SOLUTION TO HW9 - MATH 425 - SPRING 12 - S. FRIEDLAND (1)

p' 45-46 P1. $(\epsilon T)^*(\epsilon T) = (\bar{\epsilon} T^*)(\epsilon T) = \epsilon \bar{\epsilon} T^* T$

$= |\epsilon|^2 T^* T$. Hence

$\lambda_j(|\epsilon|^2 T^* T) = |\epsilon|^2 \lambda_j(T^* T)$

So $\sigma_j(\epsilon T) = \sqrt{\lambda_j((\epsilon T)^*(\epsilon T))} = \sqrt{|\epsilon|^2 \lambda_j(T^* T)} = |\epsilon| \sqrt{\lambda_j(T^* T)}$

$= |\epsilon| \sigma_j(T)$ for $j = 1, 2, \ldots$

P3. $(UAV)^*(UAV) = V^* A^* \underbrace{U^* U}_{I} AV = V^* A^* A V = V^{-1} A^* A V$

Since $(UAV)^*(UAV)$ is similar to $A^* A$ they have the same eigenvalues. So

$\lambda_j((UAV)^*(UAV)) = \lambda_j(A^* A) \Rightarrow \sigma_j^2(UAV) = \sigma_j^2(A), j = 1, 2, \ldots$

As $\sigma_j \ge 0 \Rightarrow \sigma_j(UAV) = \sigma_j(A)$ for $j = 1, 2, \ldots$

(4) $\lambda_1(A^* A) \ge \lambda_2(A^* A) \ldots \ge \lambda_n(A^* A) > 0$ as $\det A \ne 0$

Recall that for $B = A^* A$ the eigenvalues of B^{-1} are $\frac{1}{\text{eigenvalues of } B}$. Because of the order of the eigenvalues $\lambda_1(B^{-1}) = \frac{1}{\lambda_n(B)} \ge \lambda_2(B^{-1}) = \frac{1}{\lambda_{n-1}(B)} \ldots$

Hence $\sigma_1^2(A^{-1}) = \lambda_1(B^{-1}) = \frac{1}{\lambda_n(B)} = \frac{1}{\sigma_n^2(A)} =$

As $B^{-1} = (A^{-1})(A^{-1})^*$ so $\sigma_1(A^{-1}) = \frac{1}{\sigma_n(A)}$.

(5) By choosing an orthonormal basis in U_1, U_2, V_1, V_2 and representing T in these bases we deduce that T is represented by a matrix $A = \operatorname{diag}(A_1, A_2) = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix}$

$A_1 \in \mathbb{C}^{n_1 \times n_1}$, $A_2 \in \mathbb{C}^{n_2 \times n_2}$

So rank A = the number of lin. ind. row of A is clearly sum of # independent rows in A_1 and A_2 respectively.

So $\underset{\text{rank } T}{\text{rank } A} = \text{rank } A_1 + \text{rank } A_2 = \text{rank } T_1 + \text{rank } T_2$

Consider
$$AA^* = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} A_1^* & 0 \\ 0 & A_2^* \end{bmatrix} = \begin{bmatrix} A_1A_1^* & 0 \\ 0 & A_2A_2^* \end{bmatrix}$$

$A_1A_1^*$ $m_1 \times m_1$ hermitian, $A_2A_2^*$ is $m_2 \times m_2$ hermitian

Then nonzero eigenvalues of AA^* is the union of the positive eigenvalues of $A_1A_1^*$ and $A_2A_2^*$.

Hence $\{\sigma_1(T), \dots, \sigma_{\operatorname{rank}(T)}\} = \{\sigma_1(T_1), \dots, \sigma_{\operatorname{rank} T_1}(T_1)\} \cup \{\sigma_2(T_2), \dots, \sigma_{\operatorname{rank} T_2}(T_2)\}$

as <u>multisets</u>.

(9) T normal if T is represented by a normal matrix in an orthonormal basis of V.

$A = U \Delta U^*$ (normal) $\quad \Delta = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

$$AA^* = U\Delta U^* (U\Delta U^*)^* = U \Delta U^* \underbrace{(U^*)^*}_{U} \underbrace{\Delta^*}_{\bar{\Delta}} U^*$$
$$= U \Delta \bar{\Delta} U^*, \quad \Delta\bar{\Delta} = \begin{bmatrix} |\lambda_1|^2 & & 0 \\ & \ddots & \\ 0 & & |\lambda_n|^2 \end{bmatrix}$$

So $\lambda_j(AA^*) = |\lambda_j(A)|^2 \quad j=1, n.$

Hence $\sigma_j(A) = |\lambda_j(A)|$

<u>p'54</u> (1) $P^2 = P \Rightarrow Px = \lambda x \quad P^2x = \lambda^2 x = Px = \lambda x$

$\Rightarrow \lambda^2 - \lambda = 0 \qquad \lambda(\lambda-1) = 0 \Rightarrow \lambda = 0$ or $\lambda = 1$.

Let $u_1 \dots u_k$ be lin. independent columns of H, i.e. $k = \operatorname{rank} P$. Then $PV = \operatorname{span}(v_1, \dots, v_k)$

Let $u_{k+1}, \dots, u_n$ be a basis in the kernel of P.

So $Pu_j = 0$ for $j = k+1, n$. ($\operatorname{rank} P + \ker P = n$).

$v_1, \dots, v_k$ are lin. ind. eigenvectors of P

Note that $\underline{v}_j = \underline{P}\underline{e}_{i_j}$ where $\underline{v}_j$ is the i_j column of $\underline{P}$

and $\underline{e}_i = (\delta_{i1}, \dots, \delta_{in})^T$ is the ith vector in the standard basis of $\mathbb{C}^n$.

So

$$\underline{P}v_j = P^2 e_{i_j} = P e_{i_j} = v_j.$$

~~As $V =$~~ We claim that $v_1, \dots v_n$ lin. indep.

Suppose that

$$(1)\ \sum_{i=1}^{n} a_i v_i = 0 \quad \text{. Then } \underline{0} = \underline{P}\underline{0} = \underline{P}\left(\sum_{i=1}^{n} a_i v_i\right)$$

$$= \sum_{i=1}^{n} a_i \cdot P v_i = \sum_{i=1}^{k} a_i P v_i = \sum_{i=1}^{k} a_i v_i$$

as $\underline{P}\underline{v}_j = 0$ for $j > k$ — as $P\underline{v}_i = \underline{v}_i$ for $i = 1, \dots, k$.

Since $v_1, \dots, v_k$ lin. ind. $\Rightarrow a_1 = \dots = a_k = 0$

Hence (1) yields that $\sum_{i=k+1}^{n} a_i \underline{v}_i = 0$

As $\underline{v}_{k+1}, \dots, \underline{v}_n$ lin. ind $\Rightarrow$ $a_{k+1} = \dots = a_n = 0$. So $\underline{v}_1, \dots, \underline{v}_n$ lin. ind.

Since $v_1, \dots, v_n$ are eigenvectors of $\underline{P}$ it follows that $\underline{P}$ is diagonable. More precisely in $\underline{v}_1, \dots, \underline{v}_n$ $\underline{P}$ is represented by the matrix

$$\begin{bmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ 0 & & & & \ddots \; 0 \end{bmatrix}.$$

(3) (a) $A = U \Sigma V^T$ (b) $A^T = V \Sigma^T U^T$, U, V orthogonal

$$\Sigma = \begin{bmatrix} \sigma_1 \ddots \sigma_r & 0 \\ 0 & 0 \end{bmatrix} \qquad \Sigma^T = \begin{bmatrix} \sigma_1 \ddots \sigma_r & 0 \\ 0 & 0 \end{bmatrix}$$

So (b) is the singular value decomposition of A^T

In particular $\sigma_i(A) = \sigma_i(A^T)$

(b) Let $\operatorname{rank} A = r$ and consider the reduced SVD of A:

$$A = U_r \Sigma_r V_r^T$$

$$\Sigma_r = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \end{bmatrix} \in \mathbb{R}^{r\times r}, \; U_r \in \mathbb{R}^{m\times r}, \; V_r \in \mathbb{R}^{n\times r}$$

$$U_r^T U_r = V_r^T V_r = I_r$$

Then $A^{\dagger} = V_r \Sigma_r^{-1} U_r^T$

Note that $A^T = V_r \Sigma_r^T U_r^T = V_r \Sigma_r U_r^T$ is the reduced SVD of A^T.

So $(A^T)^{\dagger} = U_r \Sigma_r^{-1} V_r^T = (V_r \Sigma_r^{-1} U_r^T)^T = (A^{\dagger})^T$

(c) No. Let (a) $A = \underline{u}\,\underline{v}^T$ (b) $B = \underline{x}\,\underline{y}^T$, $\underline{u}, \underline{v}, \underline{x}, \underline{y} \neq 0$

Suppose that $y^T\underline{u} \neq 0 \Rightarrow \cancel{BA = 0}$

~~So~~ $\cancel{(BA)^{\dagger} = 0}$

~~The k~~ Assume that $\|\underline{u}\| = \|\underline{v}\| = \|\underline{x}\| \, \|\underline{y}\| = 1$

Then (a) and (b) is the reduced SVD of A an B respectively and $\sigma_1(A) = \sigma_1(B) = 1$ if $\Sigma_1(A) = \Sigma_2(A) = 1$

Hence $A^{\dagger} = \underline{v}\,\underline{u}^T$, $B^{\dagger} = \underline{y}\,\underline{x}^T$

$A^{\dagger} B^{\dagger} = \underline{v} \underbrace{(\underline{u}^T \underline{y})}_{\text{number}} \underline{x}^T$, Now $BA = \underline{x}(\underline{y}^T \underline{u})\underline{v}^T$

So $\sigma_1(BA) = |y^T\underline{u}|$ Hence $(BA)^{\dagger} = \frac{1}{(y^T\underline{u})} \underline{v}\,\underline{x}^T$

See <u>Quiz 8</u>. Hence $A^{\dagger} B^{\dagger} \neq (BA)^{\dagger}$ in general.