Combinatorial Optimization
Summary of Lectures

MCS 521, Spring 2012
LCD-grad 33023,
MWF 11–11:50, Thaft Hall 219
Instructor: Shmuel Friedland
Office: 1215 SEO, phone: 413-2159, *e:mail*: friedlan@uic.edu,
web: http://www.math.uic.edu/∼friedlan

Last update April 24, 2012

1 Week 1: 1/9-1/13, 2012

1.1 January 9, 2012

[1]: Described the assignment problem for n people to n jobs. $a_{ij} \in [0,1]$ the capability of person i to do person j. Let $\sigma : [n] := \{1, 2, \dots, \} \to [n]$ be a permutation on $[n]$. Denote by S_n the set of permutations on $[n]$. Then the optimal assignment problem is $\max\{\sum_{i\in[n]} a_{i\sigma(i)}, \sigma \in S_n\}$.

Let $\Omega_n \subset \mathbb{R}_+^{n\times n}$ be the set of doubly stochastic matrices, i.e. for each $B \in \Omega_n$ B and $B^\top$ are stochastic matrices. So $B \in \Omega_n$ iff all entries of B are nonnegative and each row and column sum is 1. Then the above maximum is equal to the linear $\max\{\mathrm{tr}(AB), B \in \Omega_n\}$.

Did p'326 in [1].

1.2 January 11, 2012

Finished the proof of Theorem A.1 pages 326-327 in [1].

1.3 January 13, 2012

Pages 327–328 in [1].

2 Week 2: 1/18-1/20, 2012

2.1 January 18, 2012

Pages 329–331 in [1].

2.2 January 20, 2012

Pages 331–333 in [1]. Started §2.1 in [1], pages 9-11.

3 Week 3: 1/23-1/27, 2012

3.1 January 23, 2012

Pages 11-15, (up to sorting the edges by cost), in [1].

3.2 January 25, 2012

Proved that the complexity of Kruskal algorithm is $O(m \log m)$. Started section on MST and Linear Programming pages 15-17.

3.3 January 27, 2012

Students solved HW problems in class. Finished MST and Linear Programming pages 15-17.

4 Week 4: 1/30-2/3, 2012

4.1 January 30 and February 1, 2012

§2.2 pages 19-27.

4.2 February 3, 2012

Discussed Ford-Bellman algorithm, scan notion, pages 28-29. Discussed acyclic digraph. Showed that for any acyclic digraph $G = (V, E)$ one can rename the vertices as $v_1, \ldots, v_n$ such that if $v_i v_j \in E$ then $i < j$. Showed that for acyclic graphs the shortest path algorithm needs one path. Pages 30 to 31 (top).

Discussed strongly regular graph and reduced graph, whose vertices are the strongly connected components of G. See for more details Section 6.1 in my book (in preparation), pages 281-288.

Started to discuss Dijkstra algorithm on page 31.

5 Week 5: 2/6-2/10, 2012

5.1 February 6, 2012

Finished Dijkstra algorithm, pages 31–33. Started Chapter 3: Maximum flow problems. Pages 37–38.

5.2 February 8, 2012

Students solved HW problems. Continued Maximum flow problems. Pages 39–40.

5.3 February 10, 2012

Covered pages 40–44 up to and including statement of Theorem 3.10.

6 Week 6: 2/13-2/17, 2012

6.1 February 13, 2012

Finished §3.2, pages 44–45. Started matching in bipartite graphs, §3.3, page 47.

6.2 February 15, 2012

Did "Biparttie Matchings and Covers in §3.3, pages 47-49. Discussed the notion of #P-complete. Example: counting the number of perfect matches in a bipartite graph with perfect match. Introduced the notion of the permanent of A. Showed that any regular bipartite graph has a perfect matching. For more details see slides of my lectures:

http://homepages.math.uic.edu/∼friedlan/AIM8Dec11verB.pdf
http://homepages.math.uic.edu/∼friedlan/uwatercolNov09.pdf

6.3 February 17, 2012

Students solved HW problems. Discussed Hall's marriage theorem. Stated this theorem in terms of permanents of nonnegative matrices.

7 Week 7: 2/20-2/24, 2012

7.1 February 20, 2012

Stated the precise version of Hall's marriage theorem: Let $G = (V_1 \cup V_2, E)$ be a balanced bipartite graph, i.e. $|V_1| = |V_2|$. Then G has a perfect matching if and only if for each subset $P \subseteq V_1$ the inequality $|P| \leq |N(P)|$ holds. (Here $N(P)$ are all the neighbors of P.)

Proved that that the bipartite graph induced by a doubly stochastic matrix has a perfect matching. Deduced from this fact Birkhoff's theorem that any doubly stochastic matrix is a convex combination of permutation matrices. Stated the van der Waerden permanental conjecture: the permanent of any $n \times n$ doubly stochastic matrix is not less then the permanent of the doubly stochastic matrix whose all entries are equal to $\frac{1}{n}$. So the permanent of any $n \times n$ doubly stochastic matrix is at least $\frac{n!}{n^n} > e^{-n}$. Showed that it implies that any d regular bipartite graphs on $2n$ vertices has at least $(\frac{d}{e})^n$ perfect matches, where $e = 2.71....$

7.2 February 22, 2012

Discussed Optimal Closure in a Digraph, pages 49–50. Started to discuss Elimination of Sports Teams. Formulated the conditions (3.5). Stated the problem of existence of a specific maximum flow pages 51–52.

7.3 February 24, 2012

Finished the subsection of "Elimination of Sports Teams". Started subsection "Flow Feasibility Problems". Covered pages 53–54.

8 Week 8: 2/29-3/02, 2012

8.1 February 27, 2012

Covered pages 55–56.

8.2 February 29, 2012

Covered pages 57–58.

8.3 March 2, 2012

Started to discuss §5.1. Did pages 127–131 up to and including the formulation of Theorem 5.3.

9 Week 9: 3/05-3/09, 2012

9.1 March 5, 2012

Pages 131–133. Started the proof Tutte-Berge Formula on p' 133.

9.2 March 7, 2012

Finished the proof Tutte-Berge Formula. Started to discuss §5.2, pages 135–136.

9.3 March 9, 2012

Discusses pages 137–139.

10 Week 10: 3/12-3/16, 2012

10.1 March 12, 2012

Finished §5.2. Started §5.3.

10.2 March 14, 2012

Students solved HW problems.

Stated the minimum-weight perfect matching problem as a linear programming problem. Started to explain the Hungarian algorithm using the dual and original problems.

10.3 March 16

Discussed subsection on pages 147–155.

11 Week 11: 3/26-3/30, 2012

Finished §5.2, §5.3. Covered §5.4 up to and including page 177.

12 Week 12: 4/2–4/6, 2012

12.1 April 2 – 4

Finished Chapter 5. Started §6.1 pp. 199–200. Stated Caratheodory's theorem, but did not prove it.

12.2 April 6

Students solved HW problems. Finished the proof Caratheodory's theorem. Discussed separation theorem for compact convex sets. Started to prove the separation theorem for polytopes, Proposition 6.2, pages 200—201.

13 Week 13: 4/09–4/13, 2012

13.1 April 9-11

Finished §6.1. Started §6.2. Covered pages 203–205.

13.2 April 13

Proved in details Proposition 6.7, 6.8, 6.9, Theorems 6.10 and 6.11.

14 Week 14: 4/16–4/20, 2012

14.1 April 16

Proved Theorems 6.12 and 6.13. Started the proof of "2nd proof of the perfect matching polytope Theorem".

14.2 April 18

Finished the proof of "2nd proof of the perfect matching polytope Theorem". Did page 211-212 up to and included Proposition 6.15.

14.3 April 20

Students solved HW problems. Did pages 212 – 214 up to Theorem 6.20.

15 Week 15: 4/23–4/27, 2012

15.1 April 23

Did pages 218–221, Theorem 6.27.

References

[1] W.J. Cook, W.H. Cunningham, W.R. Pulleyblank, A.Schrijver, *Combinatorial Optimization*, Wiley, 1998.