BANACH LIMIT(S) USING HAHN-BANACH THEOREM

Denote by $\ell^\infty(\mathbb{N})$ the collection of all bounded real, or complex valued, sequences

$$\ell^\infty(\mathbb{N}) = \{\mathbf{x} = (x_1, x_2, \dots) \mid \sup_n |x_n| < \infty\}$$

which forms a vector space with respect to pointwise addition and multiplication by a scalar. It is equipped with the *norm*

$$\|\mathbf{x}\| = \sup_{n\in\mathbb{N}} |x_n|.$$

This means that $\|\cdot\| : \ell^\infty(\mathbb{N}) \to [0, \infty)$ satisfies

- $\|\mathbf{x}\| \geq 0$ and $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = \mathbf{0}$.
- $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for any $\mathbf{x}, \mathbf{y} \in \ell^\infty(\mathbb{N})$.
- $\|c \cdot \mathbf{x}\| = |c| \cdot \|\mathbf{x}\|$.

The normed space $\ell^\infty(\mathbb{N})$ is complete with respect to the metric $\|\mathbf{x} - \mathbf{y}\|$, so it is a *Banach space*. Denote by S the shift map on $\ell^\infty(\mathbb{N})$, given by

$$S : (x_1, x_2, x_3, \dots) \mapsto (x_2, x_3, x_4, \dots).$$

Theorem 1 (Banach Limts)**.** *There exists a map $\Phi : \ell^\infty(\mathbb{N}) \to \mathbb{R}$ such that*

(1) *Φ is linear: $\Phi(\mathbf{x} + \mathbf{y}) = \Phi(\mathbf{x}) + \Phi(\mathbf{y})$ and $\Phi(c \cdot \mathbf{x}) = c \cdot \Phi(\mathbf{x})$.*
(2) *Φ is positive: $\Phi(\mathbf{x}) \geq 0$ for every $\mathbf{x} \in \ell^\infty(\mathbb{N})$ with $x_n \geq 0$.*
(3) *Φ is normalized: $\Phi(\mathbf{1}) = 1$ where $\mathbf{1} = (1, 1, \dots)$.*
(4) *Φ is shift invariant: $\Phi(S\mathbf{x}) = \Phi(\mathbf{x})$.*

The above properties on the functional Φ imply the following:

(5) *Φ has norm one: $|\Phi(\mathbf{x})| \leq \|x\|$ for every $\mathbf{x} \in \ell^\infty(\mathbb{N})$.*
(6) *Φ extends* lim *on the subspace of convergent sequences:*

$$\lim_{n\to\infty}(x_n) = c \qquad \Longrightarrow \qquad \Phi(\mathbf{x}) = c.$$

(7) *For any $\mathbf{x} \in \ell^\infty(\mathbb{N})$:*

$$\liminf_{n\to\infty}(x_n) \leq \Phi(\mathbf{x}) \leq \limsup_{n\to\infty}(x_n).$$

Proof. Consider the linear subspace of $\ell^\infty(\mathbb{N})$ given by

$$W = \{S\mathbf{x} - \mathbf{x} \mid \mathbf{x} \in \ell^\infty(\mathbb{N})\}.$$

We claim that for any $\mathbf{y} \in W$ and $c \in \mathbb{R}$

$$\|\mathbf{y} + c\mathbf{1}\| \geq |c|.$$

Indeed, if $\mathbf{z} = S\mathbf{x} - \mathbf{x} + c\mathbf{1}$, then $z_n = x_{n+1} - x_n + c$ and

$$\begin{aligned} \|\mathbf{z}\| &\geq \frac{1}{N}\sum_{n=1}^{N} |z_n| \geq \frac{1}{N}\left|\sum_{n=1}^{N} z_n\right| \\ &= \left|\frac{1}{N}(x_{N+1} - x_1) + c\right| \quad \longrightarrow \quad |c| \end{aligned}$$

as $N \to \infty$, because $|x_{N+1} - x_1| \leq 2\|\mathbf{x}\|$ is bounded.

In particular, it follows that $\mathbb{R}\cdot\mathbf{1}$ has trivial intersection with W (and its closure). Define a linear functional Φ_0 on the linear space $W \oplus \mathbb{R}\mathbf{1}$ by

$$\Phi_0(\mathbf{y} + c\mathbf{1}) = c.$$

We just established that $|\Phi_0(\mathbf{z})| \leq \|\mathbf{z}\|$ for every $\mathbf{z} \in W \oplus \mathbb{R}\mathbf{1}$.

Using **Hahn-Banach theorem** there exists a linear functional Φ on $\ell^\infty(\mathbb{N})$ extending Φ_0 from $W \oplus \mathbb{R}\mathbf{1}$ to all of $\ell^\infty(\mathbb{N})$ and such that

$$|\Phi(\mathbf{x})| \leq \|\mathbf{x}\| \qquad (\mathbf{x} \in \ell^\infty(\mathbb{N})).$$

Such a functional satisfies (1) and (3) by construction. For every $\mathbf{x}$ we have

$$\Phi(S\mathbf{x}) - \Phi(\mathbf{x}) = \Phi(S\mathbf{x} - \mathbf{x}) = \Phi_0(S\mathbf{x} - \mathbf{x}) = 0$$

proving (4). For a positive $\mathbf{x}$ write $\mathbf{x} = c \cdot \mathbf{y}$ where $c = \|\mathbf{x}\|$ and $y_n \in [0,1]$ for all $n \in \mathbb{N}$. Then $\|\mathbf{1} - \mathbf{y}\| \leq 1$. Hence

$$1 - \Phi(\mathbf{y}) = \Phi(\mathbf{1} - \mathbf{y}) \leq \|\mathbf{1} - \mathbf{y}\| \leq 1$$

implying

$$\Phi(\mathbf{y}) \geq 0, \qquad \Phi(\mathbf{x}) = c \cdot \Phi(\mathbf{y}) \geq 0.$$

This shows (3). Property (4) was obtained by construction. We turn to (7). Given a bounded $\mathbf{x} = (x_n)_{n=1}^\infty$ and any α, β with

$$\alpha < \liminf_{n\to\infty}(x_n), \qquad \limsup_{n\to\infty}(x_n) < \beta$$

there is N so that $\alpha < x_n < \beta$ for all $n > N$. This means that

$$\mathbf{y} = S^N\mathbf{x} - \alpha \cdot \mathbf{1}, \qquad \beta \cdot \mathbf{1} - S^N\mathbf{x}$$

are positive bounded sequences. Hence

$$0 \leq \Phi(\mathbf{y}) = \Phi(S^N\mathbf{x}) - \alpha = \Phi(\mathbf{x}) - \alpha \qquad \Longrightarrow \qquad \alpha < \Phi(\mathbf{x})$$

and

$$0 \leq \Phi(\mathbf{z}) = \beta - \Phi(S^N\mathbf{x}) = \beta - \Phi(\mathbf{x}) \qquad \Longrightarrow \qquad \Phi(\mathbf{x}) < \beta.$$

Since α, β could be chosen arbitrarily close to $\liminf(x_n)$, $\limsup(x_n)$, respectively, property (7) follows. Property (6) is a direct consequence of (7). □

Remark 2. There are infinitely many (in fact, $2^{2^{\aleph_0}}$-many) such *Banach Limits*. It is also possible to construct a positive, normalized, *multiplicative* (but not shift-invariant) linear functional, i.e. Φ satisfying (1), (2), (3) and

(8) Φ is multiplicative: $\Phi((x_n \cdot y_n)_{n=1}^\infty) = \Phi(\mathbf{x}) \cdot \Phi(\mathbf{y})$.

Exercise 3. Show:

- Conditions (1), (2), (3) together are equivalent to (1), (3), (5) taken together.
- Conditions (1),(2),(3),(4),(8) are inconsistent (look at $\mathbf{x} = (1, 0, 1, 0, \dots)$).