Essay 3
A Flat Function

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For this essay you will explore some properties of the function

$$flat(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

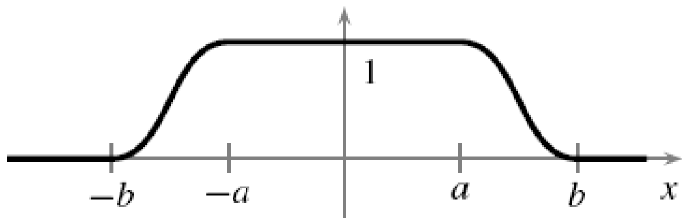
The audience for this essay is to be advanced undergraduate mathematics majors like yourself. You should use a tone suitable for a mathematics magazine for mathematics majors. You should use a format like:

I. Introduction.
II. Discussion.
III. Properties.
  A. Lemma.
    Proof.
  B. Theorem.
    Proof.
    ⋮
VI. . . . Summary.
References.
1. . . .
2. . . .
⋮

$flat$ is a $\mathbf{C}^{\infty}(\mathbb{R})$ function, meaning that $flat\colon \mathbb{R} \to \mathbb{R}$ and possesses derivatives of all orders. The function $flat$ is an example of a function whose Maclaurin series (Taylor series at 0) does not converge to it. $flat$ and all its derivatives have value 0 at $x = 0$ and so its Maclaurin series is the identically 0 series, which corresponds to the identically 0 function. $flat$ is clearly not identically 0. For instance, $flat(1) = \frac{1}{e}$.

Give proofs of the properties mentioned above, and discuss one or more additional topics related to the function. Some suggestions are given below, or you may find some on your own.

Suggested topic 1. By taking integrals of products of shifts of pieces of $flat$ and the 0 function, normalizing and then taking products of shifted reflections, it is possible to produce a $\mathbf{C}^\infty(\mathbb{R})$ function with range $[0,1]$ and value one on an interval $[-a,a]$ and value zero outside $(-b,b)$ for $0 < a < b$. This is called a $\mathbf{C}^\infty$ bump function (at 0).



A $\mathbf{C}^\infty$ bump function at 0. From Tu [1], p. 130.

Demonstrate how to construct a bump function like that above from $flat$.

Suggested topic 2. You may know of the mathematical structures known as *rings*. One of the simplest rings is the ring of integers, $\mathbb{Z}$. The $\mathbf{C}^\infty(\mathbb{R})$ functions also form a ring. These rings are, in fact, *commutative rings* since multiplication in them is commutative. Rings contain special subsets called *ideals*.

For any subset S of a ring R, the intersection of all ideals of R containing S is an ideal I of R. This ideal is the smallest ideal of R containing S. It is denoted $\langle S \rangle$, and is called the *ideal generated by* S. See Wikipedia [3] for further details.

A desirable property of a ring is being *Noetherian*, which may be defined as having the property that any ideal is *finitely generated*, meaning that the ideal is $\langle S \rangle$ for some finite subset S of the ring. Being Noetherian means that the ring shares certain properties of the ring of integers, and is therefore "nice" in some sense. See Wikipedia [2].

The ring of $\mathbf{C}^\infty(\mathbb{R})$ functions turns out *not* to be Noetherian. This is because the ideal generated by the derivatives of $flat$ is not finitely generated. A proof of this would be great, but a discussion of its plausibility would suffice.

References

1. Tu, Loring. Bump Functions and Partitions of Unity. Ch. 13 of *An Introduction to Manifolds*, Springer, New York, 2008.

2. Noetherian ring. (2010, October 13). In Wikipedia, The Free Encyclopedia. Retrieved 19:36, October 21, 2010, from
http://en.wikipedia.org/w/index.php?title=Noetherian_ring&oldid=390563549.

3. Ideal (ring theory). (2010, August 24). In Wikipedia, The Free Encyclopedia. Retrieved 15:47, October 22, 2010, from
http://en.wikipedia.org/w/index.php?title=Ideal_(ring_theory)&oldid=380799258.