1. Three.VI.2.9

In each of the following examples, we do not need to change the first vector at all. The formula for the second vector is:

\[
\bar{w}_2 = \bar{v}_2 - \text{proj}_{\bar{v}_1} \bar{v}_2 = \bar{v}_2 - \frac{\bar{v}_2 \cdot \bar{v}_1}{\bar{v}_1 \cdot \bar{v}_1} \bar{v}_1
\]

where \( v_1 \) and \( v_2 \) are the vectors in the original basis, and \( w_1 = v_1 \) and \( w_2 \) are the vectors in the orthogonal basis given by the Gram-Schmidt process. The correct answers for \( \bar{w}_2 \) are:

a) \( \left( \begin{array}{c} 1/2 \\ -1/2 \end{array} \right) \)

b) \( \left( \begin{array}{c} -1 \\ 0 \end{array} \right) \)

c) \( \left( \begin{array}{c} -1 \\ 0 \end{array} \right) \)

2. Three.VI.3.19

Suppose that \( \bar{v} \cdot \bar{w}_i \) for each \( i = 1, \ldots, n \). Then we will show that \( \bar{v} \cdot (\sum_{i=1}^{n} c_i \bar{w}_i) = 0 \).

To see this, simply note that

\[
\bar{v} \cdot (\sum_{i=1}^{n} c_i \bar{w}_i) = \sum_{i=1}^{n} c_i \bar{v} \bar{w}_i = 0
\]

3. Least squares 1

I will use the following notation:

\[
\bar{x} = \left( \begin{array}{c} -1 \\ 0 \\ 1 \\ 2 \end{array} \right)
\]

and:

\[
\bar{y} = \left( \begin{array}{c} 0 \\ 1 \\ 3 \\ 9 \end{array} \right)
\]

In this case, the line of best fit is given by \( y = \alpha + \beta x \) where

\[
\beta = \frac{\bar{x} \cdot \bar{y} - 1/4 \cdot \sum x_i \sum y_i}{\bar{x} \cdot \bar{x} - a/4(\sum x_i)^2}
\]

Particularly, we see in this case that \( \beta = 4 \). \( \alpha \) is given by

\[
\alpha = 1/4 \sum y_i - \beta \sum x_i
\]
In this case, we get $\alpha = -0.75$.

4. Least squares 2

In the case that we have an overdetermined system (as is the case here). The least squares solution is given by any solution to the equation:

$$A^T\bar{x} = A^Tb$$

So, in this case, we first note that

$$A^T A = \begin{pmatrix} 1 & -2 & 2 \\ 1 & 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -2 & 3 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 9 & -7 \\ -7 & 11 \end{pmatrix}$$

and

$$A^Tb = \begin{pmatrix} 1 & -2 & 2 \\ 1 & 3 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

The least squares solution works out to be

$$\begin{pmatrix} 1.66 \\ 1.42 \end{pmatrix}$$

So, since this is not an actual solution (you should check!), we can see that there can be no exact simultaneous solution to all three equations.

5. Least squares 3

One can find the least squares solution to the problem by finding the least squares solution to

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 4 \\ 4 \end{pmatrix}$$

We know that finding a least squares solution to this system corresponds to finding a solution to the system where we multiply both sides by $A^T$:

$$\begin{pmatrix} 4 & 6 & 14 \\ 6 & 14 & 36 \\ 14 & 36 & 98 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 13 \\ 22 \\ 54 \end{pmatrix}$$

Luckily, the matrix on the left is invertible, and after multiplying both sides by the inverse, we see

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2.75 \\ -0.25 \\ 0.25 \end{pmatrix}$$
So, the quadratic function which fits the data best, in the sense of least squared errors is

\[ y = 2.75 - .25x + .25x^2 \]