# Combinatorial Optimization
# Summary of Lectures

MCS 521, Fall 2017
LCD-grad 38317,
Tu&Th 4–5:15, Lincoln Hall 201
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## 1 Week 1: 8/28-9/1, 2017

### 1.1 August 29, 2017

Discussed very briefly an example from combinatorial optmiization: Maximal flow problem on directed graph. The full problem will be discussed later when we cover Chapter 3 in [1].

Started to discuss *polyhedron in* $\mathbb{R}^n$ given by the set of inequalities $A\mathbf{x} \le \mathbf{b}$. Here $A$ is an $m \times n$ matrix, $\mathbf{x} = (x_1, \ldots, x_n)^\top \in \mathbb{R}^n$ and $\mathbf{b} = (b_1, \ldots, b_m)^\top \in \mathbb{R}^m$:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(n-1)} & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2(n-1)} & a_{2n} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{m(n-1)} & a_{mn} \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

The equation $a_1x_1 + \cdots + a_nx_n = b$ is a hyperplane in $\mathbb{R}^n$. For $n = 2$ it is a line, for $n = 3$ it is a plane. The inequality $a_1x_1 + \cdots + a_nx_n \le b$ is called a half-space. For $n = 2$ is a half (left ot right plane). So polyhedron $A\mathbf{x} \le \mathbf{b}$ is a convex region determined by a number of half-planes. Stated Theorem A.1 (page 326) in [1]. Started to prove it by induction. Proved for $n = 1$. Showed that for induction step one needs to consider only the where at least one variable appears with different signs in $m$ inequlities. By relabeling the variables and inequalities we arrive to the case : the Fourier-Motzkin elimination [1, page 326]. Pointed out that computationally the Fourier-Motzkin elimination method, unlike Gauss elimination os not efficient.

### 1.2 August 31, 2017

Finished the proof of Theorem A.1, Farkas's Lemma for Inequaities. Proved Corollary A.2, Corollary A.3, Weak duality: Theorem A.4, Duality Theorem A.5.

# 2 Week 2: 9/5 - 9/7, 2017

## 2.1 September 5, 2017

Stated Corollary A.6, page 330. (Its proof is one of the excercises I assigned.) Stated and proved th complementary slackeness. (Page 330 - top of the page 331.)

Started Section 2.1. Did pages 9 - 12 up to "Validiy of MST algorithm".

## 2.2 September 7, 2017

Did "Validity of MST algorithms", pages 12 - 14. Disucssed "Efficiency of Minimum Spanning Tree Algorithms", pages 14-15.

# 3 Week 3: 9/12 - 9/14, 2017

## 3.1 September 12, 2017

Finished subsection: "Minimum Spanning Trees and Linear Programming", pages 15-17 in [1].

## 3.2 September 14, 2017

Started §2.2. Did: Proposition 2.9; showed the existence of a feasible potential implies that $G$ does not have negative cost cycles; described Ford's algorithm and example in Table 2.1; discussed briefly example of Table 2.2; explained the Ford-Bellman algorithm; stated Proposition 2.16.

# 4 Week 4: 9/19 - 9/21, 2017

## 4.1 September 19, 2017

Proved Proposition 2.10, Theorem 2.11, Theorem 2.12, Proposition 2.17. Defined acyclic graph. Explained how to find topological sort in acyclic graph.

## 4.2 September 21, 2017

Finished §2.2. Started §3.2. Discussed Figure 3.1.

# 5 Week 5: 9/26 - 9/28, 2017

## 5.1 September 26, 2017

Stated formally the definition of a feasible flow: Equations (3.1)-(3.3) on page 38. Proved Proposition 3.1, Proposition 3.2, Proposition 3.3, Corollary 3.4. Stated Theorem 3.5. Defined x-incimenting and x-augmetning path, (page 41). Proved Corollary 3.8. Defined the graph $G(x)$ induced by the flow $x$, page 42.

## 5.2 September 28, 2017

Finished Section 3.2.

# 6 Week 6: 10/3–10/5, 2017

## 6.1 October 3, 2017

Started §3.3. Did: Bipartite matchings and covers, and optimal closure in digraphs: pages 47–50 in [1].

## 6.2 October 5, 2017

Did "Elimination of Sports Teams", page 50 - 53. Started "Flow Feasibitlity Problems". Did page 53 and started page 54. Stated the necessary and sufficient condition $a(N(C)) \geq b(C)$ for all $C \subseteq Q$. Will prove this condition next time.

# 7 Week 7: 10/10–10/12, 2017

## 7.1 October 10, 2017

Proved the necessary and sufficient condition $a(N(C)) \geq b(C)$ for all $C \subseteq Q$ on page 54. Proved Theorem 3.15, Corollary 3.16 and Theorem 3.17.

## 7.2 October 12, 2017

Stated Theorem 3.18 and outlined its proof; proved Theorem 3.19; discussed the form of the dual problem (3.10); proved Theorem 3.20; and gave a proof without using the minimal cut theorem.

# 8 Week 8: 10/17–10/19, 2017

## 8.1 October 17, 2017

Started §4.1. Covered pages 91–94 up to Theorem 4.2.

## 8.2 October 19, 2017

Proved Theorems 4.3, 4.4, 4.5. (We are skipping Applications to Rectilinear Graph Drawing.)
Started §3.5. Did pages 71–74. Stated "Node Identification Minimum Cut Algorithm".

# 9 Week 9: 10/24–10/26, 2017

## 9.1 October 24, 2017

Stated and proved Lemma 3.37 and Theorem 3.36. Discussed Random Contraction Algorithm on page 76. Stated and proved Theorem 3.38.

## 9.2 October 26, 2017

Finished §3.5.1 and §3.5.2.

# 10 Week 10: 10/31–11/2, 2017

**Lectures by M. Aliabadi**
**See: http://homepages.math.uic.edu/∼friedlan/Aliabadi25-1Oct.pdf**.
Matchings: maximum and maximal matchings, perfect matchings, M-alternating path, M-augmenting path. Symmetric difference of two graphs on the same set of vertices, symmetric difference of two matchings. Stated [1, Theorem 5.1]. Discussed Hall's matching condition [1, Problem 3.21]. Corollary 11 [1, Problem 3.23]. König's Theorem [1, Theorem 3.14]. Stochastic and doubly stochastic matrices. Birkhoff's Theorem [1, Theorem 5.12, Theorem 6.12].

# 11 Week 11: 11/7–11/9, 2017

## 11.1 November 7, 2017

§5.1- Did pages 127 – 133, up to Lemma 5.5.

## 11.2 November 9, 2017

Finished §5.1. Started §5.2. Did pages 134–136 until "The Biparttie Case".

# 12 Week 12: 11/14–11/16, 2017

Finished §5.2. §5.3: Did pages 144-150, pages 161-163.

# 13 Week 13: 11/21, 2017

Did pages 166-173 in §5.4.

# 14 Week 14: 11/28-11/30, 2017

## 14.1 November 28

Did pages 1 - 5 in my notes "A Crash Course on Semidefintie Ptogramming"
http://homepages.math.uic.edu/∼friedlan/SDPNov17.pdf
(Finished the proof of Proposition 2.3)

## 14.2 November 30

Did Proposition 2.4, Corollary 2.5, §2.3, Stated Theorem 2.9, Lemma 2.10, Stated Theorem 2.8. Started thr proof of Theorem 2.8. upto the first paragraph on page 8.

# 15 Week 15: 12/05-12/07, 2017

## 15.1 December 5

From my notes "A Crash Course on Semidefintie Ptogramming" http://homepages.math.uic.edu/∼friedlan/SDPNov17.pdf

Finished §2. From §3.1: Proved Lemma 3.1. (Did page until Lemma 3.2.) Did page 12 until Theorem 3.3. Explained why existence of positive definite feasible solutions of the prime and dual SDP yields upper and lower bounds for $\alpha = \beta$. This is explained in (3.17).

## References


[1] W.J. Cook, W.H. Cunningham, W.R. Pulleyblank, A.Schrijver, *Combinatorial Optimization*, Wiley, 1998.