3-Tensors: ranks and approximations

Shmuel Friedland
University of Illinois at Chicago

Workshop on Algorithms for Modern Massive Data Sets,
Stanford-Yahoo, June 21-24, 2006

Last version June 27, 2006

1 Outline of the talk

- 3-Tensors
- Inequalities for the rank of 3-tensor
- Rank 3-tensor characterization
- Generic rank of 3-tensor
- An example
- Algebraic geometry & tensor rank
- Maximal tensor rank
- Max.& gen. rank upper estimates
- Theoretical bounds & explanation
- Numerical results
- Results and conjectures
- k-rank approximation of 3-tensors
- Algorithms using SVD

2 3-Tensors

$\mathbb{F} = \mathbb{R}, \mathbb{C}$.

3-Tensor Space $\mathbb{F}^{m_1 \times m_2 \times m_3} := \mathbb{F}^{m_1} \times \mathbb{F}^{m_2} \times \mathbb{F}^{m_3}$

Tensor $\mathcal{T} = [t_{ijk}]_{i=j=k=1}^{m_1, m_2, m_3}$ or simply $\mathcal{T} = [t_{ijk}]$.

Abstractly $\mathbf{U} := \mathbf{U}_1 \otimes \mathbf{U}_2 \otimes \mathbf{U}_3$

$\dim \mathbf{U}_i = m_i, i = 1, 2, 3$, $\dim \mathbf{U} = m_1 m_2 m_3$

Tensor $\tau \in \mathbf{U}_1 \otimes \mathbf{U}_2 \otimes \mathbf{U}_3$

Rank one tensor $t_{ijk} = x_i y_j z_k$,

$(i, j, k) = (1, 1, 1), \ldots, (m_1, m_2, m_3)$

or decomposable tensor $\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z}$

$[\mathbf{u}_{1,j}, \ldots, \mathbf{u}_{m_j,j}]$ basis of $\mathbf{U}_j$ $j = 1, 2, 3$

$\mathbf{u}_{i_1,1} \otimes \mathbf{u}_{i_2,2} \otimes \mathbf{u}_{i_3,3}, i_j = 1, \ldots, m_j, j = 1, 2, 3,$

basis of $\mathbf{U}$

$\tau = \sum_{i_1=i_2=i_3=1}^{m_1, m_2, m_3} t_{i_1 i_2 i_2} \mathbf{u}_{i_1,1} \otimes \mathbf{u}_{i_2,2} \otimes \mathbf{u}_{i_3,3}$

Rank τ denoted $\mathbf{rank}\,\tau$ is the minimal k:

$\tau = \sum_{i=1}^{k} \mathbf{x}_i \otimes \mathbf{y}_i \otimes \mathbf{z}_i$ (CANDEC, PARFAC)

3 Inequalities for the rank of 3-tensor

For $\mathcal{T} = \mathcal{T} = [t_{ijk}]$ let

$T_{k,3} := [t_{ijk}]_{i,j=1}^{m_1,m_2} \in \mathbb{F}^{m_1 \times m_2}, k = 1, \ldots, m_3$

$\mathcal{T} = \sum_{k=1}^{m_3} T_k \mathbf{e}_{k,3}$ (convenient notation)

$R_3 := \dim \mathrm{span}(T_{1,3}, \ldots, T_{m_3,3})$.

Claim $\mathrm{rank}\, \mathcal{T} \geq R_3$

Reason $\mathbf{U}_1 \otimes \mathbf{U}_2 \sim \mathbb{F}^{m_1 \times m_2} \equiv \mathbb{F}^{m_1 m_2}$

View $\mathbf{U}_1 \otimes \mathbf{U}_2 \otimes \mathbf{U}_3$ as
$((\mathbf{U}_1 \otimes \mathbf{U}_2) \otimes \mathbf{U}_3) \sim \mathbb{F}^{m_1 m_2 \times m_3}$

So $\mathcal{T}$ is viewed as $A \in \mathbb{F}^{m_1 m_2 \times m_3}$, $R_3 = \mathrm{rank}\, A$

$\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z}$ viewed as $(\mathbf{x} \otimes \mathbf{y}) \otimes \mathbf{z} \in \mathbb{F}^{m_1 m_2 \times m_3}$,

$\mathrm{rank}(\mathbf{x} \otimes \mathbf{y}) \otimes \mathbf{z} = 1$ if $\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z} \neq 0$

So any CANDEC of $\mathcal{T}$ induces a decomposition of A as a sum of rank one matrices

Similarly one can define R_1, R_2

$\mathrm{rank}\, \mathcal{T} \geq \max(R_1, R_2, R_3)$ (WELL KNOWN)

Note: R_1, R_2, R_3 are easily computable.

4 Rank 3-tensor characterization

OBS: $\exists \mathbf{U}_i \subset \mathbb{F}^{m_i}, \dim \mathbf{U}_i = R_i, i = 1, 2, 3$ s.t. $\mathcal{T} \in \mathbf{U}_1 \otimes \mathbf{U}_2 \otimes \mathbf{U}_3$.

PRP : For $\mathcal{T} = \mathcal{T} = [t_{ijk}]$ let $T_{k,3} := [t_{ijk}]_{i,j=1}^{m_1,m_2} \in \mathbb{F}^{m_1 \times m_2}, k = 1, \ldots, m_3$. Then $\operatorname{rank} \mathcal{T}$ dimension of subspace $L \subset \mathbb{F}^{m_1 \times m_2}$ spanned by rank one matrices containing $T_{1,3}, \ldots, T_{m_3,3}$.

PRF: Suppose $\mathcal{T} = \sum_{i=1}^{p} \mathbf{x}_i \otimes \mathbf{y}_i \otimes \mathbf{z}_i$ (1). Write $\mathbf{z}_i = \sum_{j=1}^{m_3} z_{ij} \mathbf{e}_{j,3}$ then each $T_{k,3} \in \operatorname{span}(\mathbf{x}_1 \otimes \mathbf{y}_1, \ldots, \mathbf{x}_p \otimes \mathbf{y}_p)$.

Vise versa suppose $T_{k,3} = \sum_{i=1}^{p} a_{ki} \mathbf{x}_i \otimes \mathbf{y}_i, k = 1, \ldots, m_3$.

Then (1) holds with $\mathbf{z}_i := \sum_{k=1}^{m_3} a_{ki} \mathbf{e}_{k,3}$.

5 Generic rank of 3-tensor

Basic results of algebraic geometry imply:

THM 1: A randomly chosen tensor $\mathcal{T} \in \mathbb{C}^{m_1 \times m_2 \times m_3}$ with probability one has a fixed rank denoted by $\mathrm{grank}(m_1, m_2, m_3)$, called the generic rank.

That is, there exists an algebraic variety
$\mathbf{X} \subsetneq \mathbb{C}^{m_1 \times m_2 \times m_3}$ such that for any
$\mathcal{T} \in \mathbb{C}^{m_1 \times m_2 \times m_3} \backslash \mathbf{X}$
$\mathrm{rank}\, \mathcal{T} = \mathrm{grank}(m_1, m_2, m_3)$.

RMK: Usually there exist a subvariety $\mathbf{Y} \subsetneq \mathbf{X}$ such that for any $\mathcal{T} \in \mathbf{Y}$ $\mathrm{rank}\, \mathcal{T} > \mathrm{grank}(m_1, m_2, m_3)$.

RMK: $\mathrm{grank}(m_1, m_2, m_3)$ is easily computable (See later)

RMK: For $\mathcal{T} \in \mathbb{R}^{m_1 \times m_2 \times m_3}$ usually there will be open $\mathbf{Z}$, (semi-algebraic set), where
$\mathrm{rank}\, \mathcal{T} > \mathrm{grank}(m_1, m_2, m_3)$ for $\mathcal{T} \in \mathbf{Z}$.

6 An example

Claim: $\mathrm{grank}(m, m, 2) = m$ for any $m \geq 2$.

Proof: $\tau = \mathcal{T} = [t_{ijk}]_{i,j,k}^{m,m,2}$, in standard bases of $\mathbf{U}_1 = \mathbf{U}_2 = \mathbb{C}^m$, $\mathbf{U}_3 = \mathbb{C}^2$ is represented by $A := [t_{ij1}], B := [t_{ij2}] \in \mathbb{C}^{m \times m}$.

If we change the bases of $\mathbf{U}_1 = \mathbb{C}^m$, $\mathbf{U}_2 = \mathbb{C}^m$ using matrices P, Q then τ represented by

$A' = PAQ^\top, B' = PBQ^\top$

For randomly chosen $\mathcal{T}$, A is invertible and $A^{-1}B$ is diagonable over $\mathbb{C}$, (that defines $\mathbf{X}$). So

$A^{-1}B = \sum_{i=1}^m \lambda_i \mathbf{u}_{i,2} \otimes \mathbf{u}_{i,2}$,

$I_m = \sum_{i=1}^m \mathbf{u}_{i,2} \otimes \mathbf{u}_{i,2}$.

Choose a new basis in $[\mathbf{u}_{1,1}, \ldots, \mathbf{u}_{m,1}]$ in $\mathbf{U}_1 = \mathbb{C}^m$ given by A^{-1} and leave other bases as is. Then in new bases $\mathcal{T}$ represented by

$\mathcal{T}' = I_m \mathbf{e}_{1,3} + A^{-1}B\mathbf{e}_{2,3} =$
$\sum_{i=1}^m \mathbf{u}_{i,2} \otimes \mathbf{u}_{i,2} \otimes \mathbf{e}_{1,3} + \lambda_i \mathbf{u}_{i,2} \otimes \mathbf{u}_{i,2} \otimes \mathbf{e}_{2,3} =$
$\sum_{i=1}^m \mathbf{u}_{i,2} \otimes \mathbf{u}_{i,3} \otimes (\mathbf{e}_{1,3} + \lambda_i \mathbf{e}_{2,3})$.

So $\mathrm{rank}\, \mathcal{T} \leq m$. Easy $R_1 = R_2 = m$ for $\mathcal{T}'$.

Hence $\mathrm{rank}\, \tau = m$.

If B is not diagonable then $\mathbf{rank}\,\tau > m$ (over $\mathbb{C}$).

The variety of all $B \in \mathbb{C}^{m\times m}$ which are not diagonable is essentially the variety of all complex matrices with one eigenvalue of multiplicity 2. Hence its codimesnion is 1.

The case $\mathbb{R}^{2\times 2\times 2}$

$0 \neq \tau = \mathcal{T} = [t_{ijk}] \in \mathbb{R}^{2\times 2\times 2}$
$\mathcal{T} = A\mathbf{e}_1 + B\mathbf{e}_2, A, B \in \mathbb{R}^{2\times 2}$.

Suppose A invertible

If $A^{-1}B$ has two distinct real eigenvalues, or $A^{-1}B = aI_2$ then $\mathbf{rank}\,\tau = 2$.

If $A^{-1}B$ has two distinct complex eigenvalues or it is not diagonable $\mathbf{rank}\,\tau = 3$.

If the subspace spanned by A, B does not contain an invertible matrix then $\mathbf{rank}\,\tau = 1, 2$.

(This can happen if either $\dim \operatorname{span}(A\mathbb{R}^2, B\mathbb{R}^2) = 1$ or $\dim \operatorname{span}(A^\top\mathbb{R}^2, B^\top\mathbb{R}^2) = 1$.)

For example $\tau = \mathbf{u}\otimes\mathbf{v}\otimes\mathbf{e}_1 + \mathbf{u}\otimes\mathbf{w}\otimes\mathbf{e}_2, \mathbf{u} \neq 0$

If $\mathbf{v}, \mathbf{w}$ linearly independent $\mathbf{rank}\,\tau = 2$

7 Algebraic geometry & tensor rank

View tensor one rank matrices as the map

$f : \mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3} \longrightarrow \mathbb{C}^{m_1 \times m_2 \times m_3}$

$f(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z}$ note

$(a\mathbf{x}, b\mathbf{y}, c\mathbf{z}) \mapsto (abc)\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z}$, 2-parameters lost

$f_k : (\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^k \longrightarrow \mathbb{C}^{m_1 \times m_2 \times m_3}$

$f_k(\mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1, \ldots, \mathbf{x}_k, \mathbf{y}_k, \mathbf{z}_k) :=$
$\sum_{i=1}^{k} f(\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i)$.

$f_k((\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^k)$ the irreducible quasi-variety of all 3-tensors of rank k at most. I.e. there exists an irreducible variety $\mathbf{X}_k$ and a strict subvariety $\mathbf{Z}_k \subsetneqq \mathbf{X}_k$ s.t.
$f_k((\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^k) = \mathbf{X}_k \backslash \mathbf{Z}_k$

$\dim_{\mathbb{C}} X_k$=the maximal rank of the Jacobian matrix of $J(f_k)(\mathbf{x}_1, \ldots, \mathbf{z}_k)$, which is equal to $\dim_{\mathbb{C}} X_k$ for any random choice of $(\mathbf{x}_1, \ldots, \mathbf{z}_k)$.

THM 2: $\mathrm{rank}\, J(f_k) = \dim \mathrm{span}\{\mathbf{e}_{i_1,1} \otimes \mathbf{x}_{l,2} \otimes \mathbf{x}_{l,3}, \mathbf{x}_{l,1} \otimes \mathbf{e}_{i_2,2} \otimes \mathbf{x}_{l,3}, \mathbf{x}_{l,1} \otimes \mathbf{x}_{l,2} \otimes \mathbf{e}_{i_3,3}, i_j = 1, \ldots, m_j, j = 1, 2, 3, l = 1, \ldots, k\}$.

COR : $r(k, m_1, m_2, m_3) := \dim X_k$ dimension of the subspace given in THM 2, for a randomly chosen $\mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1, \ldots, \mathbf{x}_k, \mathbf{y}_k, \mathbf{z}_k$.

COR : $\mathrm{grank}(m_1, m_2, m_3)$ minimal k s.t. $\dim X_k = m_1 m_2 m_3$.

COR : For $k = 1, \ldots, \mathrm{grank}(m_1, m_2, m_3) - 1$ $\dim \mathbf{X}_k < \dim \mathbf{X}_{k+1}$. Furthermore $\dim \mathbf{X}_k = m_1 m_2 m_3$ for $k \geq \mathrm{grank}(m_1, m_2, m_3)$.

CLM:
$k := \mathrm{grank}(m_1, m_2, m_3) \geq \lceil \frac{m_1 m_2 m_3}{m_1+m_2+m_3-2} \rceil$ and (1): $\mathbf{X}_k = \mathbb{C}^{m_1 \times m_2 \times m_3} \backslash \mathbf{X}$

PRF Fact: Any quasi-variety in $\mathbb{C}^m$ of dimension m is of the form $\mathbb{C}^m \backslash \mathbf{X}$ for some subvariety $X \subsetneq \mathbb{C}^m$. Hence (1).

Each factor $\mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z}$ has $m_1 + m_2 + m_3 - 2$ parameters. If all the parameters are independent we need at least $\lceil \frac{m_1 m_2 m_3}{m_1+m_2+m_3-2} \rceil$ to obtain $m_1 m_2 m_3$ parameters of $\mathbb{C}^{m_1 \times m_2 \times m_3}$.

$\mathrm{grank}(m_1, m_2, m_3) \geq \mathrm{grank}(l_1, l_2, l_3)$ for $m_1 \geq l_1, m_2 \geq l_2, m_3 \geq l_3$

8 Maximal tensor rank

Lemma: $f_{k-1}((\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^{k-1}) \subsetneq f_k((\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^{k})$ for $k = 1, \ldots, \mathrm{mrank}(m_1, m_2, m_3)$ and $f_k((\mathbb{C}^{m_1} \times \mathbb{C}^{m_2} \times \mathbb{C}^{m_3})^{k}) = \mathbb{C}^{m_1 \times m_2 \times m_3}$ for $k \geq \mathrm{mrank}(m_1, m_2, m_3)$.

$\mathrm{mrank}(m_1, m_2, m_3)$ maximal (tensor) rank

(of $\mathcal{T} \in \mathbb{C}^{m_1 \times m_2 \times m_3}$)

$\mathrm{grank}(m_1, m_2, m_3) \leq \mathrm{mrank}(m_1, m_2, m_3)$ (usually $<$)

$\mathrm{mrank}(m_1, m_2, m_3) \geq \mathrm{mrank}(l_1, l_2, l_3)$ for $m_1 \geq l_1, m_2 \geq l_2, m_3 \geq l_3$

The computation of $\mathrm{grank}(m_1, m_2, m_3)$ difficult,

probably NP-hard

9 Max.& gen. rank upper estimates

THM 3: Any subspace $L \subset \mathbb{C}^{m\times n}$ $\dim L = (m-k)(n-k)+1$ has A s.t. $1 \le \operatorname{rank} A \le k$.

Generic L has exactly $\gamma_{k,m,n} :=$ $\prod_{j=0}^{n-k-1} \frac{\binom{m+j}{m-k}}{\binom{m-k+j}{m-k}} = \prod_{j=0}^{n-k-1} \frac{(m+j)!\, j!}{(k+j)!\,(m-k+j)!}$, which span L.

COR:for $m \ge 2$: $\operatorname{grank}(m,m,2) = m$

$m+1 \le \operatorname{mrank}(m,m,2) \le 2m-1$

for $m, n \ge 3$:

$\operatorname{grank}(n,m,m) \le m+(n-2)(m-\lfloor\sqrt{n-1}\rfloor)$ if $m \ge 2\lfloor\sqrt{n-1}\rfloor$

$\operatorname{grank}(n,m,m) \le n(m-\lfloor\sqrt{n-1}\rfloor)$ if $m < 2\lfloor\sqrt{n-1}\rfloor < 2(m-1)$,

$\operatorname{grank}(n,m,m) = n$ if $n \in [m^2-2m+2, m^2-1]$

$\operatorname{mrank}(n,m,m) \le \sum_{i=1}^{\lfloor\sqrt{n-1}\rfloor}(2i-1)(m-i+1) + (m-\lfloor\sqrt{n-1}\rfloor^2)(m-\lfloor\sqrt{n-1}\rfloor)$

10 Theoret. bounds & explanations

$4 \leq \mathrm{grank}(3,3,3) \leq 5, \mathrm{mrank}(3,3,3) \leq 7$
$\mathrm{grank}(3,3,4) = 5 \leq \mathrm{mrank}(3,3,4) \leq 9$
$\mathrm{grank}(3,3,5) = 5 \leq \mathrm{mrank}(3,3,5) \leq 13$
$6 \leq \mathrm{grank}(3,4,4) \leq 7, \mathrm{mrank}(3,4,4) \leq 10$
$7 \leq \mathrm{grank}(4,4,4) \leq 10, \mathrm{mrank}(4,4,4) \leq 13$
$8 \leq \mathrm{grank}(4,4,5) \leq 10, \mathrm{mrank}(4,4,5) \leq 15$
$7 \leq \mathrm{grank}(3,5,5) \leq 9, \mathrm{mrank}(3,5,5) \leq 13$
$9 \leq \mathrm{grank}(4,5,5) \leq 13, \mathrm{mrank}(4,5,5) \leq 16$
$10 \leq \mathrm{grank}(5,5,5) \leq 14, \mathrm{mrank}(5,5,5) \leq 20$

Expl. $\mathrm{grank}(3,3,4) \geq \frac{3\times3\times4}{3+3+4-2} = 4.5$ Consider a generic subspace spanned by $T_{1,3}, \ldots, T_{4,3} \in \mathbb{C}^{3\times3}$. Add generic T_5. $\mathrm{span}(T_{1,3}, \ldots, T_{4,3}, T_5)$ contains 6 rank 1 matrices spanning L. So $\mathrm{grank}(3,3,4) \leq 5$.

If $\mathrm{span}(T_{1,3}, \ldots, T_{4,3}, T_5)$ a invertible matrix then $\det\ \mathrm{T}_{1,3} \neq 0$ and $\det\ \mathrm{T}_{k,3} = 0, k > 1$, so $\mathrm{mrank}(3,3,4) \leq 3 + 3 \times 2 = 9$.

Fact: $\mathrm{mrank}(3,3,4) = 5$

11 Numerical results

Normalize: $2 \leq m_1 \leq m_2 \leq m_3 (\leq 10)$

$\text{grank}(m_1, m_2, m_3) = m_3$ for

$m_3 = (m_1 - 1)(m_2 - 1) + 1, \ldots, m_1 m_2$ (a)

($\text{mrank}(m_1, m_2, m_3) = m_1 m_2, m_3 \geq m_1 m_2$)

$\text{grank}(m_1, m_2, m_3) - \lceil \frac{m_1 m_2 m_3}{m_1 + m_2 + m_3 - 2} \rceil$ for

$m_3 = m_2, \ldots, (m_2 - 1)(m_3 - 1) + 1$ $(*)$

nonincreasing sequence

(m_1, m_2, m_3) exceptional if for $(*)$

$\text{grank}(m_1, m_2, m_3) - \lceil \frac{m_1 m_2 m_3}{m_1 + m_2 + m_3 - 2} \rceil > 0$

exceptional triples $(3, 2p + 1, 2p + 1)$ $p = 1. \ldots, 8$.

Excess equal 1 (See conjecture below)

For nonexceptional (m_1, m_2, m_3)

$r(k, m_1, m_2, m_3) = k(m_1 + m_2 + m_3 - 2)$ for

$k < \text{grank}(m_1, m_2, m_3)$. Hence generic rank k

tensor is represented in a finite number of ways

($N(k, m_1, m_2, m_3)$) as k-tensor. (True if

$\text{grank}(m_1, m_2, m_3) = \frac{m_1 m_2 m_3}{m_1 + m_2 + m_3 - 2}$)

Probably not in a unique way!

12 Results and Conjectures

THM [5]: Normalize: $2 \le m_1 \le m_2 \le m_3$ then $\mathrm{grank}(m_1, m_2, m_3) = m_3$ for $m_3 = (m_1 - 1)(m_2 - 1) + 1, \ldots, m_1 m_2$ (a)

CONJ:

- $\mathrm{grank}(3, 2p, 2p) = 3p \quad (= \lceil \frac{12p^2}{4p+1} \rceil)$ for $p = 1, \ldots,$ (nonexceptional triples)
- $\mathrm{grank}(3, 2p+1, 2p+1) = 3p + 2$ $(= \lceil \frac{3(2p+1)^2}{4p+3} \rceil + 1)$ for $p = 1, \ldots$ (exceptional triple)

(The conjecture was verified numerically for $p \le 8$)

Bold Conjecture: The only exceptional triples are $(3, 2p+1, 2p+1)$ for $p \in \mathbb{N}$

Remark: Compare to the results in this lecture to [1] (see corrections in [2])

13 k-rank approximation of 3-tensors

Fundamental problem in applications:

Approximate well and fast $\mathcal{T} \in \mathbb{C}^{m_1 \times m_2 \times m_3}$ by rank k 3-tensor.

Probably the best rank k approximation of $\mathcal{T}$ is a hard optimization problem.

Can we use SVD (Linear Algebra) to simplify the approximation problem?

$\mathbb{C}^{m_1 \times m_2 \times m_3}$ has a standard inner product $\langle s_{ijk}, t_{ijk} \rangle := \sum_{i,j,k} s_{ijk} \bar{t}_{ijk}$ induced by the standard inner products on $\mathbb{C}^{m_p}, p = 1, 2, 3$.

Given subspaces $\mathbf{U}_i \subset \mathbb{C}^{m_i}$, $\dim \mathbf{U}_i = l_i, i = 1, 2, 3$, let $\mathbf{V} = \mathbf{U}_1 \otimes \mathbf{U}_2 \otimes \mathbf{U}_3$, orthogonal projection $P_{\mathbf{V}}(\mathcal{T})$ is given by having an orthonormal basis $\mathbf{f}_{1,p}, \ldots, \mathbf{f}_{p,m_p} \in \mathbb{C}^{m_p}$, $\mathrm{span}(\mathbf{f}_{1,p}, \ldots, \mathbf{f}_{l_p,p}) = \mathbf{U}_p$

Auxiliary Problem (AP): Find $\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3$ each of dimension k with maximal $||P_{\mathbf{V}}(\mathcal{T})||$.

14 Algorithms using SVD

Approximation to (AP):

View $\mathcal{T}$ as a matrix in $\mathbb{F}^{m_1 m_2 \times m_3}$. Then $\mathcal{T}$ has best rank approximation:

$\sum_{i=1}^{k} A_i \otimes \mathbf{w}_i, A_i \in \mathbb{F}^{m_1 \times m_2}, \mathbf{w}_i \in \mathbb{C}^{m_3}$

Then $\mathbf{U}_3 = \mathbf{span}(\mathbf{w}_1, \ldots, \mathbf{w}_k)$.

a.Similarly find $\mathbf{U}_1, \mathbf{U}_2$.

b. The first left k singular vectors of $[A_1 \ \ldots A_k], [A_1^\top \ \ldots A_k^\top]$ span $\mathbf{U}_1, \mathbf{U}_2$ respectively

Next possible step: find best k-rank k-approximation of $P_{\mathrm{V}}(\mathcal{T})$ in $\mathbf{V}$.

15 Acknowledgement

Makio Tamura programmed the software for computing $\mathrm{grank}(m_1, m_2, m_3)$ and $r(k, m_1, m_2, m_3)$.

Random vectors were chosen to be vectors with integer entries in [-99,99] using Matlab.

References

[1] M.V. Catalisano, A.V. Geramita and A. Gimigliano, Ranks of tensors, secant varieties of Segre varieties and fat points. *Linear Algebra Appl.* 355 (2002), 263–285.

[2] M.V. Catalisano, A.V. Geramita and A. Gimigliano, A. Erratum to: ”Ranks of tensors, secant varieties of Segre varieties and fat points”, [Linear Algebra Appl. 355 (2002), 263–285], *Linear Algebra Appl.* 367 (2003), 347–348.

[3] R. Coppi and S. Bolasco, Editors, *MULTIWAY DATA ANALYSIS*, Elsevier Science Publishers, (North-Holland), 1989.

[4] J. Carrol and J. Chang, Analysis of individual differences in multidimensional scaling via N-way generalization of ”Eckhart-Young” decomposition, *Psychometrika* 9 (1970), 267-283.

[5] S. Friedland, 3-Tensors: ranks and approximations, *in preparation*.

[6] S. Friedland and C. Krattenthaler, 2-adic valuations of certain ratios of products of factorials and applications, arXiv.math.NT/0508498 v2, 26 Aug. 2005.

[7] S. Friedland and A. Torokhti, Generalized rank-constrained matrix approximations, arXiv.math.OC/0603674 v1, 29 Mar. 2006.

[8] J. Harris and L.W. Tu, On symmetric and skew-symmetric determinantal varieties, *Topology* **23** (1984), 71–84.

[9] J.B. Kruskal, Three-Way Arrays: Rank and Uniqueness of Trilinear Decompositions, with Applications to Arithemtic Complexity and Statistics, *Linear Algebra Appl.* 18 (1977), 95-138.

[10] L. de Lathauwer, B. de Moor, and J. Vandewalle, Computation of the canonical decomposition by means of a simulatenous generalized Schur decomposition, *SIAM J. Matrix Anal. Appl.* 26 (2004), 295-327.

[11] L.R. Tucker, Some mathematical notes of three-mode factor analysis, *Psychometrika*, 31 (1966), 279-311.