Math 425 - Spring 2013, Quiz 9, April 5, 2013, Instructor: S. Friedland

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Show all work. **Unjustified** answer yields no credit.

1. (4 pts) Let $A \in H_n$. Assume that A is nonnegative definite. Show that A has a unique nonnegative definite square root. I.e., there exists a unique $B \in H_n$ which is nonnegative definite such that $A = B^2$.

2. (4 pts) Let $A, B \in H_n$. Assume that A is positive definite. Show that AB is diagonable, (similar to a diagonal matrix), and its eigenvalues are real.

3. (2 pts) Show that part 2 is false if A is only only nonnegative definite. (**Hint:** Set $n = 2$ and assume that A is a diagonal matrix.)

1. Suppose that $B \in H_n$, $B^2 = A$.
Let $\lambda_1(B) \ge \dots \ge \lambda_n(B)$ be the eigenvalues of B.
So $B\underline{x} = \lambda_i(B)\underline{x}$, $\underline{x} \ne 0 \Rightarrow B^2\underline{x} = \lambda_i(B)B\underline{x}$
$= \lambda_i(B)^2\underline{x}$. I.e. $A\underline{x} = \lambda_i(B)^2\underline{x}$.
Hence the eigenvalues of B are $\pm\sqrt{\lambda(A)}$.
$B \ge 0$ means that $\lambda_i(B) = \sqrt{\lambda_i(A)}$.
So the eigenspace of B corresponding to $\sqrt{\lambda(A)}$ is the eigenspace of A corresponding to A. Hence B unique.

2. AB is similar $A^{-1/2}(A^{1/2}A^{1/2})BA^{1/2}$
$= A^{1/2}BA^{1/2} = C$ But C is Hermitian. Hence
C is diagonable with real eigenvalues. Hence AB is diagonable with real eigenvalues

3. $\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{B} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = C$

C has only zero eigenvalues but not diagonable
as $Q\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}Q^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.