $v_2 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ corresponding to $\lambda_2 = 4$

$$\begin{bmatrix}5 & 4\\ 4 & 5\end{bmatrix} - \begin{bmatrix}9 & 0\\ 0 & 9\end{bmatrix} = \begin{bmatrix}-4 & 4\\ 4 & -4\end{bmatrix} \quad \text{eigenvector } \begin{bmatrix}1\\1\end{bmatrix}$$

So $v_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\1\end{bmatrix}$ $\quad v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\-1\end{bmatrix}$. $\quad u_1 = v_1,\ u_2 = v_2,\ u_3 = v_3$

As A positive definite then the SVD is the spectral decomposition

$$A = U \Delta U^T \qquad U = \begin{bmatrix}0 & 1 & 0\\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\end{bmatrix}$$

$$\Delta = \begin{bmatrix}3 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}$$

3. a. 2a rank $A = 1$, 2b rank $A = 2$, 2c rank $A = 2$, 3c rank $A = 3$

b. 2a A itself, 2b $\sigma_1 u_1 v_1^T = 3 \frac{1}{\sqrt{2}}\begin{bmatrix}2\\-1\end{bmatrix} \frac{1}{\sqrt{5}}\begin{bmatrix}1 & -2\end{bmatrix}$

$$= \frac{3}{\sqrt{10}}\begin{bmatrix}2 & -4\\ -1 & 2\end{bmatrix}$$

2c $\sigma_1 u_1 v_1^T = 4 \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix} = 2\begin{bmatrix}1 & 1\\ 1 & 1\\ 0 & 0\end{bmatrix}$

2d $\sigma_1 u_1 v_1^T = 3 \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\1\end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 1 & 1\end{bmatrix}$

$$= \frac{3}{2}\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 1\\ 0 & 1 & 1\end{bmatrix}$$

4 Rank 1 $\quad \sigma_1 \underline{u}_1 \underline{v}_1^T = 30 \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \\ 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$

$$= 30 \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \\ \frac{4}{15} & \frac{8}{15} & \frac{8}{15} \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 12 & 12 \\ 8 & 16 & 16 \\ 0 & 0 & 0 \end{bmatrix}$$

~~Rank 2~~ $\quad \sigma_1 \underline{u}_1 \underline{v}^T + \sigma_2 \underline{u}_2 \underline{v}_2^T$

$$\begin{bmatrix} 6 & 12 & 12 \\ 8 & 16 & 16 \\ 0 & 0 & 0 \end{bmatrix} + 15 \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \\ 0 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

$$- \begin{bmatrix} 6 & 12 & 12 \\ 8 & 16 & 16 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -8 & -4 & 8 \\ 6 & 3 & -6 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 8 & 20 \\ 14 & 19 & 10 \\ 0 & 0 & 0 \end{bmatrix}$$

5. For $A = U\Sigma V^T$, $A^T = V\Sigma^T U^T$

$R(A^T) = \text{span}(\underline{v}_1, \dots, \underline{v}_r)$ where $r = \text{rank } A$.

~~So o.n. basis of A^T is~~ $\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)^T, \left(-\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)^T$

$N(A)$ is spanned by $v_{r+1}, \dots, v_n$ in this case $r=2$

$\underline{v}_3$, $\cancel{\underline{v}_4}$ $= \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)^T$.

6. $A = U\Delta U^T$, U orthogonal. $\Delta = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$.

So $AA^T = A^2 = U\Delta^2 U^T$ Hence if $|\lambda_1| \geq \dots \geq |\lambda_n|$

$\sigma_1^2 = \lambda_1^2 = |\lambda_1|^2 \geq \dots \geq \sigma_n^2 = \lambda_n^2 \Rightarrow \sigma_i = |\lambda_i|$.

Actually Problem 9 p.45 is more general than this problem.