

## Math 414. List 7 - Improper Integrals.

Instructor: Alex Furman, 511 SEO, email: [furman@math.uic.edu](mailto:furman@math.uic.edu).

**Definition 1.** Let  $f : [a, \infty) \rightarrow \mathbf{R}$  be a function which is Riemann integrable on  $[a, b]$  for every  $b > a$ . Then

$$\int_a^\infty f dx = \lim_{b \rightarrow \infty} \int_a^b f dx$$

meaning that if the limit exists then the integral on the left is said *to converge* and its value is the limit. Similarly, one defines  $\int_{-\infty}^b$ ,  $\int_{-\infty}^\infty$ ; and for functions  $f$  which are unbounded on  $(a, b]$  but bounded and integrable on every  $[a + \epsilon, b]$  define

$$\int_a^b f dx = \lim_{\epsilon \rightarrow 0} \int_{a+\epsilon}^b f dx.$$

**Problem 2.** Prove that if  $f$  is integrable on  $[a, b]$  then

$$\lim_{\epsilon \rightarrow 0} \int_{a+\epsilon}^b f dx = \int_a^b f dx = \lim_{\epsilon \rightarrow 0} \int_a^{b-\epsilon} f dx$$

**Remark 3.** Verify, that for  $f \geq 0$  which is integrable on finite intervals the integral  $\int_a^\infty f dx$  converges if and only if  $\int_a^b f dx$  are bounded in  $b$ . This is denoted  $\int_a^\infty f dx < +\infty$  (as a notation for *convergence of the integral* this makes sense only for non-positive functions). Similarly, for the other improper integrals.

**Problem 4** (Integral Criterion for Positive series/functions). Let  $f : [0, \infty) \rightarrow [0, \infty)$  be a non-increasing function. Prove that

$$\sum_{n=1}^{\infty} f(n) < \infty \quad \text{if and only if} \quad \int_0^{\infty} f(x) dx < +\infty.$$

Deduce the following series converge if and only if  $\alpha > 1$ :

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \quad \sum_{n=2}^{\infty} \frac{1}{n(\log n)^\alpha} \quad \sum_{n=2}^{\infty} \frac{1}{n \log n (\log \log n)^\alpha}$$

Reprove the criterion “for monotonic  $a_n$  the series  $\sum a_n$  converges iff  $\sum 10^n a_{10^n}$  converges” using the integral criterion above and the substitution  $u = 10^x$ .

**Theorem 5.** (Cauchy Criterion) *An integral  $\int_a^\infty f dx$  converges if and only if for every  $\epsilon > 0$  there exists a  $B$  so that*

$$\forall b_1, b_2 \geq B, \quad \left| \int_{b_1}^{b_2} f dx \right| < \epsilon.$$

(Absolute Convergence implies Convergence): *It follows that if  $f$  is integrable on  $[a, b]$  for every  $b$ , and  $\int_a^\infty |f(x)| dx < \infty$  then  $\int_a^\infty f dx$  converges.*

**Problem 6.** Check the convergence of the following integrals (address both  $\infty$  and 0 issues):

$$\int_0^\infty \frac{\sin x}{x+x^2} dx, \quad \int_0^\infty \sin\left(\frac{1}{x^2}\right) dx, \quad \Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx \quad (\alpha > 0).$$