BANACH LIMIT(S) USING HAHN-BANACH THEOREM

Denote by $\ell^\infty(N)$ the collection of all bounded real, or complex valued, sequences

$$
\ell^\infty(N) = \{ x = (x_1, x_2, \ldots) \mid \sup_n |x_n| < \infty \}
$$

which forms a vector space with respect to pointwise addition and multiplication by a scalar. It is equipped with the norm

$$
\|x\| = \sup_{n \in \mathbb{N}} |x_n|.
$$

This means that $\| \cdot \|: \ell^\infty(N) \to [0, \infty)$ satisfies

- $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$.
- $\|x + y\| \leq \|x\| + \|y\|$ for any $x, y \in \ell^\infty(N)$.
- $\|c \cdot x\| = |c| \cdot \|x\|$.

The normed space $\ell^\infty(N)$ is complete with respect to the metric $\|x - y\|$, so it is a Banach space. Denote by $S$ the shift map on $\ell^\infty(N)$, given by

$$
S: (x_1, x_2, x_3, \ldots) \mapsto (x_2, x_3, x_4, \ldots).
$$

**Theorem 1 (Banach Limits).** There exists a map $\Phi: \ell^\infty(N) \to \mathbb{R}$ such that

1. $\Phi$ is linear: $\Phi(x + y) = \Phi(x) + \Phi(y)$ and $\Phi(c \cdot x) = c \cdot \Phi(x)$.
2. $\Phi$ is positive: $\Phi(x) \geq 0$ for every $x \in \ell^\infty(N)$ with $x_n \geq 0$.
3. $\Phi$ is normalized: $\Phi(1) = 1$ where $1 = (1, 1, \ldots)$.
4. $\Phi$ is shift invariant: $\Phi(Sx) = \Phi(x)$.

The above properties on the functional $\Phi$ imply the following:

5. $\Phi$ has norm one: $|\Phi(x)| \leq \|x\|$ for every $x \in \ell^\infty(N)$.
6. $\Phi$ extends lim on the subspace of convergent sequences:

   $$
   \lim_{n \to \infty} (x_n) = c \implies \Phi(x) = c.
   $$

7. For any $x \in \ell^\infty(N)$:

   $$
   \liminf_{n \to \infty} (x_n) \leq \Phi(x) \leq \limsup_{n \to \infty} (x_n).
   $$

**Proof.** Consider the linear subspace of $\ell^\infty(N)$ given by

$$
W = \{ Sx - x \mid x \in \ell^\infty(N) \}.
$$

We claim that for any $y \in W$ and $c \in \mathbb{R}$

$$
\|y + c1\| \geq |c|.
$$

Indeed, if $z = Sx - x + c1$, then $z_n = x_{n+1} - x_n + c$ and

$$
\|z\| \geq \frac{1}{N} \sum_{n=1}^{N} |z_n| \geq \frac{1}{N} \left| \sum_{n=1}^{N} z_n \right| = \left| \frac{1}{N} (x_{N+1} - x_1) + c \right| \to |c|
$$

as $N \to \infty$, because $|x_{N+1} - x_1| \leq 2\|x\|$ is bounded.
In particular, it follows that $\mathbb{R} \cdot 1$ has trivial intersection with $W$ (and its closure). Define a linear functional $\Phi_0$ on the linear space $W \oplus \mathbb{R}1$ by

$$\Phi_0(y + c1) = c.$$ 

We just established that $|\Phi_0(z)| \leq \|z\|$ for every $z \in W \oplus \mathbb{R}1$.

Using Hahn-Banach theorem there exists a linear functional $\Phi$ on $\ell^\infty(N)$ extending $\Phi_0$ from $W \oplus \mathbb{R}1$ to all of $\ell^\infty(N)$ and such that

$$|\Phi(x)| \leq \|x\| \quad (x \in \ell^\infty(N)).$$

Such a functional satisfies (1) and (3) by construction. For every $x$ we have

$$\Phi(Sx) - \Phi(x) = \Phi(Sx - x) = \Phi_0(Sx - x) = 0$$

proving (4). For a positive $x$ write $x = c \cdot y$ where $c = \|x\|$ and $y_n \in [0, 1]$ for all $n \in N$. Then $\|1 - y\| \leq 1$. Hence

$$1 - \Phi(y) = \Phi(1 - y) \leq \|1 - y\| \leq 1$$

implying

$$\Phi(y) \geq 0, \quad \Phi(x) = c \cdot \Phi(y) \geq 0.$$ 

This shows (3). Property (4) was obtained by construction. We turn to (7). Given a bounded $x = (x_n)_{n=1}^\infty$ and any $\alpha, \beta$ with

$$\alpha < \liminf_{n \to \infty} (x_n), \quad \limsup_{n \to \infty} (x_n) < \beta$$

there is $N$ so that $\alpha < x_n < \beta$ for all $n > N$. This means that

$$y = S^N x - \alpha \cdot 1, \quad \beta \cdot 1 - S^N x$$

are positive bounded sequences. Hence

$$0 \leq \Phi(y) = \Phi(S^N x) - \alpha = \Phi(x) - \alpha \quad \implies \quad \alpha < \Phi(x)$$

and

$$0 \leq \Phi(z) = \beta - \Phi(S^N x) = \beta - \Phi(x) \quad \implies \quad \Phi(x) < \beta.$$ 

Since $\alpha, \beta$ could be chosen arbitrarily close to $\liminf(x_n), \limsup(x_n)$, respectively, property (7) follows. Property (6) is a direct consequence of (7). □

**Remark 2.** There are infinitely many (in fact, $2^{2^{\aleph_0}}$-many) such Banach Limits. It is also possible to construct a positive, normalized, multiplicative (but not shift-invariant) linear functional, i.e. $\Phi$ satisfying (1), (2), (3) and

$$\Phi$$

(8) $\Phi$ is multiplicative: $\Phi((x_n \cdot y_n)_{n=1}^\infty) = \Phi(x) \cdot \Phi(y)$.

**Exercise 3.** Show:

- Conditions (1), (2), (3) together are equivalent to (1), (3), (5) taken together.
- Conditions (1),(2),(3),(4),(8) are inconsistent (look at $x = (1, 0, 1, 0, \ldots)$).