Eigenvalue inequalities, log-convexity and scaling: old results and new applications, a tribute to Sam Karlin

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Univ. Illinois at Chicago

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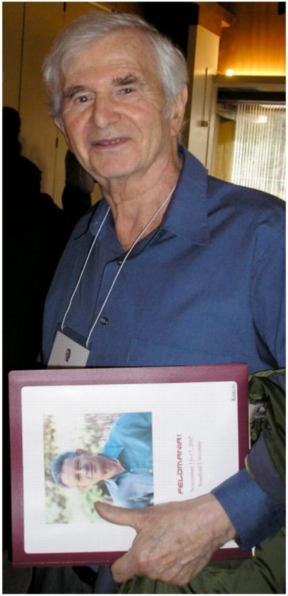

Figure: Karlin

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He died Dec. 18, 2007 at Stanford Hospital after a massive heart

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$\mathbf{z}_j = C\mathbf{z}_{j-1}, j = 1, \ldots$, i.e. $\mathbf{z}_j = C^j \mathbf{z}_0, j = 1, \ldots,$

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No species extinct if $\rho(DA) > 1$.

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$\mathbf{d} = (d_1, \ldots, d_n) > \mathbf{0}$, $D = D(\mathbf{d}) := \mathrm{diag}(d_1, \ldots, d_n)$
$\rho(D(\mathbf{d})A) \ge \rho(A) \prod_{i=1}^n d_i^{x_i(A)y_i(A)}$
If A has positive diagonal then equality holds iff $D(\mathbf{d}) = aI_n$.

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COR: $\min_{\mathbf{z}>\mathbf{0}} \sum_{i=1}^n x_i(A)y_i(A)\frac{(A\mathbf{z})_i}{z_i} = \rho(A)$
(weighted arithmetic-geometric inequality)

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$\log \rho(DA) = \sum_{i=1}^{n} \mathbf{x}_i(A)\mathbf{y}_i(A)(\log d_i + \frac{(A\mathbf{x}(DA))_i}{x_i(DA)}) \geq$
$\log \rho(A) + \sum_{i=1}^{n} \mathbf{x}_i(A)\mathbf{y}_i(A) \log d_i$

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$\max_{\boldsymbol{\mu}} \min_{\mathbf{u}\in\mathbb{R}^n} \sum_{i=1}^n \frac{(Ae^{\mathbf{u}})_i}{e^{u_i}}\mu_i = \min_{\mathbf{u}\in\mathbb{R}^n} \max_{\boldsymbol{\mu}} \sum_{i=1}^n \frac{(Ae^{\mathbf{u}})_i}{e^{u_i}}\mu_i$

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PROOF: $\mathbf{w} = (w_1, \ldots, w_n) := \mathbf{u} \circ \mathbf{v}$. Then $f_{\mathbf{w}}(\mathbf{z}) := \sum_{i=1}^n w_i \frac{\log(A\mathbf{z})_i}{z_i}$ on Π_n has unique critical point in interior of Π_n

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Example 2: $A = \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}$ always rescalable to doubly stochastic with many more solutions than in THM 3.

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Contrary to claim in FK75 I do not know how to prove THM3 for FI matrices
Reason: why $f(\mathbf{z}) := \sum_{i=1}^n w_i \log \frac{(A\mathbf{z})_i}{z_i}$ blows to ∞ on $\partial\Pi_n$, or attains minimum in the interior of Π_n?

Irreducible matrices with zero diagonal entries - FT08

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Proof:
$\sum_{i=1}^n w_i \log \frac{d_i y_i}{(AD(\mathbf{d})\mathbf{y})_i} = \sum_{i=1}^n w_i \log \frac{y_i}{(D(\mathbf{c})AD(\mathbf{d})\mathbf{y})_i} + \sum_{i=1}^n w_i \log(c_i d_i)$

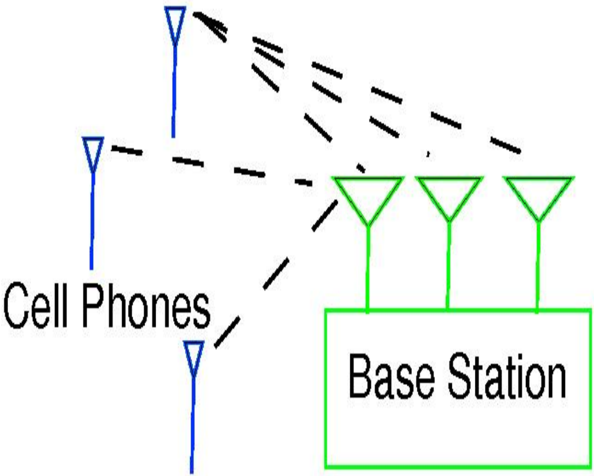

Figure: Cell phones communication

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Maximizing sum rates in Gaussian interference-limited channel

$$\max_{\mathbf{0} \leq \mathbf{p} \leq \bar{\mathbf{p}}} \sum_{i=1}^n w_i \log(1 + \gamma_i(\mathbf{p})) = \max_{\mathbf{0} \leq \mathbf{p} \leq \bar{\mathbf{p}}} \Phi_{\mathbf{w}}(\gamma(\mathbf{p})) = \Phi_{\mathbf{w}}(p^\star)$$

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$\Phi_{\mathbf{w},rel}(\gamma) := \sum_{i=1}^n w_i \log \gamma_i, \ \gamma > \mathbf{0}$
obtained by replacing $\log(1+t)$ with smaller $\log t$

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For $\mathbf{z} = (z_1, \ldots, z_n)^\top > \mathbf{0}$ let $\mathbf{z}^{-1} := (z_1^{-1}, \ldots, z_n^{-1})^\top$
$\gamma(\mathbf{p}) = \mathbf{p} \circ (F\mathbf{p} + \boldsymbol{\mu})^{-1}$, $\boldsymbol{\mu} = (\frac{\nu_1}{g_{11}}, \ldots, \frac{\nu_n}{g_{nn}})^\top$
$F = [f_{ij}] \in \mathbb{R}_+^{n\times n}$ has zero diagonal and $f_{ij} = \frac{g_{ij}}{g_{ii}}$ for $i \neq j$

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If $\sum_{j\neq i} w_j > w_i > 0$ for $i = 1, \ldots, n$
relaxed maximal problem can be solved by THM 4.

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The inverse map $P : \Gamma \to \mathbb{R}^n_+$ given

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maximization of convex function on closed unbounded convex set

Approximation methods-I

Approximation 1:
For $K \gg 1$ $\mathcal{D}_K := \{\mathbf{x} \in \mathcal{D},\ \mathbf{x} \geq -K\mathbf{1} = -K(1,\ldots,1)^\top\}$
consider $\max_{\mathbf{x}\in\mathcal{D}_K} \Phi_{\mathbf{w}}$

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Choose a few boundary points $\xi_1,\ldots,\xi_N \in \mathcal{D}$ s.t.
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$H_{j,k}(\mathbf{x}) = \mathbf{w}_{j,k}^\top\mathbf{x},\ \mathbf{w}_{j,k} = \mathbf{x}(D(e^{\xi_k})(F + \frac{1}{\bar{p}_j}\mu\mathbf{e}_j^\top)) \circ \mathbf{y}(D(e^{\xi_k})(F + \frac{1}{\bar{p}_j}\mu\mathbf{e}_j^\top))$

$\mathcal{D}(\xi_1,\ldots,\xi_N,K) = \{\mathbf{x} \in \mathbb{R}^n, H_{j,k}(\mathbf{x}) \leq H_{j,k}(\xi_k), j \in \mathcal{A}_k, k \in \langle N\rangle, \xi \geq -K\mathbf{1}\}$

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$\max_{\mathbf{x}\in\mathcal{D}(\xi_1,\ldots,\xi_N,K)} \Phi_{\mathbf{w}}(e^{\mathbf{x}}) \geq \max_{\mathbf{x}\in\mathcal{D}_K} \Phi_{\mathbf{w}}(e^{\mathbf{x}})$

Approximation methods-II

Approximation 3:

$$\max_{\mathbf{x}\in\mathcal{D}(\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_N,K)} \Phi_{\mathbf{w},rel}(e^{\mathbf{x}}) = \max_{\mathbf{x}\in\mathcal{D}(\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_N,K)} \mathbf{w}^\top \mathbf{x}$$

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E.g., divide $[\mathbf{0}, \mathbf{p}]$ by a mesh, and choose all boundary points with positive coordinates

$\xi_k = \gamma(\mathbf{p}_k)$ and $\mathcal{A}_k$ all j s.t. $p_{j,k} = \bar{p}_j$

Direct methods

Study $\max_{\mathbf{0}\le\mathbf{p}\le\bar{\mathbf{p}}} \Phi_{\mathbf{w}}(\mathbf{p}) = \Phi_{\mathbf{w}}(\mathbf{p}^{\star})$

If $w_i = 0$ then $p_i^{\star} = 0$.

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Apply gradient methods and their variations

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