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QUASI-FUCHSIAN VS NEGATIVE CURVATURE METRICS ON SURFACE GROUPS

BY

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To Benjy Weiss with gratitude and admiration

ABSTRACT

We compare two families of left-invariant metrics on a surface group $\Gamma = \pi_1(\Sigma)$ in the context of course-geometry. One family comes from Riemannian metrics of negative curvature on the surface Σ, and another from quasi-Fuchsian representations of Γ. We show that the Teichmüller space $\mathscr{T}(\Sigma)$ is the only common part of these two families, even when viewed from the coarse-geometric perspective.

1. Introduction and statement of the main result

1.A. Introduction and Background. Let Σ be a closed surface of genus at least two, and $\Gamma = \pi_1(\Sigma)$ its fundamental group. The Teichmüller space $\mathscr{T}(\Sigma)$ has several equivalent descriptions: as the moduli space of (i) complex structures, or (ii) conformal structures, or (iii) Riemannian structures of constant curvature -1 on Σ, or as (iv) the space of discrete cocompact representations $\Gamma \to \mathrm{PSL}_2(\mathbf{R})$, up to conjugation. The latter two points of view can be extended as follows:

- $\mathscr{R}(\Sigma)$—the space of all Riemannian structures of possibly variable negative curvature, up to isotopy and scaling.
- $\mathscr{QF}(\Sigma)$—the space of all convex cocompact representations

$$\Gamma = \pi_1(\Sigma) \longrightarrow \mathrm{PSL}_2(\mathbf{C}) \cong \mathrm{Isom}^+(\mathbf{H}^3),$$

up to conjugation.

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Both $\mathscr{R}(\Sigma)$ and $\mathscr{QF}(\Sigma)$ arise from convex cocompact isometric Γ-actions on CAT(-1) spaces: the Γ-action by deck transformations on the universal cover $(\tilde{\Sigma}, d_{\tilde{g}})$ in the Riemannian case, and the Γ-action on $\mathbf{H}^3$ in the quasi-Fuchsian case.

We can put these notions into an even broader context by looking at the space $\mathscr{D}_\Gamma$ of equivalence classes $[d]$ of left-invariant metrics d on Γ obtained from restricting the metric of the underlying Gromov-hyperbolic space to a Γ-orbit. Here two metrics d, d' on Γ are equivalent if they are bounded distance from each other after scaling:

$$d \sim d' \quad \text{if } \exists k, A: \quad |d'(\gamma_1, \gamma_2) - k \cdot d(\gamma_1, \gamma_2)| \leq A.$$

This perspective, introduced by the second author in [11] (see also more recent treatment in Bader–Furman [1]), allows to observe possible "geometries" of Σ from the "outside" by studying the corresponding classes $[d] \in \mathscr{D}_\Gamma$ of metrics d on Γ. The space $\mathscr{D}_\Gamma$ can be defined for a general non-elementary Gromov hyperbolic group Γ, and $\mathscr{D}_\Gamma$ contains classes of metrics on Γ from various sources, such as word metrics on Γ, Green metrics associated with symmetric generating random walks on Γ (see Blachère–Haïssinsky–Mathieu [3, 4]), Anosov representations of Γ in higher rank simple Lie groups (see Dey–Kapovich [10]), etc.

To avoid ambiguity in scaling we can normalize metrics d by the growth

$$h_d = \lim_{R \to \infty} \frac{1}{R} \log \#\{\gamma \in \Gamma \mid d(\gamma, e) < R\},$$

replacing d by $\hat{d} = h_d \cdot d$, so that $h_{\hat{d}} = 1$. For $\delta \in \mathscr{D}_\Gamma$ we can define:

- Marked Length Spectrum $\ell_\delta : \Gamma \to \mathbf{R}_+$ given by the limit

  $$\ell_\delta(\gamma) = \lim_{n \to \infty} \frac{\hat{d}(\gamma^n, e)}{n}$$

  where $\delta = [d]$ and $\hat{d} = h_d \cdot d$. Note that ℓ_δ is constant on conjugacy classes, so we can write it as $\ell_\delta : \mathcal{C}_\Gamma \to \mathbf{R}_+$.
- Patterson–Sullivan-like Γ-invariant measure class $[\nu_\delta^{\mathrm{PS}}]$ on $\partial\Gamma$ (see Coorneart [7], and [11, 1]).
- Bowen–Margulis–Sullivan-like Γ-invariant Radon measure m_δ^{BMS} on the space $\partial^{(2)}\Gamma$ of distinct pairs (ξ, η) of points on $\partial\Gamma$ (see [11, 1]).

In [11] (see also Bader–Furman [1]), it was shown that each $\delta \in \mathscr{D}_\Gamma$ is determined by each of these objects. Furthermore, extending a prior work of Bader–Muchnik [2], Garncarek [12] showed that for each $\delta \in \mathscr{D}_\Gamma$ the quasi-regular unitary Γ-representation

$$\pi_\delta : \Gamma \longrightarrow U(\partial\Gamma, \nu_\delta^{\mathrm{PS}})$$

is irreducible, and that the map $\mathscr{D}_\Gamma \longrightarrow \hat{\Gamma}$, $\delta \mapsto \pi_\delta$, is also injective. Thus $\mathscr{D}_\Gamma$ can be embedded into any one of the following spaces:

$$\mathbf{R}_+^{\mathcal{C}_\Gamma}, \quad \mathrm{Prob}(\partial\Gamma), \quad \mathrm{Meas}_\Gamma(\partial^{(2)}\Gamma), \quad \hat{\Gamma}.$$

The space $\mathscr{D}_\Gamma$ is also equipped with a natural metric: given two classes $\delta = [d]$, $\delta' = [d']$ in $\mathscr{D}_\Gamma$ we can define the (log) Lipschitz distance by

$$\rho_{\mathrm{Lip}}(\delta, \delta') := \log\left(\inf\left\{\frac{K}{k} \mid \exists A,\, k \cdot d - A \leq d' \leq K \cdot d + A\right\}\right).$$

It is clear from the definition that $\rho_{\mathrm{Lip}}(-,-)$ is symmetric and satisfies the triangle inequality. One can see that for any $a, b \in \Gamma \setminus \{e\}$ one has

$$\left|\log\left(\frac{\ell_\delta(a)}{\ell_\delta(b)} : \frac{\ell_{\delta'}(a)}{\ell_{\delta'}(b)}\right)\right| \leq \rho_{\mathrm{Lip}}(\delta, \delta').$$

This shows that $\rho_{\mathrm{Lip}}(\delta, \delta') = 0$ implies $\ell_\delta = \ell_{\delta'}$, which occurs only when $\delta = \delta'$. So $\rho_{\mathrm{Lip}}(,)$ is indeed a metric on $\mathscr{D}_\Gamma$ (see also a recent work of Cantrell–Tanaka [6] for a more detailed picture).

1.B. Riemannian and quasi-Fuchsian structures on surfaces. In this paper we focus on surface group $\Gamma = \pi_1(\Sigma)$ and two specific sources for $\delta \in \mathscr{D}_\Gamma$: namely $\mathscr{R}(\Sigma)$ and $\mathscr{QF}(\Sigma)$.

For the case of negatively curved Riemannian metric g on Σ, fix $x \in \tilde{\Sigma}$ and consider the metric on Γ

$$d_{g,x}(\gamma_1, \gamma_2) := d_{\tilde{g}}(\gamma_1 x, \gamma_2 x).$$

Since $|d_{g,x} - d_{g,x'}| \leq d(x, x')$ the class $[d_{g,x}]$ does not depend on the choice of $x \in \tilde{\Sigma}$, and we can denote this class by $\delta_g = [d_{g,x}]$. Note that $h_{d_{g,x}}$ is the topological entropy of the geodesic flow on the unit tangent bundle $T^1\Sigma$ to Σ, and we assume that all $g \in \mathscr{R}(\Sigma)$ are normalized so that $h_{d_{g,x}} = 1$. We have a map

$$i : \mathscr{R}(\Sigma) \longrightarrow \mathscr{D}_\Gamma\,. \tag{1.1}$$

The Marked Length Spectrum Rigidity Conjecture, that for surfaces was proved by Otal [14] and Croke [8], asserts that a Riemannian structure g of variable negative curvature on a surface Σ is uniquely determined by the function $\ell_g : \mathcal{C}_\Gamma \to \mathbf{R}$. As a consequence, we obtain:

PROPOSITION 1.1: *The map* $\mathscr{R}(\Sigma)\longrightarrow\mathscr{D}_\Gamma$, $i: g\mapsto\delta_g$, *is injective.*

Our second source of examples are quasi-Fuchsian representations. For $q\in\mathscr{QF}(\Sigma)$ choose a representation $\pi:\Gamma\to\mathrm{Isom}^+(\mathbf{H}^3)\cong\mathrm{PSL}_2(\mathbf{C})$ in this class and a point $y\in\mathbf{H}^3$ and consider the metric on Γ:

$$d_{\pi,y}(\gamma_1,\gamma_2):=d_{\mathbf{H}^3}(\pi(\gamma_1).y,\pi(\gamma_2).y).$$

The class $[d_{\pi,y}]$ does not depend on the choice of $y\in\mathbf{H}^3$ and remains unchanged if π is replaced by a conjugate $\gamma\mapsto g\pi(\gamma)g^{-1}$; thus we write δ_q for $[d_{\pi,y}]$. This gives a well defined map

$$j:\mathscr{QF}(\Sigma)\longrightarrow\mathscr{D}_\Gamma\,. \tag{1.2}$$

One can deduce from a work of Burger [5] (or Dal'bo–Kim [9]) the following.

PROPOSITION 1.2: *The map* $\mathscr{QF}(\Sigma)\longrightarrow\mathscr{D}_\Gamma$, $j:q\mapsto\delta_q$, *is injective.*

Hence one might view each of $\mathscr{R}(\Sigma)$ and $\mathscr{QF}(\Sigma)$ as being embedded in $\mathscr{D}_\Gamma$.

Remark 1.3: We note in passing that the uniformization theorem allows us to view $\mathscr{R}(\Sigma)$ as a bundle over $\mathscr{T}(\Sigma)$ with fibers that can be identified with the positive cone $C^\infty_+(\Sigma)/\mathbf{R}_+$; in particular $\mathscr{R}(\Sigma)$ is connected. One can show that the map (1.1) is continuous, and so the image $i(\mathscr{R}(\Sigma))$ in $\mathscr{D}_\Gamma$ is connected.

Ahlfors and Bers showed that $\mathscr{QF}(\Sigma)$ can be identified with $\mathscr{T}(\Sigma)\times\mathscr{T}(\Sigma)$, and is in particular connected. The map (1.2) can be shown to be continuous; hence the image $j(\mathscr{QF}(\Sigma))$ is a connected subset of $\mathscr{D}_\Gamma$.

It is natural to wonder whether the intersection

$$i(\mathscr{R}(\Sigma))\cap j(\mathscr{QF}(\Sigma))\subset\mathscr{D}_\Gamma$$

contains anything except for the image of $\mathscr{T}(\Sigma)$. In other words, is it true that given a quasi-Fuchsian representation $\pi:\Gamma\longrightarrow\mathrm{PSL}_2(\mathbf{C})$ and a negatively curved metric g on the surface Σ, there exist constants k,A and points $x\in\tilde\Sigma$, $y\in\mathbf{H}^3$, so that

$$k\cdot d_{\tilde g}(\gamma.x,x)-A\le d_{\mathbf{H}^3}(\pi(\gamma).y,y)\le k\cdot d_{\tilde g}(\gamma.x,x)+A\quad(\gamma\in\Gamma)$$

only if g has constant curvature, π is conjugate to a subgroup of $\mathrm{PSL}_2(\mathbf{R})$, and (Σ,g) and π represent the same point in $\mathscr{T}(\Sigma)$?

Our main result answers this affirmatively.

Theorem A: *The images of* $\mathscr{R}(\Sigma)$ *and* $\mathscr{QF}(\Sigma)$ *in* $\mathscr{D}_\Gamma$ *have only* $\mathscr{T}(\Sigma)$ *in common. Moreover, for any* $q \in \mathscr{QF}(\Sigma) \setminus \mathscr{T}(\Sigma)$ *there is* $\alpha_q > 0$ *so that*

$$\rho_{\mathrm{Lip}}(\delta_q, \delta_g) \geq \alpha_q > 0$$

for all $g \in \mathscr{R}(\Sigma)$.

The following natural question remains open.

Question 1.4: Is it true that for any $g \in \mathscr{R}(\Sigma) \setminus \mathscr{T}(\Sigma)$ there is $\beta_g > 0$ so that

$$\rho_{\mathrm{Lip}}(\delta_q, \delta_g) \geq \beta_g > 0$$

for all $q \in \mathscr{QF}(\Sigma)$?

Acknowledgements. This note is dedicated to Benjy Weiss on the occasion of his 80th birthday. His profound contributions to Ergodic Theory and Dynamics, breadth of his interests and originality of his ideas are an inspiration to us and many, many others.

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2. Length inequalities for negatively curved surfaces

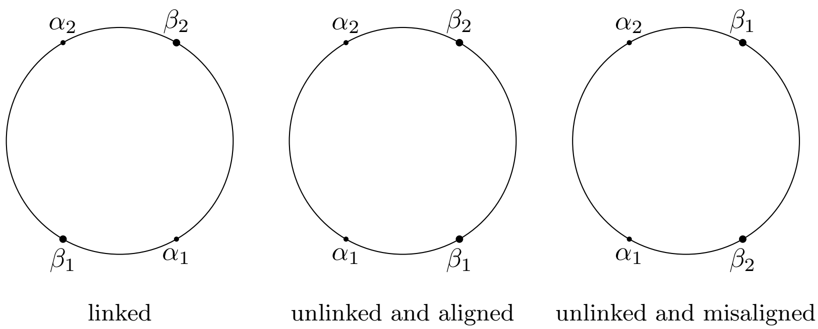
Consider the topological picture first. Let Σ be a closed surface of genus at least two, $\Gamma = \pi_1(\Sigma)$ the corresponding surface group, that acts on the universal cover $\tilde{\Sigma}$ by deck transformations. This action extends to the action of Γ on the boundary circle $\partial\tilde{\Sigma}$, which is also the Gromov boundary $\partial\Gamma$ of Γ. Every $\gamma \neq 1$ in Γ has two fixed points on the topological circle $\partial\tilde{\Sigma}$: a repelling point γ^- and an attracting point γ^+. We shall consider a pair $a, b \in \Gamma$ where a^-, a^+, b^-, b^+ are four distinct points on the circle.

Let $A = (\alpha_1, \alpha_2)$ and $B = (\beta_1, \beta_2)$ be two ordered pairs on a circle C, where all four points are distinct. The action of $\mathrm{Homeo}(C)$ on such pairs has 3 orbits corresponding to 3 possible relative positions of the two pairs A, B:

- The pairs are **linked**, meaning that β_1 and β_2 lie in distinct arcs defined by $\{\alpha_1, \alpha_2\}$—connected components of $C \setminus \{\alpha_1, \alpha_2\}$. The relation of being linked is symmetric: A is linked with B iff B is linked with A.

The order within the pairs $A = (\alpha_1, \alpha_2)$ and $B = (\beta_1, \beta_2)$ does not change the status of being linked. We say that disjoint pairs A and B are **unlinked** if they are not linked.

- The pairs A and B are **unlinked and aligned**, if in the arc $\widehat{\alpha_1, \alpha_2}$ determined by $\{\alpha_1, \alpha_2\}$ on C containing β_1 and β_2 one has linear order $\alpha_1 < \beta_1 < \beta_2 < \alpha_2$. We note that A is unlinked and aligned with B iff B is unlinked and aligned with A. In this case flipping the order in both pairs A and B simultaneously does not change the status of being aligned.
- The pairs A and B are **unlinked and misaligned**, if in the arc $\widehat{\alpha_1, \alpha_2}$ determined by $\{\alpha_1, \alpha_2\}$ on C containing β_1 and β_2 one has linear order $\alpha_1 < \beta_2 < \beta_1 < \alpha_2$. We note that A is unlinked and misaligned with B iff B is unlinked and misaligned with A. In this case flipping the order in both of A and B simultaneously does not change the status of being misaligned. Yet flipping the order in either A or B makes the pair unlinked and aligned.



Let us now choose a negatively curved Riemannian metric g on Σ, and let $\tilde{g}$ be its lift to $\tilde{\Sigma}$. Denote by $d_{\tilde{g}}$ the corresponding distance on $\tilde{\Sigma}$, and by $\ell_g : \Gamma \to [0, \infty)$ the associated **stable length**

$$\ell_g(\gamma) := \lim \frac{1}{n} d_{\tilde{g}}(\gamma^n.p, p)$$

where $p \in \tilde{\Sigma}$ is arbitrary.

THEOREM 2.1: *Let $a, b \in \Gamma$ be non-trivial elements with distinct fixed points a^-, a^+, b^-, b^+ on the boundary circle $\partial\Gamma$. Then:*

(1) *If (a^-, a^+) and (b^-, b^+) are linked, then*

$$\ell_g(ab) < \ell_g(a) + \ell_g(b).$$

(2) *If (a^-, a^+) and (b^-, b^+) are unlinked and aligned, then*

$$\ell_g(ab) > \ell_g(a) + \ell_g(b).$$

(3) *If (a^-, a^+) and (b^-, b^+) are unlinked and misaligned, then*

$$\ell_g(a^{-1}b) > \ell_g(a) + \ell_g(b).$$

Proof. First recall that in the case of negatively curved manifolds, such as (Σ, g), the stable length $\ell_g(\gamma)$ can also be defined as the **minimal translation length**

$$\ell_g(\gamma) = \inf_{p \in \tilde{\Sigma}} d_{\tilde{g}}(\gamma.p, p).$$

Moreover, when $\ell_g(\gamma) > 0$, which is the case of any non-trivial $\gamma \neq 1$, the inf is attained and the set

$$\mathrm{Ax}_\gamma := \{p \in \tilde{\Sigma} \mid d_{\tilde{g}}(\gamma.p, p) = \ell_g(\gamma)\}$$

is the geodesic line (γ^-, γ^+) in $\tilde{\Sigma}$. It is called the **axis** of γ.

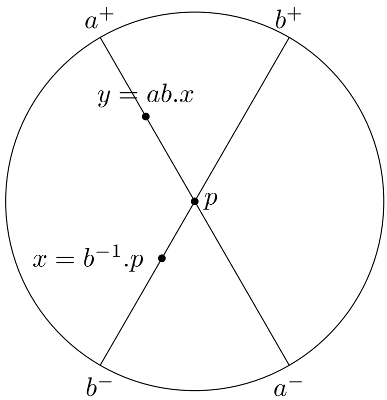
Elementary topology of the disc $\tilde{\Sigma}$ implies that when (a^-, a^+) and (b^-, b^+) are linked, the axes Ax_a and Ax_b must intersect in $\tilde{\Sigma}$. Due to negative curvature the intersection is a singleton: $\mathrm{Ax}_a \cap \mathrm{Ax}_b = \{p\}$. Since $p \in \mathrm{Ax}_b$, we have $x = b^{-1}.p \in \mathrm{Ax}_b$. Similarly, we have p and $y = a.p$ are in Ax_a as well. To prove part (1) we use the triangle inequality to obtain for $x = b^{-1}.p$:

$$\begin{aligned}\ell_g(ab) \leq d_{\tilde{g}}(x, ab.x) &< d_{\tilde{g}}(x, b.x) + d_{\tilde{g}}(b.x, ab.x)\\ &= d_{\tilde{g}}(b^{-1}.p, p) + d_{\tilde{g}}(p, a.p) = \ell_g(b) + \ell_g(a).\end{aligned}$$

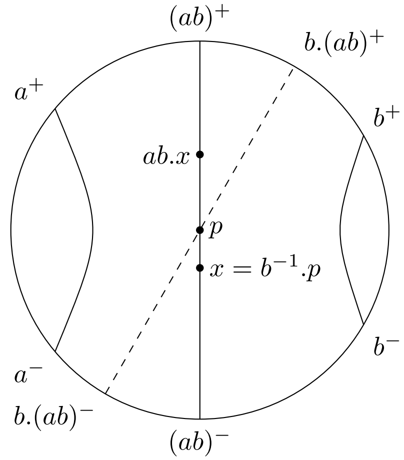
We observe that the second inequality is strict and will sharpen it in the proof of Theorem A below.

In the case where the pairs (a^-, a^+) and (b^-, b^+) are unlinked and aligned, we remind ourselves of the definition, that a^-, a^+ define an arc $\widehat{a^-a^+}$ on the boundary circle containing both b^- and b^+, which can be equipped with a linear order (anti-clockwise in the figure) so that

$$a^- < b^- < b^+ < a^+.$$



linked



unlinked and aligned

The action of b on the arc/interval from b^+ to a^+ is decreasing towards the fixed point b^+, while the action of a is increasing towards a^+. Thus ab maps this interval into itself, and therefore the attracting point $(ab)^+$ satisfies $b^+ < (ab)^+ < a^+$. Moreover, we have

$$b^+ = b.b^+ < b.(ab)^+ < (ab)^+.$$

Since the repelling fixed point of an element is the attracting fixed point of its inverse, the same argument gives $a^- < (ab)^- < b^-$. We claim that $a^- < b.(ab)^- < (ab)^-$. Indeed, in the linear order on the arc $\widehat{b^+b^-}$ that contains $a^\pm$ so that $b^+ < a^+$, $a^- < b^-$ the map b is decreasing, and thus $\xi = b.(ab)^- < (ab)^-$. Since $a.\xi = (ab).(ab)^- = (ab)^- > \xi$ we deduce that $a^- < \xi < (ab)^-$. Hence

$$a^- < b.(ab)^- < (ab)^-.$$

We conclude that the pair $((ab)^-, (ab)^+)$ is linked with its image under b. Denote by p the intersection of Ax_{ab} and $b.\mathrm{Ax}_{ab}$ in $\tilde{\Sigma}$, and let $x = b^{-1}.p$. Since $p \in b.\mathrm{Ax}_{ab}$ we have $x \in \mathrm{Ax}_{ab}$ and $ab.x \in \mathrm{Ax}_{ab}$ as well. Thus the points x, $p = b.x$, $ab.x = a.p$ lie on the geodesic line Ax_{ab}, and in fact in this linear order. This can be seen by inspecting the projections of these points to Ax_a and Ax_b, making use of the assumption that the pairs are aligned. Hence

$$\ell_g(ab) = d_{\tilde{g}}(x, ab.x) = d_{\tilde{g}}(x, b.x) + d_{\tilde{g}}(p, a.p) > \ell_g(b) + \ell_g(a).$$

The strict inequality here occurs because $p \notin \mathrm{Ax}_a$ and $x \notin \mathrm{Ax}_b$. This proves statement (2).

Statement (3) follows from (2) by replacing a by a^{-1}. This completes the proof of Theorem 2.1. ∎

3. Spiraling of the boundary of a quasi-Fuchsian embedding

Let $\Gamma = \pi_1(\Sigma)$ be a surface group, and $q \in \mathscr{QF}(\Sigma)$ be defined by a representation $\pi : \Gamma \to \mathrm{PSL}_2(\mathbf{C})$. For $\gamma \in \Gamma$ the element $g = \pi(\gamma) \in \mathrm{PSL}_2(\mathbf{C})$ has two preimages $\pm\hat{g}$ in $\mathrm{SL}_2(\mathbf{C})$. Since the traces $\pm\,\mathrm{tr}(\hat{g})$ are invariant under conjugation, we can denote them by $\pm\,\mathrm{tr}_q(\gamma)$. The following is a particular case of a lemma of Vinberg [15] (see [13, Corollary 3.2.5]).

Lemma 3.1: *Let $\Gamma = \pi_1(\Sigma)$ be a surface group, and $q \in \mathscr{QF}(\Sigma) \setminus \mathscr{T}(\Sigma)$. Then there exists $\gamma \in \Gamma$ with $\pm\,\mathrm{tr}_q(\gamma) \in \mathbf{C} \setminus \mathbf{R}$.*

Let $\pi : \Gamma \to \mathrm{PSL}_2(\mathbf{C})$ be a quasi-Fuchsian representation. There exists a Γ-equivariant continuous map

$$\phi : \partial\Gamma \longrightarrow \mathbb{P}^1_{\mathbf{C}}, \quad \phi \circ \gamma = \pi(\gamma) \circ \phi$$

that is a homeomorphism between the topological circle $\partial\Gamma$ and the Jordan curve on the sphere $\mathbb{P}^1_{\mathbf{C}}$ formed by the limit set $L_{\pi(\Gamma)}$ of $\pi(\Gamma)$.

Proposition 3.2: *Let $q \in \mathscr{QF}(\Sigma) \setminus \mathscr{T}(\Sigma)$ be given by a quasi-Fuchsian representation $\pi : \Gamma \longrightarrow \mathrm{PSL}_2(\mathbf{C})$. Then there exists an isometrically embedded hyperbolic plane $\mathbf{H}^2 \subset \mathbf{H}^3$ and a sequence $\xi_1, \xi_2, \dots \to \xi_* \in \partial\Gamma$ whose cyclic order with respect to the circle $\partial\Gamma$ is*

$$\xi_1, \xi_2, \xi_3, \xi_4, \dots, \xi_*$$

and whose images $\phi(\xi_n) \in \mathbb{P}^1_{\mathbf{C}}$ lie on the boundary circle $\partial\mathbf{H}^2$ in the following cyclic order:

$$\phi(\xi_1), \phi(\xi_3), \phi(\xi_5), \dots, \phi(\xi_*), \dots, \phi(\xi_6), \phi(\xi_4), \phi(\xi_2).$$

In particular, we have:

- *(ξ_1, ξ_4) and (ξ_2, ξ_3) are unlinked and aligned in $\partial\Gamma$, while $(\phi(\xi_1), \phi(\xi_4))$ and $(\phi(\xi_2), \phi(\xi_3))$ are linked in $\partial\mathbf{H}^2$.*
- *(ξ_1, ξ_3) and (ξ_2, ξ_4) are linked in $\partial\Gamma$, while $(\phi(\xi_1),\phi(\xi_3))$ and $(\phi(\xi_2),\phi(\xi_4))$ are unlinked and aligned in $\partial\mathbf{H}^2$.*

Proof. Fix an element $\gamma \in \Gamma$ with $\pm\,\mathrm{tr}_q(\gamma) \in \mathbf{C} \setminus \mathbf{R}$ as in Lemma 3.1. Note that γ must be hyperbolic, and denote by ξ_* the attracting point $\gamma^+ \in \partial\Gamma$. At the same time $\pi(\gamma) \in \mathrm{PSL}_2(\mathbf{C})$ is loxodromic with an attracting point $\phi(\gamma^+)$. Identifying $\mathbb{P}^1_{\mathbf{C}}$ with $\mathbf{C} \cup \{\infty\}$ and replacing $\pi : \Gamma \longrightarrow \mathrm{PSL}_2(\mathbf{C})$ by an appropriate conjugate we may assume $\phi(\gamma^+) = \infty$ and $\phi(\gamma^-) = 0$. Then the action of $\pi(\gamma)$ on $\mathbf{C}$ is given by the linear map

$$z \mapsto (\lambda e^{2\pi i\theta}) \cdot z \quad \text{with } \lambda > 1,\ \theta \in \mathbf{R} \setminus \mathbf{Z}.$$

Identify $\partial\Gamma \setminus \{\gamma^+\}$ with $\mathbf{R}$ so that γ^- corresponds to $0 \in \mathbf{R}$. With a slight abuse of notation we write γ and ϕ for the corresponding homeomorphism of $\mathbf{R}$, and an equivariant injective continuous map $\mathbf{R} \longrightarrow \mathbf{C}$. Note that $\gamma(0) = 0$, γ is strictly increasing on $[0, \infty)$ (and strictly decreasing on $(-\infty, 0]$), while ϕ satisfies

$$\phi(\gamma(t)) = \lambda e^{2\pi i\theta} \cdot \phi(t) \tag{3.1}$$

and

$$|\phi(t)| \to \infty \quad \text{as } |t| \to \infty.$$

Since $\phi(t) \neq 0$ for all $t \in (0, \infty)$ there exist continuous functions $r : (0, \infty) \to (0, \infty)$ and $s : (0, \infty) \to \mathbf{R}$ so that

$$\phi(t) = r(t) \cdot e^{2\pi i \cdot s(t)} \quad (t > 0).$$

Thus (3.1) implies that

$$r(\gamma^n(t)) = \lambda^n \cdot r(t), \quad s(\gamma^n(t)) = s(t) + n\Theta$$

where $\Theta \in \theta + \mathbf{Z}$. Note that the assumption that $q \in \mathscr{QF}(\Sigma) \setminus \mathscr{T}(\Sigma)$ gives $\theta \notin \mathbf{Z}$ (Lemma 3.1), implying $\Theta \neq 0$.

Fix $t_0 > 0$ and use points $t_n = \gamma^n(t_0)$, $n \in \mathbf{Z}$, to partition the ray $(0, \infty)$. Let

$$R_0 = \max\{r(t) \mid 0 \le t \le t_0\}, \quad r_0 = \min\{r(t) \mid t_0 \le t < \infty\}.$$

Then $|\phi(t)| = r(t) \le \lambda^n \cdot R_0$ for all $t \in [0, t_n]$, and $|\phi(t)| = r(t) \ge \lambda^n \cdot r_0$ for all $t \ge t_n$. We can now choose integers $n(1) < m(1) < n(2) < m(2) < \cdots$ so that

$$(m(k) - n(k)) \cdot |\Theta| > 1, \quad \lambda^{n(k+1) - m(k)} > R_0/r_0$$

for all $k \in \mathbf{N}$. The first condition guarantees that $s(t_{m(k)}) > s(t_{n(k)}) + 1$ and therefore there exist

$$\xi_k \in [t_{n(k)}, t_{m(k)}] \quad \text{with } e^{2\pi i \cdot s(\xi_k)} = (-1)^k.$$

Thus $\phi(\xi_k) = (-1)^k r(\xi_k)$ lie on the real line $\mathbf{R} \subset \mathbf{C}$ on both sides of $0 \in \mathbf{R}$ in alternating order. Since $\xi_k \leq t_{m(k)} < t_{n(k+1)} \leq \xi_{k+1}$ we also have

$$|\phi(\xi_k)| = r(\xi_k) \leq \lambda^{m(k)} \cdot R_0 < \lambda^{n(k+1)} \cdot r_0 \leq r(\xi_{k+1}) = |\phi(\xi_{k+1})|.$$

Thus the sequence $\{|\phi(\xi_k)|\}$ is monotonically increasing. In particular, we have

$$\cdots < \phi(\xi_5) < \phi(\xi_3) < \phi(\xi_1) < 0 < \phi(\xi_2) < \phi(\xi_4) < \phi(\xi_6) < \cdots$$

on $\mathbf{R} \subset \mathbf{C}$. Recalling that $\phi(\xi_*) = \infty$ we get the required cyclic order. ∎

4. Proof of Theorem A

Let us first recall two general well-known facts, one related to CAT(-1) spaces (X, d_X), and another to Gromov hyperbolic groups Γ acting on their boundary $\partial\Gamma$. We will apply them to $X = \mathbf{H}^3$ and to the surface group $\Gamma = \pi_1(\Sigma)$.

Recall that given a point $p \in \mathbf{H}^3$ and a pair of distinct boundary points $\xi \neq \eta \in \partial\mathbf{H}^3$ the following limit exists:

$$\mathrm{B}_p(\xi, \eta) = \lim_{x \to \xi,\ y \to \eta} (d_{\mathbf{H}^3}(p, x) + d_{\mathbf{H}^3}(p, y) - d_{\mathbf{H}^3}(x, y)).$$

Triangle inequality implies that $\mathrm{B}_p(\xi, \eta) \geq 0$. Crucial for our purposes is the fact that the strict inequality occurs unless p lies on the geodesic line (ξ, η):

$$\mathrm{B}_p(\xi, \eta) > 0 \quad \Longleftrightarrow \quad p \notin (\xi, \eta).$$

The second fact is a consequence of the topological transitivity of the geodesic flow on the unit tangent bundle to the surface. It can be used to show that for any $\xi \neq \eta$ in $\partial\Gamma$ there exists an infinite sequence $\{\gamma_n\}$ in Γ so that

$$\xi = \lim_{n \to \infty} \gamma_n^+, \quad \eta = \lim_{n \to \infty} \gamma_n^-$$

where $\gamma_n^-, \gamma_n^+ \in \partial\Gamma$ denote the repelling and the attracting points of $\gamma_n \in \Gamma$.

With these observations we can proceed to the proof of Theorem A. Using Proposition 3.2, let us pick (ξ_1, ξ_4) and (ξ_2, ξ_3) that are unlinked and aligned in $\partial\Gamma$ while $(\phi(\xi_1), \phi(\xi_4))$ and $(\phi(\xi_2), \phi(\xi_3))$ are linked in a copy of a hyperbolic plane $\partial\mathbf{H}^2$ contained in the hyperbolic space $\mathbf{H}^3$. Let $p \in \mathbf{H}^3$ denote the intersection of the linked geodesic lines $(\phi(\xi_1), \phi(\xi_4))$ and $(\phi(\xi_2), \phi(\xi_3))$. Since these two geodesic lines are distinct, $p \notin (\phi(\xi_2), \phi(\xi_4))$, and therefore, using the first fact, we obtain

$$\delta = \mathrm{B}_p(\phi(\xi_2), \phi(\xi_4)) > 0.$$

We can now use the second fact, and find sequences $\{a_n\}$ and $\{b_n\}$ in Γ, so that

$$a_n^- \longrightarrow \xi_1, \quad a_n^+ \longrightarrow \xi_4, \quad b_n^- \longrightarrow \xi_2, \quad b_n^+ \longrightarrow \xi_3.$$

Denote $A_n = \pi(a_n)$ and $B_n = \pi(b_n)$ the corresponding elements in $\mathrm{PSL}_2(\mathbf{C})$. Note that $\phi(a_n^\pm) = A_n^\pm$ and $\phi(b_n^\pm) = B_n^\pm$ are the repelling/attracting points in $\partial\mathbf{H}^3$. Upon replacing a_n, b_n by their powers, we may assume that

$$\ell_{\mathbf{H}^3}(A_n) \longrightarrow \infty, \quad \ell_{\mathbf{H}^3}(B_n) \longrightarrow \infty.$$

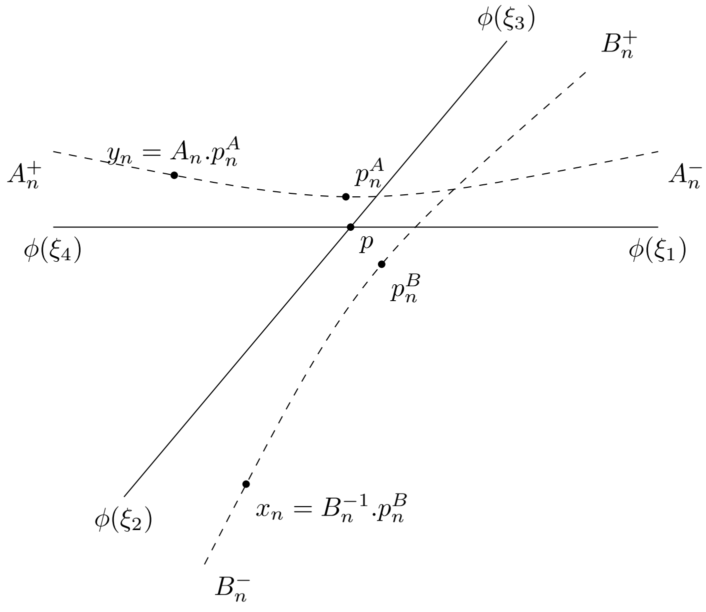
Let p_n^A denote the projection of point p to the geodesic line $(\phi(a_n^-), \phi(a_n^+))$ which is the axis Ax_{A_n} in $\mathbf{H}^3$. Since $\phi : \partial\Gamma \longrightarrow \partial\mathbf{H}^3$ is continuous,

$$A_n^- = \phi(a_n^-) \longrightarrow \phi(\xi_1) \quad \text{and} \quad A_n^+ = \phi(a_n^+) \longrightarrow \phi(\xi_4).$$

This implies

$$d_{\mathbf{H}^3}(p_n^A, p) \longrightarrow 0.$$

Similarly, denoting by $p_n^B \in \mathbf{H}^3$ the projection of p to the geodesic line $(\phi(\xi_2), \phi(\xi_3))$ which is the axis Ax_{B_n} in $\mathbf{H}^3$, we get $d_{\mathbf{H}^3}(p_n^B, p) \longrightarrow 0$.



Now consider the points $x_n = B_n^{-1}.p_n^B$ and $y_n = A_n.p_n^A$. Since $p_n^b = B^n.x_n$ and x_n are on the axis Ax_{B_n} of B_n we have $d_{\mathbf{H}^3}(p_n^b, x_n) = \ell_{\mathbf{H}^3}(B_n)$ and

$$|d_{\mathbf{H}^3}(p, x_n) - \ell_{\mathbf{H}^3}(B_n)| \leq d_{\mathbf{H}^3}(p, p_n^B) \longrightarrow 0. \tag{4.1}$$

Similarly,

$$|d_{\mathbf{H}^3}(p, y_n) - \ell_{\mathbf{H}^3}(A_n)| \leq d_{\mathbf{H}^3}(p, p_n^A) \longrightarrow 0. \tag{4.2}$$

Hence

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} A_n^+ = \phi(\xi_2), \quad \lim_{n\to\infty} y_n = \lim_{n\to\infty} B_n^- = \phi(\xi_4).$$

Therefore

$$\lim_{n\to\infty}(d_{\mathbf{H}^3}(x_n, p) + d_{\mathbf{H}^3}(p, y_n) - d_{\mathbf{H}^3}(x_n, y_n)) = \mathrm{B}_p(\phi(\xi_2), \phi(\xi_4)) = \delta > 0. \tag{4.3}$$

We also have

$$\begin{aligned} d_{\mathbf{H}^3}((A_n B_n).x_n, y_n) &= d_{\mathbf{H}^3}(A_n.p_n^B, y_n) \\ &= d_{\mathbf{H}^3}(A_n.p_n^B, A_n.p_n^A) = d_{\mathbf{H}^3}(p_n^B, p_n^A) \\ &\leq d_{\mathbf{H}^3}(p_n^B, p) + d_{\mathbf{H}^3}(p, p_n^A) \longrightarrow 0. \end{aligned}$$

Using (4.1), (4.2), (4.3) we deduce

$$\lim_{n\to\infty}(\ell_{\mathbf{H}^3}(A_n) + \ell_{\mathbf{H}^3}(B_n) - d_{\mathbf{H}^3}(A_n B_n.x_n, x_n)) = \delta.$$

Since $\ell_{\mathbf{H}^3}(A_n B_n) \leq d_{\mathbf{H}^3}(A_n B_n.x_n, x_n)$, it follows that

$$\liminf_{n\to\infty}(\ell_{\mathbf{H}^3}(A_n) + \ell_{\mathbf{H}^3}(B_n) - \ell_{\mathbf{H}^3}(A_n B_n)) \geq \delta.$$

The latter fact can be rewritten as

$$\liminf_{n\to\infty}(\ell_q(a_n) + \ell_q(b_n) - \ell_q(a_n b_n)) \geq \delta.$$

Recall that (ξ_1, ξ_4) and (ξ_2, ξ_3) are unlinked and aligned in $\partial\Gamma$, and are approximated by (a_n^-, a_n^+) and (b_n^-, b_n^+) respectively. Thus, we can find $k \in \mathbf{N}$ large enough, so that the pair of elements $a = a_k$, $b = b_k$ satisfy

$$\ell_q(a) + \ell_q(b) - \ell_q(ab) > \frac{1}{2}\delta$$

while (a^-, a^+) and (b^-, b^+) are unlinked and aligned. By Theorem 2.1 the latter condition implies that for every $g \in \mathscr{R}(\Sigma)$ we have

$$\ell_g(a) + \ell_g(b) - \ell_g(ab) < 0.$$

Thus

$$\frac{\ell_q(a)+\ell_q(b)}{\ell_q(ab)} : \frac{\ell_g(a)+\ell_g(b)}{\ell_g(ab)} > \frac{\ell_q(a)+\ell_q(b)}{\ell_q(ab)} > 1+\frac{\delta}{2\ell_q(ab)}.$$

We deduce that for every $g \in \mathscr{R}(\Sigma)$ we get

$$\rho_{\mathrm{Lip}}(\delta_q, \delta_g) > \log(1+\frac{\delta}{2\ell_q(ab)}) > 0.$$

This completes the proof of Theorem A.

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