MATH 215, Practice Final Exam, 2010-04-28

Problem 1. Prove by induction that

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2n-1)\cdot(2n+1)}=\frac{n}{2n+1}.$$

Proof. Statement $P(n)$ says that "the above identity holds for n".

Base case: $\frac{1}{(2\cdot 1-1)(2\cdot 1+1)}=\frac{1}{3}=\frac{1}{1\cdot 2+1}$. So $P(1)$ is true.

Induction step $P(n)\implies P(n+1)$:

$$\begin{aligned}
&\frac{1}{1\cdot 3}+\quad\dots\quad+\frac{1}{(2n-1)\cdot(2n+1)}+\frac{1}{(2(n+1)-1)\cdot(2(n+1)+1)}\\
\text{by}\quad P(n)\quad &=\frac{n}{2n+1}+\frac{1}{(2(n+1)-1)\cdot(2(n+1)+1)}\\
&=\frac{n}{2n+1}+\frac{1}{(2n+1)(2n+3)}=\frac{2n^2+3n+1}{(2n+1)(2n+3)}\\
&=\frac{(2n+1)(n+1)}{(2n+1)(2n+3)}=\frac{n+1}{2n+3}=\frac{n+1}{2(n+1)+1}.
\end{aligned}$$

Problem 2. For each of the following statements decide whether they are True or False and give a short argument if True, or counter example if False.

(a) $\forall n\in\mathbf{Z},\exists m\in\mathbf{Z},\ n+m\equiv 1 \mod 2$.
True. For any $n\in\mathbf{Z}$ one may take $m=n+1$. Then $n+m=2n+1$ which is odd.

(b) $\forall n\in\mathbf{Z},\exists m\in\mathbf{Z},\ (2n+1)^2=2m-1$.
True. For any $n\in\mathbf{Z}$, we have

$$(2n+1)^2=4n^2+4n+1=2(2n^2+2n+1)-1.$$

So given n, $m=2n^2+2n+1$ satisfies $(2n+1)^2=2m-1$.

(c) $\exists n\in\mathbf{Z},\forall m>n,\ m^2>100m$.
True. Indeed for $m>100$ one has $m^2>100m$. So $n=101$ satisfies the above.

Problem 3. Let $X=\{1,2,3,4\}$ and $Y=\{a,b,c,d\}$.

(a) Give an example of a bijection $f : X \to Y$ and an example of a function $g : X \to Y$ which is not bijective.
Define $f : X \to Y$ by: $f(1) = d$, $f(2) = c$, $f(3) = b$, $f(4) = a$. It is a bijection.
Define $g : X \to Y$ by: $g(1) = d$, $g(2) = a$, $g(3) = b$, $g(4) = a$. It is not injective ($g(2) = g(4)$), and not surjective (c is not in the image).

(b) How many functions $X \to Y$ are there?
There are $4^4 = 256$ functions from X to Y.

(c) How many bijections $X \to Y$ are there?
There are $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ bijections.

Problem 4. Solve the following congruencies (show all your work):

(a) $65x \equiv 14 \mod 70$.
Let's find a solution to the Diophantine equation

$$65x + 70y = 14. \tag{1}$$

As $g = (65, 70) = 5$ does not divide 14, we conclude that there are no solutions to the Diophantine equation, and no solutions to the congruence equation.

(b) $65x \equiv 15 \mod 70$.
Consider the Diophantine equation

$$65x + 70y = 15. \tag{2}$$

We have $g = (65, 70) = 5$, which divides 15. Dividing throughout by 5, replace (2) by an equivalent equation

$$13x + 14y = 3. \tag{3}$$

To find solutions for this, first consider the equation

$$13t + 14s = 1. \tag{4}$$

We can use Euclid's Algorithm, or just guess $t = -1$, $s = 1$ as a solution to (4). Multiplying by 3, we get $x = -3$, $y = 3$ as a solution to (3) (and also to (2)). Hence

$$x \equiv -3 \equiv 11 \mod 14 \qquad \text{solves} \qquad 13x \equiv 3 \mod 14.$$

In fact, this is a unique solution to the congruency (the uniqueness is related to the fact that 13 and 14 are coprime). Equivalently, one may say that $x = 11 + 14k$ with $k \in \mathbf{Z}$ are *all* integers for which there is a corresponding y (namely $y = 3 - 13k$) solving (3) and therefore also (2).

Going back to the congruency equation

$$65x \equiv 14 \mod 70$$

note that the family $x = 11 + 14k$, $k \in \mathbf{Z}$, of integer solutions which gave a single congruence $\mod 14$, gives 5 $(= g)$ congruencies $\mod 70$. Namely

$$x \equiv 11, \text{ or } 25, \text{ or } 39, \text{ or } 53, \text{ or } 67 \mod 70.$$

Problem 5. Sketch the graph of $f(x) = x^2 + 1$. Find

(a) $\overrightarrow{f}([0,1]) = \{x^2 + 1 \mid 0 \le x \le 1\} = [1,2]$.

(b) $\overleftarrow{f}([-2,1]) = \{x \mid -2 \le x^2 + 1 \le 1\} = \{0\}$.

(c) $\overleftarrow{f}([0,3]) = \{x \mid 0 \le x^2 + 1 \le 3\} = \{x \mid x^2 \le 2\} = [-\sqrt{2}, \sqrt{2}]$.

(d) $\overrightarrow{f}([1,2]) = \{x^2 + 1 \mid 1 \le x \le 2\} = [2,5]$.

Problem 6. Prove by truth table *and* illustrate by Venn diagram that

$$A \setminus (B \setminus C) \subseteq (A \setminus B) \cup C.$$

Problem 7. Compute or give a simpler expression for:

(a) $7! \cdot \sum_{n=0}^{7} \frac{10^n}{n!\,(7-n)!}$

$$= \sum_{n=0}^{7} \frac{7!}{n!\,(7-n)!} \cdot 10^n = \sum_{n=0}^{7} \binom{7}{n} \cdot 10^n \cdot 1^{7-n} = (10+1)^7 = 11^7.$$

(b) The number of ways 20 students could sit in a classroom with 25 chairs is

$$25 \cdot 24 \cdot \dots \cdot 22 \cdot 21 = \frac{25!}{5!}.$$

Problem 8. What is $R_5(13^{2010} + 8^{503})$? Recall that $R_5(n)$ denotes the remainder after the division of n by 5.

Let's compute congruencies $\mod 5$. Since $13 \equiv 3 \mod 5$ and $8 \equiv 3 \mod 5$ it suffices to look at $3^n \mod 5$. We have

$$3 \equiv 3, \quad 3^2 = 9 \equiv 4, \quad 3^3 \equiv 3^2 \cdot 3 \equiv 4 \cdot 3 \equiv 2, \quad 3^4 \equiv 3^3 \cdot 3 =\equiv 2 \cdot 3 \equiv 1.$$

Thus $3^4 \equiv 1 \mod 5$. Since $2010 = 4 \cdot 502 + 2$ we get

$$13^{2010} \equiv 3^{2010} \equiv 3^{2008} \cdot 3^2 \equiv 1 \cdot 9 \equiv 4 \mod 5.$$

Since $3^4 \equiv 1 \mod 5$ and $503 = 4 \cdot 125 + 3$ we get

$$8^{503} \equiv 3^{503} \equiv (3^4)^{125} \cdot 3^3 \equiv 1 \cdot 27 \equiv 2 \mod 5.$$

Finally $13^{2010} + 8^{503} \equiv (4 + 2) \equiv 1 \mod 5$. So $R_5(13^{2010} + 8^{503}) = 1$.