Problem 1. Consider the matrix

\[
A = \begin{bmatrix}
-2 & -4 \\
1 & -6
\end{bmatrix}
\]

a. Find the characteristic polynomial.
b. Find the eigenvalues and corresponding linear independent eigenvectors.
c. Find the general solution to the system of differential equations \( \frac{dx}{dt} = Ax \).

Hints for solution: \( \det(A - zI) = (-2 - z)(-6 - z) + 4 = z^2 + 8z + 16 = (z + 4)^2 \).

There one double eigenvalue \( \lambda_1 = \lambda_2 = -4 \).

To find a basis in the eigenvector space we need to find a basis in the null space of \( A + 4I \). It is a vector \((2, 1)^\top\).

Since there is only one eigenvalue and one independent eigenvector, the Jordan canonical form of \( A \) is \[
\begin{bmatrix}
-4 & 1 \\
0 & -4
\end{bmatrix}.
\]

Hence the minimal polynomial of \( A \) is equal to the characteristic polynomial of \( A \): \( \psi(z) = (z + 4)^2 \).

The general solution of system of differential equations is \( e^{At}x_0 \).


Since \( \psi(z) = (z + 4)^2 \), there are two components of \( A \): \( Z_{10}, Z_{11} \).

To find them we need to find the corresponding two Lagrange-Sylvester polynomials, see pages 67-68 of my notes.

\( Z_{10} = \phi_{10}(A), Z_{11} = \phi_{11}(A) \). The polynomial \( \phi_{10}, \phi_{11} \) are of degree 1 at most, (degree of the minimal polynomial minus one). They are characterized by the following data:

\( \phi_{10}(-4) = 1, \phi_{10}'(-4) = 0 \). So \( \phi_{10} = a(z + 4) + b \). Hence \( 1 = \phi_{10}(-4) = b \). \( 0 = \phi_{10}'(-4) = a \).

a. So \( \phi_{10} = 1 \) and \( Z_{10} = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \).

Since \( \phi_{11}(-4) = 0 \) it follows that \( \phi_{11}(z) = c(z + 4) \). \( \phi_{11}'(-4) = 1 \Rightarrow c = 1 \). So \( \phi_{11}(z) = z + 4 \) and \( Z_{11} = \phi_{11}(A) = A + 4I = A = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \). (4.14) implies \( e^{At} = e^{-4t}Z_{10} + te^{-4t}Z_{11} = e^{-4t}\begin{bmatrix} 1 + 2t & -4t \\ t & 1 - 2t \end{bmatrix} \). So \( x(t) = e^{-4t}\begin{bmatrix} 1 + 2t & -4t \\ t & 1 - 2t \end{bmatrix}(c_1, c_2)^\top \).

Problem 2. Let \( A \in \mathbb{C}^{n \times n} \). Assume that \( A \) has only two linearly independent eigenvectors.

1. Show that \( A \) has a has at most two distinct eigenvalues.

2. Assume that \( n = 3 \). Write down all possible Jordan canonical forms of \( A \).

3. Let \( n = 3 \) and assume that \( A^2 \) is diagonalizable. Suppose that the trace of \( A \) is equal to 3. What is the Jordan canonical form of \( A \)?

Hints for solution: Each Jordan block \( J_k(\lambda) \in \mathbb{C}^{k \times k} \), see page 48 of Math 425 Notes, has exactly one linearly independent eigenvalue. Hence the number of linearly independent eigenvectors of \( A \) is the number of Jordan blocks in the Jordan canonical form (JCF) of \( A \). Hence the given \( A \) has two Jordan blocks in its JCF. Each \( J_k(\lambda) \) has one eigenvalues of multiplicity \( k \). Hence the JCF of \( A \) has at most two distinct eigenvalues, i.e. \( A \) has has at most two distinct eigenvalues.
Suppose that \( n = 3 \). As \( A \) has two Jordan blocks one must be of order 2 and one of order 1. Hence the Jordan canonical form if \( A \) is 
\[
B = \begin{bmatrix}
\lambda & 1 & 0 \\
0 & \lambda & 0 \\
0 & 0 & \mu
\end{bmatrix}.
\]

Note that \( A^2 \) is similar to \( B^2 = \begin{bmatrix}
\lambda^2 & 2\lambda & 0 \\
0 & \lambda^2 & 0 \\
0 & 0 & \mu^2
\end{bmatrix} \). It is easy to show that \( C := \begin{bmatrix}
\lambda^2 & 2\lambda \\
0 & \lambda^2
\end{bmatrix} \) has only one eigenvalue \( \lambda^2 \). So \( C \) is diagonalizable, if and only if it similar to a diagonal matrix \( \lambda^2 I_2 \). Any matrix similar to \( cI_2 \) is of the form \( cI_2 \). Hence \( C \) diagonalizable if and only if \( \lambda = 0 \). Now the trace of \( A \) is \( 2\lambda + \mu = \mu = 3 \). Hence the JCF of \( A \) is 
\[
B = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 3
\end{bmatrix}.
\]

**Problem 3.** Let \( A = \begin{bmatrix}
3 & -1 & 1 \\
7 & -5 & 1 \\
6 & -6 & 2
\end{bmatrix} \)

(a) Show that the characteristic and the minimal polynomial of \( A \) are equal to \( (z-2)^2(z+4) \).
(b) Find the components of \( A \).
(c) Find the formula for \( A^l \) for any integer \( l \) using the components of \( A \).

**Hints for solution:**
Expand \( \det(zI - A) \) be the first row and show that it is equal to \( (z-2)^2(z+4) \). The minimal polynomial of \( A \) is either \((z-2)(z+4)\) or \((z-2)^2(z+4)\). Show that the product \((A - 2I)(A + 4I) \neq 0 \). For example multiply the first row of \( A - 2I \) by \((-1, -1, 1)\) time the first column of \( A + 4I \) to see that \((1, 1, 1)\) entry of \((A - 2I)(A + 4I) \) different from 0. Hence the minimal polynomial of \( A \) is \((z-2)^2(z+4) \).

There are 3 \( Z \) components of \( A \): \( Z_{10}, Z_{11} \) corresponding to \( z = 2 \) and \( Z_{20} \) corresponding to \( z = -4 \). So \( Z_{10} = \phi_{10}(A), Z_{11} = \phi_{11}(A), Z_{20} = \phi_{20}(A) \). The 3 polynomials \( \phi_{10}, \phi_{11}, \phi_{20} \) are of degree 2 at most.

\( \phi_{10}(2) = 1, \phi_{10}'(2) = 0, \phi_{10}(-4) = 0 \). Hence \( \phi_{10} = (z + 4)(a(z - 2) + b) \). \( 1 = \phi_{10}(2) = 6b \to b = \frac{1}{6} \). \( 0 = \phi_{10}'(2) = b + 6a \to a = -\frac{1}{36} \). So \( \phi_{10} = \frac{1}{36}(z + 4)(-z + 8) \). So \( Z_{10} = \frac{1}{36}(A + 4I)(-A + 8I) \).

\( \phi_{11}(2) = 0, \phi_{11}'(2) = 1, \phi_{11}(-4) = 0 \). Hence \( \phi_{10} = c(z - 2)(z + 4) \).

\[
1 = \phi_{11}'(2) = 6c \Rightarrow c = \frac{1}{6} \Rightarrow \phi_{11}(z) = \frac{1}{6}(z - 2)(z + 4) \Rightarrow Z_{11} = \frac{1}{6}(A - 2I)(A + 4I).
\]

\( \phi_{20}(2) = 0, \phi_{20}'(2) = 0, \phi_{20}(-4) = 1 \). So \( \phi_{20}(z) = d(z - 2)^2 \). \( 1 = \phi_{20}(-4) = d36 \). \( \phi_{20} = \frac{1}{36}(z - 2)^2, Z_{20} = \frac{1}{36}(A - 2I)^2 \).

Use formula (4.3) on page with \( f(z) = z^l \) to deduce that \( A^l = 2^l Z_{10} + l^2 - 1 Z_{11} + (-4)^l Z_{20} \).

**Problem 5.** Suppose that \( A \in \mathbb{C}^{n \times n} \) and the minimal polynomial of \( A \) is \((z - \alpha)(z - \beta)^2(z - \gamma)^3 \) where \( \alpha, \beta, \gamma \) are three distinct complex numbers.

(a) Write down the general form of the characteristic polynomial of \( A \).
(b) What is the condition for \( A \) to be power stable, i.e. \( \lim_{l \to \infty} A^l = 0 \).
(c) What is the condition for \( A \) to be power convergent, i.e. \( \lim_{l \to \infty} A^l = B \).

**Hints for solution:**
\[\det(zI - A) = (z - \alpha)^a(z - \beta)^b(z - \gamma)^c \] where \( a, b, c \) are integers satisfying \( a \geq 1, b \geq 2, c \geq 3 \).

Use Theorem 4.5, page 70, to deduce. \( A \) is power stable if and only if \(|\alpha| < 1, |\beta| < 1, |\gamma| < 1 \). \( A \) is power convergent if either \(|\alpha| < 1 \) or \( \alpha = 1 \). (Note that all the Jordan blocks corresponding to \( \alpha \) are of order 1.) Since there is a Jordan block of order 2 corresponding to \( \beta \)
Problem 6. Let \( T : V \to V \) be a linear transformation on a finite dimensional vector space \( V \) over a field \( F \).

a. Let \( U \) be a nontrivial subspace of \( V \), i.e. \( 0 < \dim U < \dim V \). Define the quotient space \( \tilde{V} := V/U \) and show that it is a vector space over \( F \) of dimension \( \dim V - \dim U \).

b. Suppose that \( U \) is \( T \)-invariant. Show that \( T \) induces a linear transformation \( \bar{T} : \tilde{V} \to \tilde{V} \).

c. Let \( \psi, \tilde{\psi} \) be the minimal polynomials of \( T, \bar{T} \) respectively. Show that \( \psi \) divides \( \tilde{\psi} \).

Hints for solution: Definition: A quotient space \( \tilde{V} := V/U \) is the set of all cosets \([x] := x + U = \{y \in V : y - x \in U\}\) for any \( x \in V \). If \( \{u_1, \ldots, u_m\} \) is a basis for \( U \) and \( \{v_1, \ldots, v_n\} \) a basis for \( V \) then \([u_{m+1}], \ldots, [u_n]\) is a basis of \( V/U \). Hence \( \dim V/U = \dim V - \dim U \).

Suppose that \( TU \subseteq U \), i.e. \( U \) is invariant under \( T \). Then \( T[x] = T(x + U) = Tx + TU \subseteq Tx + U \). So \([Tx]\) is well defined, i.e. \([Tx] = [Ty]\) for any \( y \in x + U \). Hence \( T \) induces an linear operator on \( V/U \). Define this operator as \( \bar{T} : \tilde{V} \to \tilde{V} \). Since \( \psi(T) = 0 \) it follows that \( \tilde{\psi} = 0 \). Hence the minimal polynomial of \( \bar{T} \) must divide \( \psi \).

Problem 7.

1. Let \( B = [b_{ij}]_{i,j=1}^{n} \in \mathbb{R}^{n \times n} \) be a symmetric matrix. Show that \( \langle x, y \rangle := y^\top B x \) is an inner product on \( \mathbb{R}^n \) if and only if \( B \) is positive definite.

2. \( A \in \mathbb{R}^{n \times n} \). Is \( B = A^\top A \) nonnegative definite? When \( A \) is positive definite? Justify.

3. Let \( A = [a_{ij}]_{i,j=1}^{n} \in \mathbb{R}^{n \times n} \). Assume that \( a_{ij} \in \{0, 2\} \) for \( i, j = 1, \ldots, n \) and \( n \geq 2 \).

Show that \( |\det A| \leq 2^n n^2 \). Can equality hold for some matrix \( A \)?

Hints for solution: Clearly \( y^\top B x \) is a bilinear form for an \( B \in \mathbb{R}^{n \times n} \). If \( B \) is symmetric then \( \langle x, y \rangle = y^\top B x = (y^\top B x)^\top = x^\top B^\top y = x^\top By = \langle x, y \rangle \) is symmetric. Then \( \langle x, x \rangle > 0 \) for any \( x \neq 0 \) if and only if \( x^\top \) \( x > 0 \) for \( x \neq 0 \), i.e. \( B \) is positive definite.

\( B = A^\top A \Rightarrow x^\top A^\top A x = (Ax)^\top (Ax) \geq 0 \). Hence \( B \) is nonnegative definite. \( B \) is positive definite yields that \( Ax = 0 \iff x = 0 \). So the columns of \( A \) are linearly independent, i.e. \( \text{rank } A = n \).

Hadamard’s inequality, page 13 of Math 425 notes, yields that \( |\det A| \) does not exceed the products of the norms of all the columns of \( A \). Since \( a_{ij} \in \{0, 2\} \) the norm of each column is at most \( \sqrt{1 + 4 + \ldots + 4} = \sqrt{4n} = 2\sqrt{n} \). So \( |\det A| \leq (2\sqrt{n})^n = 2^n n^2 \). Equality will hold if and only if the columns are orthogonal to each other. This is impossible since all the columns are positive.

Problem 8. Let \( B = [b_{ij}]_{i,j=1}^{n} \in \mathbb{R}^{n \times n} \) be a real symmetric matrix. Denote by \( A = [a_{ij}]_{i,j=1}^{n-1} \) the real symmetric matrix obtained from \( B \) by deleting the \( j \)-th row and column.

1. Show the Cauchy interlacing inequalities

\[ \lambda_i(B) \geq \lambda_i(A) \geq \lambda_{i+1}(B), \quad \text{for } i = 1, \ldots, n-1. \]

2. Show that inequality \( \lambda_1(B) + \lambda_n(A) \leq \lambda_1(A) + b_{jj} \).

Hints for solution: Note that if \( U \) is a subspace of \( \mathbb{R}^n \) orthogonal to \( e_j = (\delta_{1j}, \ldots, \delta_{nj})^\top \), i.e. each vector \( x = (x_1, \ldots, x_n) \) satisfies \( x_i \) then by letting \( y = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)^\top \in \mathbb{R}^{n-1} \) we get \( x^\top B x = y^\top A y \). Theorem 2.36, page 21 yields

\[ \lambda_i(B) = \max_{U \subset \mathbb{R}^n, \dim U = i} \min_{0 \neq x \in U} \frac{x^\top B x}{x^\top x} \geq \max_{V \subset \mathbb{R}^{n-1}, \dim V = i} \min_{0 \neq y \in V} \frac{x^\top B x}{x^\top x} = \lambda_i(A). \]

Consider the following equalities and inequalities

\[ -\lambda_{n-k+1}(B) = \lambda_k(-B) \geq \lambda_k(-A) = -\lambda_{n-1-k+1}(A) \] to deduce \( \lambda_i(A) \geq \lambda_{i+1}(B) \).
Note that trace $B = b_{11} + \ldots + b_{nn} = \lambda_1(B) + \ldots + \lambda_n(B)$, trace $A = a_{11} + \ldots + a_{(n-1)(n-1)} = \lambda_1(A) + \ldots + \lambda_n(A)$. Also trace $B$ is trace $A$ plus $b_{jj}$. So $\sum_{k=1}^n \lambda_k(B) = b_{jj} + \sum_{k=1}^{n-1} \lambda_k(A)$. Hence

$$\lambda_1(B) + \lambda_n(B) - (\lambda_1(A) + b_{jj}) = \sum_{k=2}^{n-1} (\lambda_k(A) - \lambda_k(B)) \leq 0.$$

**Problem 9.** Let $A = [a_{ij}]_{i,j=1}^{n} \in \mathbb{R}^{m \times n}$. Recall that the singular value decomposition of $A$ is given as $A = U \Sigma V^\top$, where $U \in \mathbb{R}^{m \times n}, V \in \mathbb{R}^{n \times n}$ are orthogonal matrices, and $\Sigma \in \mathbb{R}^{m \times n}$ is a diagonal matrix with the nonnegative diagonal entries $\sigma_1(A) \geq \sigma_2(A) \geq \ldots$, which are called the singular values of $A$. Show

1. $\sigma_1(A) = \max_{\|x\| = 1} \frac{\|Ax\|}{\|x\|}$. Here $\|x\| = \sqrt{x^\top x}$. **Hint:** $\|Ax\|^2 = x^\top A^\top Ax$. 

2. Assume that the absolute value of each entry of $A$ is bounded above by $a$, i.e. $|a_{ij}| \leq a$ for all $i, j$. Show that $\sigma_1(A) \leq a\sqrt{mn}$. Give an example of $A$, for which equality holds. **Hint:** Use part 1 and the Cauchy-Schwarz inequality to estimate $|(Ax)_i|$. 

3. Let $A^\dagger$ be the Moore-Penrose inverse of $A$. Assume that rank $A = k$. What is the formula of $\sigma_1(A^\dagger)$ in terms of $\sigma_1(A), \ldots, \sigma_k(A)$? (Justify !)

**Hints for solution:** The maximum characterization of the the first eigenvalue of $B = A^\top A$, page 232 of Math 320 notes, yields that $\lambda_1(B) = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$. Here $\|x\| = \sqrt{x^\top x}$, where $x^\top$ is the $i-$th row of $A$. Since the absolute value of each element of $A$ is bounded above by $a$ it follows that $\|u_i\| \leq \sqrt{a^2 + \ldots + a^2} = a\sqrt{n}$. Hence $\|Ax\| = \sqrt{(Ax)^2_1 + \ldots + (Ax)^2_m} \leq a\sqrt{n} \|x\|$. The maximal characterization of $\sigma_1(A)$ yields $\sigma_1(A) \leq a\sqrt{mn}$. Take a matrix $A$ whose all entries are equal to $a$. It is a rank one matrix. $B = A^\top A$ is a rank one matrix with trace $mna$. $B$ has only positive eigenvalue, which is equal to the trace of $B$. So $\sigma_1(A)^2 = \lambda_1(B) = mna$.

The reduced SVD of $A$ is $A = U_r \Sigma_r V_r^\top$, see (2.2) page 43 of Math 425 notes. Here $\Sigma_r$ is the diagonal matrix $\text{diag}(\sigma_1(A), \ldots, \sigma_r(A))$. Then $A^\dagger = V_r \Sigma_r^{-1} U_r^\top$. As $\text{diag}(\sigma_1(A), \ldots, \sigma_r(A))^{-1} = \text{diag}(\frac{1}{\sigma_1(A)}, \ldots, \frac{1}{\sigma_r(A)})$ it follows that $\sigma_i(A^\dagger) = \frac{1}{\sigma_i(A)}$ for $i = 1, \ldots, r$. Hence $\sigma_1(A^\dagger) =$ ...