Some Mathematics for Essay 2

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Here are some values of the ζ function and an integral that we have seen are related to Essays 1 or 2. The numerical approximations were computed by Wolfram Alpha (e.g. `http://www.wolframalpha.com/input/?i=\zeta(3/2)`). The volume of the Ultra-Deluxe Set, $\zeta\left(\frac{3}{2}\right)$, "is employed in calculating the critical temperature for a Bose-Einstein condensate in a box with periodic boundary conditions, and for spin wave physics in magnetic systems," according to Wikipedia [1]. The volume of the Deluxe Set, $\zeta(3)$, is known as Apéry's constant after the mathematician that first proved it to be irrational. See Wikipedia [1] for the above and further information, and also Sondow [2].

$$\zeta\left(\frac{3}{2}\right) = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} \approx 2.612375348685488343348567567924071630 5708$$

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.644934066848226436472415166646025189218 9499$$

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.202056903159594285399738161511449 99$$

$$V_{\text{horn}} = \pi \int_1^{\infty} \frac{1}{x^{\frac{3}{2}}} dx = -\frac{2\pi}{\sqrt{x}}\bigg|_1^{\infty} = 2\pi \approx 6.283185307179586476925286766559$$

The horn volume is for the $\frac{1}{x^{\frac{3}{4}}}$ horn of revolution [3].

Note that the approximation given for $\zeta(\frac{3}{2})$ is not a sufficient argument that the UDS volume is less than 3. One proper way to get this result is to show that

$$\sum_{n=2}^{\infty} \frac{1}{n^{\frac{3}{2}}} < \int_1^{\infty} \frac{1}{x^{\frac{3}{2}}} dx, \text{ and so } \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} < 1 + \int_1^{\infty} \frac{1}{x^{\frac{3}{2}}} dx.$$

Evaluating the integral (=2) thus gives the required inequality for the UDS volume.

References

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2. Sondow, Jonathan and Weisstein, Eric W. Riemann Zeta Function.
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