Math 533 Solutions to Midterm Exam, Part I 2010

1. For each of the following statements decide if True or False, and **justify** your answer: give a brief argument if True, or a counter example if False.

   (a) Every countable set $A \subset \mathbf{R}$ has zero Lebesgue measure.

   True. For any $a \in \mathbb{R}$ one has $m(\{a\}) = 0$ (because $m(\{a\}) \leq m((a-\epsilon, a]) = \epsilon$). Therefore, for any countable set $A = \{a_n \mid n \in \mathbb{N}\}$ then $m(A) = \sum_n m(\{a_m\}) = 0$ using countable additivity.

   (b) For $A \subset \mathbf{R}$ with $m(A) = 0$ and continuous $f : \mathbf{R} \to \mathbf{R}$, the image $f(A)$ has $m(f(A)) = 0$.

   False. The Cantor Lebesgue function $f : [0,1] \to [0,1]$, that can be defined by

$$f(\sum_{n=1}^{\infty} \frac{a_n}{3^n}) = \sum_{n=1}^{\infty} \frac{\phi(a_n)}{2^n}, \qquad a_n \in \{0,1,2\}, \qquad \phi(0) = \phi(1) = 0, \ \phi(2) = 1$$

   is a continuous monotonic function that maps the Cantor set $C = \{\sum_{n=1}^{\infty} \frac{b_n}{3^n} \mid b_n \in \{0,2\}\}$ **onto** the unit interval $[0,1]$. Since $m(C) = 0$ while $m([0,1]) = 1$, this is a counter example.

   (c) If $f_n : X \to \mathbf{R}$ are measurable functions on a measure space $(X, \mathcal{M}, \mu)$ then the function $f(x) = \sup_n f_n(x)$ is a measurable.

   True. For any $a \in \mathbb{R}$ one has

$$f^{-1}((a,\infty)) = \{x \in X \mid f(x) > a\} = \{x \in X \mid \exists n \in \mathbb{N},\ f_n(x) > a\} = \bigcup_{n=1}^{\infty} \{x \in X \mid f_n(x) > a\}.$$

   Since this is a countable union of in $\mathcal{M}$, we have $f^{-1}((a,\infty)) \in \mathcal{M}$ for all $a \in \mathbb{R}$, which implies that f is measurable.

   (d) If $f_n \in L^+(X, \mathcal{M}, \mu)$ satisfy $\displaystyle\int f_n \, d\mu < \frac{1}{n^2}$, then $\displaystyle\sum_{n=1}^{\infty} f_n(x) < \infty$ for μ-a.e. $x \in X$.

   True. Let $f : X \to [0,\infty]$ be defined by $f(x) = \sum f_n(x)$. By the Monotone Convergence Theorem

$$\int_X f \, d\mu = \int_X \left(\sum_{n=1}^{\infty} f_n(x) \right) d\mu = \sum_{n=1}^{\infty} \left(\int_X f_n \, d\mu \right) < \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

   Hence $E = \{x \mid f(x) = \infty\}$ has $\mu(E) = 0$.

   (e) Define: set A is of G_δ-type. Is is true or False that every closed set A is of G_δ type?

   True. A set A is of G_δ-type if there exists a sequence $\{U_n\}_{n=1}^{\infty}$ of open sets so that $A = \bigcap_{n=1}^{\infty} U_n$.

   We claim that for any set A, the closure is $\bar{A} = \bigcap_{n=1}^{\infty} U_n$ where $U_n = \bigcup_{a \in A} (a - \frac{1}{n}, a + \frac{1}{n})$ are open sets. Indeed, the following are equivalent conditions

   1. $x \in \bigcap_{n=1}^{\infty} U_n$,
   2. for every $n \in \mathbb{N}$ there exists $a_n \in A$ with $|a_n - x| < \frac{1}{n}$,
   3. x is a limit of points $a_n \in A$,
   4. $x \in \bar{A}$.

   Hence for a closed set A one has $A = \bar{A} = \bigcap U_n$ – a G_δ-type.

2. Answer briefly the following questions:

(a) State Fatou's Lemma.

Fatou's Lemma. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $L^+(X, \mathcal{M}, \mu)$. Then

$$\int_X \left(\liminf_{n\to\infty} f_n(x) \right) \, d\mu(x) \leq \liminf_{n\to\infty} \left(\int_X f_n \, d\mu \right).$$

(b) Define: measure space $(X, \mathcal{M}, \mu)$ is *complete*.

Definition. $(X, \mathcal{M}, \mu)$ is *complete* if $A \in \mathcal{M}$ with $\mu(A) = 0$ implies $E \in \mathcal{M}$ for every $E \subset A$.

(c) Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of subsets of some space X. Express the sets

$$\begin{aligned} A &= \{x \in X \mid \{n \in \mathbb{N} \mid x \in E_n\} \text{ is infinite}\}, \\ B &= \{x \in X \mid \{n \in \mathbb{N} \mid x \notin E_n\} \text{ is finite}\} \end{aligned}$$

using $\{E_n\}$ and basic operations such as "$\bigcup$" and "$\bigcap$".

Solution.

$$A = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} E_n, \qquad B = \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} E_n.$$

(d) Define: m^* is an *outer measure* on X.

Solution. An outer measure is a function $m^* : P(X) \to [0, \infty]$ so that

1. $m^*(\emptyset) = 0$.
2. If $A \subset B$ then $m^*(A) \leq m^*(B)$.
3. For any sequence $\{A_n\}$ of subsets of X, $m^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} m^*(A_n)$.

(e) Describe the Lebesgue-Stieltjes measure m_F corresponding to the function $F(x) = \lfloor x \rfloor$ - the largest integer n with $n \leq x$.

Solution. The measure m_F is counting measure on $\mathbb{Z}$, i.e., $m_F(E) = \#(E \cap \mathbb{Z})$ for any $E \subset \mathbb{R}$. (In particular, all subsets of $\mathbb{R}$ are m_F-measurable). First we have

$$m_F((a, b]) = F(b) - F(a) = \lfloor b \rfloor - \lfloor a \rfloor = \#\{n \in \mathbb{Z} \mid a < n \leq b\} = \#(\mathbb{Z} \cap (a, b]).$$

For every integer $k \in \mathbb{Z}$ we have $m_F(\{k\}) = 1$ because $\{k\} = \bigcap_{n=1}^{\infty}(k - \frac{1}{n}, k]$ and $m_F((k - \frac{1}{n}, k]) = 1$ for all $n \geq 1$. At the same time $(k, k+1) = \bigcup_{n=1}^{\infty}(k, k+1-\frac{1}{n}]$ and since $m_F((k, k+1-\frac{1}{n}]) = 0$ we have $m_F((k, k+1)) = 0$. Hence $\mathbb{R} \setminus \mathbb{Z} = \bigcup_{k\in\mathbb{Z}}(k, k+1)$ is an m_F-null set. Since all subsets are also m_F-measurable it follows that all $E \subset \mathbb{R}$ are m_F-measurable, and $m_F(E) = m_F(E \cap \mathbb{Z}) + m_F(E \setminus \mathbb{Z}) = \#(E \cap \mathbb{Z})$.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space.
   (a) Let $f \in L^+(X, \mathcal{M}, \mu)$ with $\int_X f\, d\mu = 0$. Prove that $\mu\left(f^{-1}(0, \infty]\right) = 0$.
   (b) Let $g \in L^1(X, \mathcal{M}, \mu)$. Prove that there exist a sequence $\{A_n\}$ of measurable sets with $\mu(A_n) < \infty$, so that $g(x) = 0$ for $x \in X \setminus \bigcup_1^\infty A_n$.

<u>**Proof.**</u> **(a).** For $t > 0$ let $E_t = f^{-1}([t, \infty])$. Then $t \cdot \chi_{E_t}(x) \le f(x)$ and therefore

$$t \cdot \mu(E_t) = \int t \cdot \chi_{E_t}\, d\mu \le \int f\, d\mu = 0.$$

Hence $\mu(E_t) = 0$ for every $t > 0$. Since

$$\{x \in X \mid f(x) > 0\} = f^{-1}((0, \infty]) = \bigcup_{n=1}^{\infty} f^{-1}([\frac{1}{n}, \infty]) = \bigcup_{n=1}^{\infty} E_{1/n}$$

a countable union of μ-null sets, it follows that $\mu\left(f^{-1}(0, \infty]\right) = 0$.

(b). As before, let $E_t =$ and use $t \cdot \chi_{E_t}(x) \le |g(x)|$ to deduce

$$t \cdot \mu(E_t) = \int t \cdot \chi_{E_t}\, d\mu \le \int |g|\, d\mu < \infty.$$

Hence for $t > 0$, one has $\mu(E_t) < \infty$. Setting $A_n = E_{1/n}$ observe that

$$x \in \bigcup_{n=1}^{\infty} A_n \qquad \Longleftrightarrow \qquad |g(x)| > 0 \qquad \Longleftrightarrow \qquad g(x) \neq 0.$$

This completes the proof.

4. Let $\{E_n\}$ be a sequence of Borel subsets of $[0, 1]$. Prove that

$$m(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k) \ge \limsup_{n\to\infty} m(E_n).$$

<u>**Proof.**</u>

Let $A_n = \bigcup_{k=n}^{\infty} E_k$ and $A = \bigcap_{n=1}^{\infty} A_n$. Then $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ converge to A. Since $A_1 \subset [0, 1]$ one has $m(A_1) \le 1$. Using "continuity from above", we deduce that

$$m(A) = \lim_{n\to\infty} m(A_n).$$

For any $n \le k$ one has $A_n = E_n \cup E_{n+1} \cup \ldots \supset E_k$. Hence $m(A_n) \ge m(E_k)$, and

$$m(A_n) \ge \sup_{k \ge k} m(E_k).$$

Thus

$$m(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k) = m(A) = \lim_{n\to\infty} m(A_n) \ge \lim_{n\to\infty} \left(\sup_{k \ge n} m(E_k) \right) = \limsup_{n\to\infty} m(E_n).$$