1. Three.V.1.6

One can see that in terms of the standard basis vectors,

\[
\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]
\[
\begin{pmatrix} -2 \\ 4 \end{pmatrix} = -2 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 4 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

So, the change of basis matrix from \(D\) to \(\epsilon_2\) is:

\[
\begin{pmatrix} 2 & -2 \\ 1 & 4 \end{pmatrix}
\]

In the other direction things are a bit more challenging; first we need to calculate \(c_i\) such that

\[
c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

In other words, we are solving the system:

\[
\begin{pmatrix} 2 & -2 & 1 \\ 1 & 4 & 0 \end{pmatrix} \longrightarrow_{\rho_2 \rightarrow \rho_2 - 1/2\rho_1} \begin{pmatrix} 2 & -2 & 1 \\ 0 & 5 & -1/2 \end{pmatrix}
\]

Thus, \(c_2 = -1/10\). Substituting into row 1, we see \(c_1 = 2/5\)

Next, we have to solve the system:

\[
d_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + d_2 \begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

So, as above,

\[
\begin{pmatrix} 2 & -2 & 0 \\ 1 & 4 & 1 \end{pmatrix} \longrightarrow_{\rho_2 \rightarrow \rho_2 - 1/2\rho_1} \begin{pmatrix} 2 & -2 & 0 \\ 0 & 5 & 1 \end{pmatrix}
\]

So, we can see that \(d_2 = 1/5\). Examining row 1 then shows us that \(d_1 = 1/5\). So, the change of basis matrix is

\[
\begin{pmatrix} 2/5 & 1/5 \\ -1/10 & 1/5 \end{pmatrix}
\]

It would also be acceptable to find the inverse of the original change of basis matrix (this is essentially what we are doing); this is rather easier if you recall the trick for \(2 \times 2\) matrices.

2. Three.V.2.10

a) \[
\begin{pmatrix} 1 & 3 & 0 \\ 2 & 3 & 0 \\ 1 \end{pmatrix}
\]
has rank two, because the first two columns are obviously linearly independent.

\[
\begin{pmatrix}
  2 & 2 & 1 \\
  0 & 5 & -1
\end{pmatrix}
\]

has rank two because the first two columns are obviously linearly independent. So, the matrices are matrix equivalent.

b) Both matrices have rank two. Again, the columns of each matrix are obviously linearly independent. So, they are matrix equivalent.

c) 

\[
\begin{pmatrix}
  1 & 3 \\
  2 & 6
\end{pmatrix}
\]

The first and second columns are linearly dependent since the second column is three times the first. So, the rank of the matrix is one.

\[
\begin{pmatrix}
  1 & 3 \\
  2 & -6
\end{pmatrix}
\]

Of course, the two columns here are linearly dependent, because, for instance, the determinant is -12. So, the matrices are not matrix equivalent.

3. THREE.V.2.11

For this problem, like the previous one, we need only decide what the rank of the matrix is, and then use the standard basis vectors.

a) 

\[
\begin{pmatrix}
  2 & 1 & 0 \\
  4 & 2 & 0
\end{pmatrix}
\]

The first and second columns are linearly dependent (the first column is twice the second). So, the canonical representative is:

\[
\begin{pmatrix}
  1 & 0 & 0 \\
  0 & 0 & 0
\end{pmatrix}
\]

b) Here, the first three columns are linearly independent, since, for instance, the determinant of that 3 × 3 matrix is -3. So, the canonical representative is:

\[
\begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0
\end{pmatrix}
\]
By definition, the set consists of
\[
\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| x + 3y - z = 0 \right\}
\]
If we wish to describe this set via basis elements, we could let \( y = 0, z = 1 \), and see \( x = 1 \), while if \( y = 1, z = 0 \), then \( x = -3 \). So, we know that a basis for the subspace is given by
\[
\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} \right\}
\]
5. THREE.vI.1.8

a) Note that one simply needs any vector which lies on the line. So, apply the formula for projection to get:
\[
\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}
\]

b) Again, we need only one vector on the line, so we pick \( \begin{pmatrix} 1 \\ 3 \end{pmatrix} \). So, now the projection is given by:
\[
\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -2/5 \\ -6/5 \end{pmatrix}
\]