SOME ERGODIC PROPERTIES OF METRICS ON HYPERBOLIC GROUPS

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ABSTRACT. Let Γ be a non-elementary Gromov hyperbolic group, and $\partial\Gamma$ denote its Gromov boundary. Consider Γ-invariant proper, quasi-convex metric d on Γ, and the associated Patterson–Sullivan measure class $[\nu^{\mathrm{PS}}]$ on $\partial\Gamma$, and its square $[\nu^{\mathrm{PS}} \times \nu^{\mathrm{PS}}]$ on $\partial^2\Gamma$ – the space of distinct pairs of points on the boundary. We study ergodicity properties of the Γ-actions on $(\partial\Gamma, [\nu^{\mathrm{PS}}])$ and on $(\partial^2\Gamma, [\nu^{\mathrm{PS}} \times \nu^{\mathrm{PS}}])$.

1. Introduction and Statement of the main results

Closed Riemannian manifolds of strictly negative curvature have been extensively studied in geometry and dynamics. The geodesic flow on the unit tangent bundle to such a manifold has many invariant probability measures, among which there is a unique one of maximal entropy. This measure, constructed independently by R.Bowen and G.A.Margulis is closely related to the Patterson–Sullivan measures on the boundary of the universal covering of the manifold, acted upon by the fundamental group of the manifold.

In this paper we consider a broader class of dynamical systems $\Gamma \curvearrowright (\partial\Gamma, [\nu^{\mathrm{PS}}])$, where Γ is a Gromov hyperbolic group, $\partial\Gamma$ is its boundary, and $[\nu^{\mathrm{PS}}]$ is a Patterson–Sullivan measure class associated to a geometry on Γ taken from a large family that includes geometric context as above, word metrics, Green metrics etc. (see Setup 1.1 below). We shall prove several ergodicity properties in this broader context, generalizing some known results and obtaining some results that are new even in the context of negatively curved manifolds (Theorem 1.6, Corollaries 1.7 and 1.8).

Setup 1.1 (Coarse-geometric framework)**.**
Let Γ be a non-elementary Gromov-hyperbolic group, acting faithfully on its boundary $\partial\Gamma$. Consider a proper and cocompact isometric Γ-action on a quasi-convex Gromov-hyperbolic metric space (M, d_M), and the left-invariant pseudo-metric d on Γ, defined by restricting d_M to a Γ-orbit of some point $o \in M$:

$$d(g_1, g_2) := d_M(g_1 o, g_2 o) \qquad (g_1, g_2 \in \Gamma).$$

Note that different points $o, o' \in M$ define pseudo-metrics d, d' that differ by a bounded amount: $|d(g_1, g_2) - d'(g_1, g_2)| \leq 2d_M(o, o')$.

We denote by D_Γ the set of all possible pseudo-metrics $d : \Gamma \times \Gamma \to [0, \infty)$ obtained from such isometric Γ-actions, and let $\mathcal{D}_\Gamma = D_\Gamma / \sim$ be the space of equivalence classes of such pseudo-metrics, where $d \sim d'$ if $d - d'$ is bounded. We denote by $[d] \in \mathcal{D}_\Gamma$ the equivalence class of $d \in D_\Gamma$.

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Examples 1.2.
The reader can keep in mind the following main classes of examples:

(a) Let N be a closed connected Riemannian manifold of strictly negative sectional curvature, $\Gamma = \pi_1(N)$ its fundamental group, $M = \widetilde{N}$ its universal covering, and d_M on M the Riemannian metric lifted from the given Riemannian structure on N. We shall refer to these as **geometric examples**. These well studied objects motivate our general discussion.
(b) Non-elementary Gromov hyperbolic groups Γ, equipped with a word metric d_S associated with a choice of a finite symmetric generating set S for Γ.
(c) Convex cocompact group actions on proper $\mathrm{CAT}(-1)$ spaces. For example, quasi-fuchsian embedding of a surface group in $\mathrm{PSL}_2(\mathbb{C}) = \mathrm{Isom}_+(\mathbf{H}^3)$.
(d) Green metric d_μ associated with a symmetric, finitely supported, generating probability measure μ on any Gromov hyperbolic group Γ (see [4]).

Let Γ and (M, d_M) be as in Setup 1.1. Throughout this paper we shall use the following constructions that depend only on the equivalence class $[d] \in \mathcal{D}_\Gamma$ of $d \in D_\Gamma$. Denote by δ_Γ the **growth exponent** for (Γ, d), given by

$$\delta_\Gamma := \lim_{R\to\infty} \frac{1}{R} \log \# \{g \in \Gamma \mid d_M(go, o) < R\} . \tag{1.1}$$

The limit exists, does not depend on the choice of the base point $o \in M$, or the representative $d \in [d]$, and (since we assume Γ to be non-elementary) is strictly positive: $\delta_\Gamma > 0$ (cf. [8]).

Example 1.3. In the geometric Example 1.2.(a), δ_Γ coincides with the *topological entropy* of the geodesic flow on the unit tangent bundle T^1N; this values is achieved by the Kolmogorov-Sinai entropy of a unique invariant probability measure – the *Bowen–Margulis measure*. In the convex cocompact Example 1.2.(c), δ_Γ is usually referred to as the *critical exponent* of Γ (cf. [17]).

Generalizing the Patterson–Sullivan theory from the geometric context, one can define analogous measures in the coarse-geometric setting as weak-* limits, $s \searrow \delta_\Gamma$, of the probability measures μ_s on the compactification $\overline{M} = M \sqcup \partial M$, where μ_s are given by

$$\mu_s := \frac{1}{\sum_{h\in\Gamma} e^{-s\cdot d_M(ho,o)}} \cdot \sum_{g\in\Gamma} e^{-s\cdot d_M(go,o)} \cdot \mathrm{Dirac}_{go}. \tag{1.2}$$

It is known that any weak-* limit, $\mu = \lim_{i\to\infty} \mu_{s_i}$ as $s_i \searrow \delta_\Gamma$, is supported on the boundary $\partial M = \partial\Gamma$, has no atoms, and any two such limits μ and μ' are mutually equivalent with uniformly bounded Radon-Nikodym derivatives $\mathrm{d}\mu'/\mathrm{d}\mu \in L^\infty(\mu)$ (cf. [8]). For general $d \in \mathcal{D}_\Gamma$, one cannot claim uniqueness for the limit measures, but any weak-* limit will do for our purposes, as we are interested only in the *measure class* of such measures. This measure class, denoted $[\nu^{\mathrm{PS}}]$, is Γ-invariant. The square of this measure class $[\nu^{\mathrm{PS}} \times \nu^{\mathrm{PS}}]$ is supported on the space

$$\partial^2\Gamma := \left\{ (\xi, \eta) \in (\partial\Gamma)^2 \mid \xi \neq \eta \right\}$$

of distinct pairs of points on the boundary of Γ. This space is locally compact, but is not compact.

In the geometric context of Example 1.2.(a), $\partial^2\Gamma$ can be identified with the space $T^1M/\mathbb{R}$ of unparametrized geodesic lines in T^1M (here $M = \widetilde{N}$), and its extension by $\mathbb{R}$ can be identified with the parametrized geodesic lines, and thereby with the unit tangent bundle T^1M itself. In this context the Γ-action on $\partial^2M = \partial^2\Gamma$ preserves an infinite Radon measure, often called Bowen–Margulis–Sullivan current. This action is ergodic; this is directly related to the ergodicity of the geodesic flow on T^1N equipped with the Bowen–Margulis measure, which is usually proved via Hopf argument.

In the general coarse-geometric context of Setup 1.1, we do not have a precise analogue for the geodesic flow on T^1N, but can consider the Γ-action on the boundary $(\partial\Gamma, [\nu^{\mathrm{PS}}])$ and its double $(\partial^2\Gamma, [\nu^{\mathrm{PS}} \times \nu^{\mathrm{PS}}])$, and expect them to be ergodic. This is established in the following theorem; along the way an analogue of Bowen–Margulis–Sullivan current is constructed, denoted m^{BMS} below.

Theorem 1.4 (Double Ergodicity)**.**
The diagonal Γ-action on $(\partial^2\Gamma, [\nu^{\mathrm{PS}} \times \nu^{\mathrm{PS}}])$ is ergodic. In particular, the Γ-action on $(\partial\Gamma, [\nu^{\mathrm{PS}}])$ is ergodic. The measure class $[\nu^{\mathrm{PS}} \times \nu^{\mathrm{PS}}]$ contains a unique (up to a certain well defined scaling) infinite Radon measure m^{BMS}.

The following result is an ergodic theorem for the infinite measure preserving action $\Gamma \curvearrowright (\partial^2\Gamma, m^{\mathrm{BMS}})$. It is formulated in purely geometric terms, using the **almost Busemann cocycle** (see §2.C):

$$\sigma(g,\xi) := \limsup_{x\to\xi}\, (d_M(g^{-1}o, x) - d_M(o,x)) \qquad (\xi \in \partial\Gamma,\ g \in \Gamma).$$

Theorem 1.5 (Ergodic Theorem)**.**
For any $f \in L^1(\partial^2\Gamma, m^{\mathrm{BMS}})$ for m^{BMS}-a.e. $(\xi,\eta) \in \partial^2\Gamma$ one has

$$\lim_{T\to\infty} \frac{1}{T} \cdot \sum_{\{g\in\Gamma \ \mid\ \sigma(g,\xi)\in[0,T]\}} f(g\xi, g\eta) = \int_{\partial^2\Gamma} f \,\mathrm{d}m^{\mathrm{BMS}}.$$

The same limit is obtained if the condition $\sigma(g,\xi) \in [0,T]$ is replaced by

$$\sigma(g,\eta) \in [0,T], \qquad \text{or by} \qquad \frac{1}{2}(\sigma(g,\xi) - \sigma(g,\eta)) \in [0,T].$$

Consider the above formula for a continuous function with compact support $f \in C_c(\partial^2\Gamma)$. Note that while for sufficiently large T the set $\{g \in \Gamma \mid \sigma(g,\xi) \in [0,T]\}$ is infinite, only finitely many elements from this set contribute non-zero summands $f(g\xi, g\eta)$. Moreover, the sums grow linearly in T, so changing σ in a bounded way has no effect on the limit. This explicit formula illustrates how the BMS measure m^{BMS} can be directly derived from the metric d on Γ, and that it depends only on the class $[d] \in \mathcal{D}_\Gamma$ of equivalent metrics.

Double ergodicity stated in Theorem 1.4 can be deduced also from the following finer ergodicity property (see §2.A).

Theorem 1.6.
Given a probability measure $\nu \in [\nu^{\mathrm{PS}}]$ in the Patterson–Sullivan class, the projection

$$\mathrm{pr}_i : (\partial\Gamma \times \partial\Gamma, \nu \times \nu) \longrightarrow (\partial\Gamma, \nu), \qquad \mathrm{pr}_i(\xi_1, \xi_2) = \xi_i \qquad (i = 1, 2)$$

are relatively (SAT), and therefore are relatively metrically ergodic. In particular, the diagonal Γ action on $(\partial\Gamma \times \partial\Gamma, \nu \times \nu)$ is metrically ergodic.

The notions that appear in the above result are defined and discussed in §2.A. Here we state the following consequence.

Corollary 1.7 (Weak Mixing)**.**
The action $\Gamma \curvearrowright (\partial^2\Gamma, m^{\mathrm{BMS}})$ is weakly mixing, in the sense that for any ergodic p.m.p. Γ-space (Ω, ω) the diagonal Γ-action on $(\partial^2\Gamma \times \Omega, m^{\mathrm{BMS}} \times \omega)$ is ergodic.

The weakly mixing, hence ergodic, Γ-action on $(\partial^2\Gamma, m^{\mathrm{BMS}})$ can be extended to the Γ-action on $\partial^2\Gamma \times \mathbb{R}$ preserving the infinite measure $m^{\mathrm{BMS}} \times \mathcal{L}$, where $\mathcal{L}$ is the Lebesgue measure on $\mathbb{R}$ (see §3.B). However this extension is no longer ergodic. In fact, the Γ-action admits a finite measure fundamental domain; the scaling of m^{BMS}, mentioned in Theorem 1.4, is chosen to ensure that this fundamental domain has measure one. The measure space $(\partial^2\Gamma \times \mathbb{R}, m^{\mathrm{BMS}} \times \mathcal{L})$ with the measure-preserving action of $\Gamma \times \mathbb{R}$ defines the $\mathbb{R}$-flow, denoted $\phi^{\mathbb{R}}$, on the quotient probability space

$$(X, \mu^{\mathrm{BM}}) := (\partial^2\Gamma \times \mathbb{R}, m^{\mathrm{BMS}} \times \mathcal{L})/\Gamma.$$

This is a measurable analogue of the geodesic flow on the unit tangent bundle T^1N equipped with the Bowen–Margulis measure (hence the notation μ^{BM}) in the case of the geodesic flow on negatively curved manifolds.

In the general case, the flow $(X, \mu^{\mathrm{BM}}, \phi^{\mathbb{R}})$ is ergodic; this follows from Theorem 1.4. This extends the ergodicity results that are well known in the geometric Examples 1.2.(a). However, in contrast to the geometric examples, the flow $(X, \mu^{\mathrm{BM}}, \phi^{\mathbb{R}})$ may fail to be mixing in the general setting. In fact, in the word metric Examples 1.2.(b) the flow $(X, \mu^{\mathrm{BM}}, \phi^{\mathbb{R}})$ has a rotation quotient.

The flow $(X, m, \phi^{\mathbb{R}})$ comes with a well defined cohomology class of measurable cocycles

$$c : \mathbb{R} \times X \longrightarrow \Gamma.$$

Therefore, given any ergodic probability measure preserving action $\Gamma \curvearrowright (\Omega, \omega)$ one can define the **induced flow** $\phi_c^{\mathbb{R}}$ on $X \times \Omega$ by $\phi_c^t(x, p) = (\phi^t(x), c(t, x).p)$. This flow can also be constructed as the $\mathbb{R}$-flow on the quotient

$$(\partial^2\Gamma \times \mathbb{R} \times \Omega, m^{\mathrm{BMS}} \times \mathcal{L} \times \omega)/\Gamma.$$

Corollary 1.8.
Let $(X, \mu^{\mathrm{BM}}, \phi^{\mathbb{R}})$ be the measurable geodesic flow associated to $[d] \in \mathcal{D}_\Gamma$, let $\Gamma \curvearrowright (\Omega, \omega)$ be an ergodic measure-preserving action on a probability space. Then the induced flow $\phi_c^{\mathbb{R}}$ on $(X \times \Omega, \mu^{\mathrm{BM}} \times \omega)$ is ergodic.

If Γ is a uniform (i.e. cocompact) lattice in the rank-one simple Lie group $G = \mathrm{Isom}(\mathbf{H}_K^n)$ (where $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$ with $n = 2$) and $M = \mathbf{H}_K^n$ is its associated symmetric space, then one can deduce the above result using Moore ergodicity theorem. In this case the flow $\phi^{\mathbb{R}}$ is Bernoulli and the induced one is a K-flow, cf. [9], [11]. However, Corollary 1.8 seems to be new even for the geodesic flow on negatively curved manifolds that are not locally symmetric.

Organization of this paper. Section 2 contains some preliminaries: a discussion of abstract ergodicity properties in §2.A, our notations and conventions for Gromov hyperbolic geometries in §2.B, and a construction of an auxiliary topological almost geodesic flow in §2.C.

Section 3 describes the measurable constructions, that include the Γ-invariant measure m^{BMS}, and the geodesic flow on $\partial^2\Gamma \times \mathbb{R}/\Gamma$.

The goal of Section 5 is to prove Theorem 1.4 using Theorem 1.5 and an analogue of Hopf argument.

Section **??** contains the proof of Theorem 1.6, which uses a Lebesgue differentiation argument. This gives an alternative proof for Theorem 1.4. Corollaries 1.7 and 1.8 are deduced from Theorem 1.6.

Some remarks. This paper continues [10] and fixes two flaws in the latter. First, in the definition of the coarse-geometric context it is essential to explicitly require quasi-convexity (as in framework 1.1), because it is not implied by being quasi-isometric to a word metric (counter example: $d' = d + \sqrt{d}$ where d is a quasi-convex metric).

Secondly, it was wrongly assumed in [10] that the double ergodicity (Theorem 1.4) was known in the broad coarse-geometric context. One of the motivations for the present paper was to close this gap, adding along the way some details claimed in [10], such as the existence of the finite measure fundamental domain (Proposition 3.5).

The present paper was motivated by the beautiful work of Garncarek [12], where the double ergodicity (as in Theorem 1.4) is needed for the classification of the irreducible unitary representation $\Gamma \to \mathcal{U}(L^2(\partial\Gamma, [\nu^{\mathrm{PS}}]))$.

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2. Preliminaries

2.A. Notions of Ergodicity.

In this subsection we describe several notions of ergodicity for non-singular group actions, that are discussed in [2], [1], and in the forthcoming [3]; see also [14]. The discussion applies to any locally compact secondly countable group, but we shall focus here on the case of a general countable discrete group Γ.

Given a probability measure μ on a standard Borel space $(X, \mathcal{X})$ we denote by $[\mu]$ the measure class of μ, i.e. all probability measures μ' on $(X, \mathcal{X})$ with $\mu' \sim \mu$. A measurable action $\Gamma \times X \to X$ on a standard probability space (X, μ) is **measure-class-preserving** (or **non-singular**) if $g_*\mu \sim \mu$, i.e. if $[g_*\mu] = [\mu]$ for every $g \in \Gamma$. We shall also use the notation $\Gamma \curvearrowright (X, [\mu])$ and consider $(X, [\mu], \Gamma)$ as a **measure class preserving system**. Let $\Gamma \curvearrowright (X, [\mu])$ and $\Gamma \curvearrowright (Y, [\nu])$ be two measure-class-preserving Γ-actions. A **quotient** map is a measurable map $p : X \to Y$ with $p_*\mu \sim \nu$ so that $p \circ g = g \circ p$ μ-a.e. on X for every $g \in \Gamma$. A measurable set $A \subset X$ is called Γ-**invariant** if $\mu(gA \triangle A) = 0$ for every $g \in \Gamma$.

Definition 2.1. A measure-class-preserving action $\Gamma \curvearrowright (X, [\mu])$ is said to be:

- **ergodic** if the only Γ-invariant measurable subsets $A \subset X$ are null or conull: $\mu(A) = 0$ or $\mu(A^c) = 0$.
- **weakly mixing** if for any ergodic p.m.p. action $\Gamma \curvearrowright (\Omega, \omega)$ the diagonal Γ-action on $(X \times \Omega, \mu \times \omega)$ is ergodic.
- **metrically ergodic** if given any separable metric space (S, d) and a homomorphism $\pi : \Gamma \to \mathrm{Isom}(S, d)$, the only a.e. Γ-equivariant measurable maps $F : X \to S$ are constant ones.
- **strongly almost transitive** (SAT) if for any measurable $A \subset X$ with $\mu(A) > 0$ and any $\epsilon > 0$, there is $g \in \Gamma$ with $\mu(gA) > 1 - \epsilon$.

For a Γ-quotient map $p : (X, [\mu]) \to (Y, [\nu])$, we say that the quotient map p is

- **relatively ergodic** if the only Γ-invariant measurable sets $A \subset X$ are, up to null sets, pull-backs of Γ-invariant measurable subsets $B \subset Y$.
- **relatively metrically ergodic**, if for any measurable family $\{(S_y, d_y)\}_{y \in Y}$ of (separable) metric spaces, with a measurable family
$$\{\pi_y(g) : (S_y, d_y) \longrightarrow (S_{gy}, d_{gy})\} \qquad (g \in \Gamma,\ y \in Y)$$
of isometries with $\pi_y(gh) = \pi_{hy}(g) \circ \pi_y(h)$, the only a.e. Γ-equivariant measurable maps $\{F(x) \in S_{p(x)}\}_{x \in X}$ are pull-backs $F = f \circ p$ of measurable Γ-equivariant family $\{f(y) \in S_y\}_{y \in Y}$.
- **relatively (SAT)** if for any measurable set $A \subset X$ with $\mu(A) > 0$ and $\epsilon > 0$, there is $g \in \Gamma$ and a positive ν-measure subset $B \subset Y$ so that
$$\mu(A \cap gA \cap p^{-1}(B)) > (1 - \epsilon) \cdot \mu(p^{-1}(B)).$$

There is also a notion of relative weak mixing, but we shall not use it here. We point out that all these concepts depend only on the relevant measure-classes $[\mu]$, $[\nu]$. Let us record the following implications (see [3], or [1] for more details). For measure-class-preserving group actions one has

$$\text{(SAT)} \Longrightarrow \text{(MErg)} \Longrightarrow \text{(WM)} \Longrightarrow \text{(Erg)}, \tag{2.1}$$

and for equivariant quotient map

$$\text{(rSAT)} \Longrightarrow \text{(rMErg)} \Longrightarrow \text{(rWM)} \Longrightarrow \text{(rErg)} \tag{2.2}$$

where (SAT), (Erg), (MErg), (WM) denote strongly almost transitivity, ergodicity, metric ergodicity, weak mixing; and (rSAT), (rErg), (rMErg), (rWM) the corresponding relative notions. Let us record a direct proof of the following.

Lemma 2.2.
If measure-class preserving actions $\Gamma \curvearrowright (X, [\mu])$, $\Gamma \curvearrowright (Y, [\nu])$ are such that the projections
$$\mathrm{pr}_1 : X \times Y \to X, \qquad \mathrm{pr}_2 : X \times Y \to Y$$
are relatively (SAT), then the diagonal Γ-action on $(X \times Y, [\mu \times \nu])$ is metrically ergodic. In particular, $(X \times Y, [\mu \times \nu], \Gamma)$ is ergodic and weakly mixing.

Proof. Let (S, d) be a separable metric space, a homomorphism $\pi : \Gamma \to \mathrm{Isom}(S, d)$ and $F : X \times Y \to S$ a measurable Γ-equivariant map. Using the assumption that $X \times Y \to X$ is (rSAT), we shall prove that F descends to a measurable Γ-equivariant map $F_1 : X \to S$, i.e. almost everywhere $F(x, y) = F_1(x)$. Applying a similar argument to $X \times Y \to Y$, we will conclude that $F(x, y)$ is a.e. constant $s_0 \in S$. Since F is equivariant, s_o must be $\pi(\Gamma)$-fixed. This will show that $X \times Y$ is (MErg).

Let us now show that $F : X \times Y \to S$ descends to $X \to S$. By replacing the metric d by $\min(d, 1)$, we may assume that $d \leq 1$. Define a measurable function $\phi : X \to \mathbb{R}_+$ by

$$\phi(x) := \int_Y \int_Y d(F(x, y_1), F(x, y_2)) \, \mathrm{d}\nu(y_1) \, \mathrm{d}\nu(y_2).$$

It suffices to show that μ-a.e. $\phi(x) = 0$. If not, there is $r > 0$ so that the set

$$E := \{x \in X \mid \phi(x) > r\} \qquad \text{has} \qquad \mu(E) > 0.$$

Take $\epsilon, \rho = r/5$. Since S can be covered by countably many ρ-balls, it follows that there is $s \in S$ so that

$$A := \{(x, y) \in E \times Y \mid F(x, y) \in \mathrm{B}(s, \rho)\} \qquad \text{has} \qquad \mu \times \nu(A) > 0.$$

For a set $C \subset X \times Y$ and $x \in X$ we denote $C_x := \{y \in Y \mid (x, y) \in C\}$. It follows from the assumption that $X \times Y \to X$ is (rSAT) that there is $g \in \Gamma$ and subsets $A' \subset A$ and $E' \subset E$ of positive measure, so that $gA' \subset E \times Y$ and $\nu(gA'_x) > 1 - \epsilon$ for all $x \in E' \subset E$. However, this is a contradiction, because

$$gA' \subset gA \subset F^{-1}(\pi(g) \mathrm{B}(s, \rho)) = F^{-1}(\mathrm{B}(\pi(g)s, \rho))$$

and so for $x \in E'$ one can estimate

$$\phi(x) \leq 2\rho \cdot 1 + 1 \cdot 2\epsilon < r.$$

This completes the proof that $\Gamma \curvearrowright (X \times Y, [\mu \times \nu])$ is (MErg).

This property clearly implies ergodicity by considering two point metric space $S = \{0, 1\}$ with the trivial Γ-action.

Let us show weak-mixing. Consider an ergodic p.m.p. action $\Gamma \curvearrowright (\Omega, \omega)$. Then the unitary Γ-representation π on $L^2_0(\Omega, \omega) = L^2(\Omega, \omega) \ominus \mathbb{C}$ has no non-zero invariant vectors. Take S to be the unit sphere of $L^2_0(\Omega, \omega)$. Let $E \subset X \times Y \times \Omega$ be a Γ-invariant subset. For $(x, y) \in X \times Y$ denote $E_{x,y} := \{z \in \Omega \mid (x, y, z) \in E\}$ and note that the measurable function $X \times Y \to [0, 1]$, defined by $(x, y) \mapsto \omega(E_{x,y})$, is Γ-invariant. By ergodicity there is a constant c so that a.e. $\omega(E_{x,y}) = c$. If $c \neq 0, 1$ then $F(x, y) = (1_{E_{x,y}} - c)/\sqrt{c \cdot (1 - c)}$ is an equivariant map to S, which is impossible. Hence $c = 0$ or $c = 1$, and so E is null or conull in $X \times Y \times \Omega$. This completes the proof of the lemma. □

2.B. **Gromov hyperbolic spaces.**

Let (M, d_M) be a metric space. The *Gromov product* of $x, y \in M$ relative to $z \in M$ is defined by

$$\langle x \mid y \rangle_z := \frac{1}{2} \left(d_M(x, z) + d_M(y, z) - d_M(x, y) \right).$$

Note that $\langle x \mid y \rangle_z \geq 0$. A metric space (M, d) is *Gromov hyperbolic* if there exists a constant C so that

$$\langle x \mid z \rangle_o \geq \min \left(\langle x \mid y \rangle_o, \langle y \mid z \rangle_o \right) - C \tag{2.3}$$

for all $x, y, z, o \in M$. One consequence of this is that for some D_0 and any $D > D_0$ for any $x_1, x_2, x_3 \in M$ the following set of *almost-medians*

$$\mathrm{QM}_D(x_1, x_2, x_3) := \left\{ m \in M \mid \langle x_i \mid x_j \rangle_m < D, \ \forall i \neq j \in \{1, 2, 3\} \right\} \tag{2.4}$$

is non-empty and has uniformly bounded diameter.

A sequence $(x_i)_{i\in\mathbb{N}}$ in M is convergent at infinity if $\langle x_i \mid x_j\rangle_o \to +\infty$ as $i,j\to\infty$ for some (hence any) $o\in M$. The *boundary* ∂M of M is defined as the set of all equivalence classes $\xi=[(x_i)]$ of sequence (x_i) in M convergent at infinity, where $(x_i)\sim(x_i')$ if $\langle x_i \mid x_i'\rangle_o \to +\infty$ as $i\to\infty$ for some (hence any) $o\in M$. The fact that this is an equivalence relation and that the definition is independent of the choice of $o\in M$ follows from the hyperbolicity condition. Denote

$$\overline{M} := M\sqcup\partial M$$

and, write $x_i\to\xi\in\overline{M}$ to express $\lim d(x_i,\xi)=0$ if $\xi\in M$, and $[(x_i)]=\xi$ if $\xi\in\partial M$. Extend the notion of Gromov product to $x,y\in\overline{M}$ and $o\in M$ by

$$\langle \xi \mid \eta\rangle_o := \inf_{x_i\to\xi,\, y_i\to\eta}\ \liminf_{i\to\infty}\langle x_i \mid y_i\rangle_o. \tag{2.5}$$

Then $\langle\xi\mid\eta\rangle_o < +\infty$ iff $\xi\neq\eta$. We denote by $\partial^2 M$ the space of distinct ordered pairs at the boundary

$$\partial^2 M := \left\{(\xi,\eta)\in(\partial M)^2 \mid \xi\neq\eta\right\}.$$

Assuming, as we will, that M is a *proper* metric space, there is a topology defined on the Gromov boundary ∂M in which it is compact, and $\overline{M}=M\sqcup\partial M$ is a compactification of M containing the latter as an open dense subset.

The topology on ∂M can also be defined by a metric (or rather a family of Hölder equivalent metrics), directly related to the Gromov product, as follows. There exists $\alpha_0<1$ so that for any $\alpha\in(\alpha_0,1)$ there is a metric of the form

$$d_{\partial\Gamma}(\xi,\eta) := \alpha^{-\langle\xi|\eta\rangle_o + b(\xi,\eta)} \qquad \text{with} \qquad b(\xi,\eta)=O(1) \tag{2.6}$$

and we set $d_{\partial\Gamma}(\xi,\xi)=0$ by convention. Hereafter we fix such a metric and α, and denote by $\mathrm{B}(\xi,r)=\{\eta\in\partial\Gamma \mid d_{\partial\Gamma}(\xi,\eta)<r\}$ the r-ball around ξ.

Define the **upper Busemann** function $\beta^*: M\times M\times\partial M\to\mathbb{R}$ by

$$\beta^*(x,y;\zeta) := \limsup_{z\to\zeta}\,(d_M(x,z)-d_M(y,z)). \tag{2.7}$$

By taking the $\liminf$ we can define a similar the lower Busemann function β_*. The hyperbolicity assumption implies that $\beta^*(x,y;\zeta)-\beta_*(x,y;\zeta)$ is uniformly bounded. Hereafter we work with the upper Busemann function β^*, but as everything will take place up to a bounded error, this choice makes no essential difference. Notice that the expression

$$\beta^*(x,z;\zeta)=\beta^*(x,y;\zeta)+\beta^*(y,z;\zeta)+O(1)$$

for $x,y,z\in M$, $\zeta\in\partial M$. So $\beta^*(-.-;\zeta)$ is an **almost-cocycle**.

Remark 2.3. Note that in many situations, including Examples 1.2.(a),(c),(d), the lower and the upper Busemann function coincide $\beta_*=\beta^*$, and this function is actually a cocycle, i.e. it satisfies:

$$\beta(x,y;\zeta)+\beta(y,z;\zeta)=\beta(x,z;\zeta) \qquad (x,y,z\in M,\ \zeta\in\partial M).$$

In these situations one can also construct a second Busemann function $B: M\times\partial^2 M\to\mathbb{R}_+$, denoted $B_x(\xi,\eta)$, with the property

$$\beta(x,y;\xi)+\beta(x,y;\eta)=B_x(\xi,\eta)-B_y(\xi,\eta) \qquad (x,y\in M,\ \xi\neq\eta\in\partial M). \tag{2.8}$$

The slight simplification of our arguments in these special situation, does not justify the restriction of generality of our discussion. Hence we proceed with the almost-cocycle β^* and with the use of Gromov product $\langle - \mid - \rangle_x$ instead of $B_x(-,-)$, because one always has

$$\beta^*(x,y;\xi) + \beta^*(x,y;\eta) = \langle \xi \mid \eta \rangle_y - \langle \xi \mid \eta \rangle_x + O(1) \tag{2.9}$$

for all $x, y \in M$ and $\xi \neq \eta \in \partial M$.

For a constant C a C-**almost-geodesic** is a map $p : I \to M$ from an interval $I \subset \mathbb{R}$ to M satisfying

$$|d_M(p(t), p(s)) - |t-s|| < C \qquad (t, s \in I).$$

If $I = [a, b]$ is a finite interval, we say that p is a C-**almost-geodesic segment** connecting $x = p(a)$ to $y = p(b)$ in M. If $I = [a, \infty)$ say that p is a C-almost-geodesic **ray** connecting $p(a) \in M$ to $\xi = \lim_{t\to\infty} p(t) \in \partial M$; similarly a C-almost-geodesic **line** $p : (-\infty, +\infty) \to M$ connects two points at the boundary $\xi = \lim_{t\to-\infty} p(t)$ and $\eta = \lim_{t\to\infty} p(t)$. We say that (M, d_M) is **quasi-convex** if there exists a constant C so that every two distinct points in $\overline{M} = M \sqcup \partial M$ can be connected by C-almost-geodesic.

2.C. **A topological geodesic almost-flow.**

In the setting of Example 1.2.(a) we have the geodesic flow action of $\mathbb{R}$ on T^1N and on $T^1\widetilde{N}$. Our goal is to construct an analogue of this geodesic flow in the more general coarse-geometric setting of Setup 1.1. We will indeed establish such a p.m.p. action in the next section. In this section we consider a topological construction. Due to the fact that in coarse-geometric framework various metric properties are well behaved only up to an additive constant, our topological construction will only be an **almost-action** (or a **coarse action**).

Remark 2.4. In [16] Mineyev constructs a topological version of geodesic flow resolving various almost-actions. We decided to avoid using this machinery, as our topological almost-geodesic flow is only an auxiliary tool needed for the measurable geodesic flow discussed below.

Let $\Gamma < \mathrm{Isom}(M, d_M)$ be as in Setup 1.1. Define $\sigma : \Gamma \times \partial\Gamma \to \mathbb{R}$ by

$$\sigma(g, \xi) := \beta^*(g^{-1}o, o; \xi) = \limsup_{x\to\xi} \left(d_M(g^{-1}o, x) - d_M(o, x)\right). \tag{2.10}$$

Notice that σ is an **almost-cocycle**, namely

$$\sigma(gh, \xi) = \sigma(g, h.\xi) + \sigma(h, \xi) + O(1)$$

for all $g, h \in \Gamma$ and $\xi \in \partial\Gamma$. Hereafter $O(1)$ represents an implicit uniformly bounded quantity.

The *rough skeleton* for the almost geodesic-flow is the space $\partial^2 M \times \mathbb{R}$. It is equipped with an action of $\Phi^{\mathbb{R}}$

$$\Phi^s(\xi, \eta, t) = (\xi, \eta, t+s),$$

the flip involution

$$(\xi, \eta, t) \;\mapsto\; (\eta, \xi, -t),$$

and an **almost-action**, denoted by a star $*:\Gamma\times\partial^2 M\times\mathbb{R}\longrightarrow\partial^2 M\times\mathbb{R}$, and defined by

$$g*(\xi,\eta,t):=\left(g\xi,g\eta,t+\frac{\sigma(g,\eta)-\sigma(g,\xi)}{2}\right). \tag{2.11}$$

Since σ is an almost-cocycle, we have

$$g*(h*(\xi,\eta,t))=\Phi^{O(1)}\left(gh*(\xi,\eta,t)\right).$$

Remark 2.5. This almost-action is actually an action if σ is a cocycle. This is the case in the geometric example, where the geodesic flow $\Phi^{\mathbb{R}}$ commutes with the flip and the Γ-action on $T^1\widetilde{N}$.

For example, in the geometric Example 1.2.(a) one has a Γ-equivariant map

$$\partial^2\widetilde{N}\times\mathbb{R}\xrightarrow{\cong}T^1\widetilde{N}\longrightarrow\widetilde{N},$$

where $T^1\widetilde{N}\longrightarrow\widetilde{N}$ is the projection to the base point. The following Proposition describes the properties of an analogous construction in the general coarse-geometric framework of Setup 1.1.

Proposition 2.6.
There exist $C,D<\infty$ such that, upon choosing a base point $o\in M$, there is a map

$$\pi:\ \partial^2 M\times\mathbb{R}\longrightarrow M$$

satisfying:

(a) $\pi(\xi,\eta;-):\ \mathbb{R}\longrightarrow M$ *is a C-almost-geodesic connecting ξ to η.*
(b) $\pi(\xi,\eta,0)\in\mathrm{QM}_D(\xi,\eta,o)$ *(see Equation (2.4) for definition).*
(c) $d\left(\pi\left(g*(\xi,\eta,t)\right),g\pi(\xi,\eta,t)\right)<C$ *for all $g\in\Gamma$.*

Proof. Fix C large enough to guarantee existence of C-almost geodesics between any two points of $\overline{M}$. Choose D large enough to ensure that for any three points $\xi,\eta,\zeta\in\overline{M}$, $\mathrm{QM}_D(\xi,\eta,\zeta)$ has non-empty intersection with any C-almost-geodesic connecting any two of the three points.

For each $(\xi,\eta)\in\partial^2 M$ choose a C-almost-geodesic $\pi(\xi,\eta,-):\mathbb{R}\to M$ connecting ξ to η, and adjust its parametrization to ensure $\pi(\xi,\eta,0)\in\mathrm{QM}_D(\xi,\eta,o)$. Thus properties (a), (b) are satisfied by construction.

To show (c), consider $g\in\Gamma$, $\xi,\eta\in\partial^2\Gamma$, $t\in\mathbb{R}$. Since g is an isometry of (M,d_M), both $g\pi(\xi,\eta,-)$ and $\pi(g\xi,g\eta,-)$ are C-almost geodesics connecting $g\xi$ to $g\eta$. Hence for some $\tau_g\in\mathbb{R}$

$$d\left(g\pi(\xi,\eta,t),\pi(g\xi,g\eta,t+\tau_g)\right)=O(1).$$

Let $p\in\mathrm{QM}_D(\xi,\eta,o)$ - an almost projection of o to the geodesic line connecting ξ to η. Then $gp\in\mathrm{QM}_D(g\xi,g\eta,go)=g\,\mathrm{QM}_D(\xi,\eta,o)$ because g is an isometry of M. Similarly, choose $q\in\mathrm{QM}_D(g\xi,g\eta,o)$. Note that $|\tau_g|=d(q,gp)+O(1)$ and the sign of τ_g is determined by the order of q,gp on ξ,η. Thus we have

$$\begin{aligned}\tau_g+O(1)&=\frac{\beta^*(o,go;g\eta)+\beta^*(go,o;g\xi)}{2}\\&=\frac{\beta^*(g^{-1}o,o;\eta)-\beta^*(g^{-1}o,o;\xi)}{2}\\&=\frac{\sigma(g,\eta)-\sigma(g,\xi)}{2}.\end{aligned}$$

This proves property (c). □

2.D. **A contraction lemma.** We shall need the following well-known geometric lemma that describes some contraction dynamics on the boundary (corresponding to the dynamics on stable/unstable foliations in hyperbolic dynamics of the geodesic flow).

Lemma 2.7.
Given a compact subset $K \subset \partial^2\Gamma$ there exists C so that if $\xi, \eta, \eta' \in \partial\Gamma$ and $g \in \Gamma$ satisfy

$$(\xi, \eta), (\xi, \eta'), (g\xi, g\eta) \in K, \qquad \sigma(g, \xi) < 0$$

then, denoting $t = -\sigma(g, \xi) > 0$, we have

$$\sigma(g, \eta), \sigma(g, \eta') \in [t - C, t + C], \qquad d_{\partial\Gamma}(g\eta, g\eta') < a^{t-C}.$$

Proof. Choose

$$p \in \mathrm{QM}_D(\xi, o, \eta), \quad p' \in \mathrm{QM}_D(\xi, o, \eta'), \quad q \in \mathrm{QM}_D(g\xi, o, g\eta).$$

These points should be thought of as approximate nearest point projections of the base point $o \in M$ to the almost-geodesics lines (ξ, η), (ξ, η'), $(g\xi, g\eta)$. Let us also choose a "nearly a projection" q' of o to $(g\xi, g\eta')$, namely $q' \in \mathrm{QM}_D(g\xi, o, g\eta')$.

Compactness of K means that $d(o, p)$, $d(o, p')$, $d(o, q)$ are bounded by some $R = R(K)$, and so also

$$d(p, p') \leq d(o, p) + d(o, p') < 2R.$$

Since g is an isometry, $g^{-1}q \in \mathrm{QM}_D(\xi, g^{-1}o, \eta)$; so is "nearly a projection" of $g^{-1}o$ to (ξ, η). As $d(g^{-1}o, g^{-1}q) = d(o, q) < R$, we have

$$\sigma(g, \xi) = -d(p, q) + O(R), \qquad \sigma(g, \eta) = d(p, q) + O(R).$$

So the almost-geodesic rays $[q, \eta)$ and $[q, \eta')$ have a common initial segment $[q, p] \approx [q, p']$ of length $t - O(R)$. The same applies to almost-geodesic rays $[g^{-1}o, \eta)$ and $[g^{-1}o, \eta')$. Thus

$$\langle g\eta \mid g\eta' \rangle_o = \langle \eta \mid \eta' \rangle_{g^{-1}o} \geq t - O(R)$$

which gives the estimate in the lemma. □

3. Measure-theoretic constructions

3.A. **Patterson–Sullivan measures.**
We keep the assumption that $\Gamma < \mathrm{Isom}(M, d)$ acts properly cocompactly on a quasi-convex Gromov hyperbolic space (M, d) as in the Setup 1.1.

Proposition 3.1.
There exists a probability measure ν on $\partial\Gamma$ such that

$$\frac{\mathrm{d}g_*^{-1}\nu}{\mathrm{d}\nu}(\xi) = e^{\delta_\Gamma \cdot \sigma(g,\xi) + O(1)}, \qquad \nu(\mathrm{B}(\xi, a^t)) = e^{\delta_\Gamma \cdot t + O(1)}.$$

for $\xi \in \partial\Gamma$ and $t > 0$. Any two such measures are equivalent and have bounded Radon–Nikodym derivatives. Denote by $[\nu^{\mathrm{PS}}]$ the common measure class.

The original works of Patterson and Sullivan for $\Gamma < \mathrm{Isom}(\mathbf{H}^n)$ was extended to strictly negative curvature (cf. [17]). There is no need for a bounded additive term $O(1)$ in these settings. In [8] Cooneart has carefully analyzed the case of word metrics (Example 1.2.(b)), but the same methods apply to our more general setting 1.1, see [4].

We shall need the fact that a PS-measure ν as above is Ahlfors regular (see [4]). As a consequence it satisfies the following version of Lebesgue differentiation:

Theorem 3.2 (Lebesgue Differentiation)**.**
Given $f \in L^1(\partial\Gamma, \nu)$ for ν-a.e. $\xi \in \partial\Gamma$:

$$\lim_{\epsilon \to 0} \frac{1}{\nu(\mathrm{B}(\xi, \epsilon))} \int_{\mathrm{B}(\xi,\epsilon)} |f(\zeta) - f(\xi)| \, \mathrm{d}\nu(\zeta) = 0.$$

3.B. **A measurable version of the geodesic flow.**
Let us now establish some further properties. We start from an analogue of Sullivan's result.

Proposition 3.3.
There exists a Γ-invariant Radon measure, denoted m^{BMS}, in the measure class $[\nu^{\mathrm{PS}} \times \nu^{\mathrm{PS}}]$ on $\partial^2 M = \partial^2\Gamma$. Moreover, m^{BMS} has the form

$$\mathrm{d}m^{\mathrm{BMS}}(\xi, \eta) = e^{F(\xi,\eta)} \, \mathrm{d}\nu(\xi) \, \mathrm{d}\nu(\eta)$$

where F is a measurable function on $(\partial^2\Gamma, [\nu^{\mathrm{PS}} \times \nu^{\mathrm{PS}}])$ satisfying

$$F(\xi, \eta) = \delta_\Gamma \cdot \langle \xi \mid \eta \rangle_o + O(1).$$

Proof.
Consider the Radon measure m_o on $\partial^2 M$ defined by

$$dm_o(\xi, \eta) = e^{\delta_\Gamma \cdot \langle \xi | \eta \rangle_o} \; \mathrm{d}\nu(\xi) \; \mathrm{d}\nu(\eta).$$

It is Γ-quasi-invariant, and for $g \in \Gamma$ the log of the Radon-Nikodym derivative satisfies

$$\begin{aligned}
&\delta_\Gamma^{-1} \cdot \log \frac{\mathrm{d}g_*^{-1} m_o}{\mathrm{d}m_o}(\xi, \eta) \\
&\quad = \langle \xi \mid \eta \rangle_{g^{-1}o} - \langle \xi \mid \eta \rangle_o + \delta_\Gamma^{-1} \cdot \log \frac{\mathrm{d}g_*^{-1}\nu}{\mathrm{d}\nu}(\xi) + \delta_\Gamma^{-1} \cdot \log \frac{\mathrm{d}g_*^{-1}\nu}{\mathrm{d}\nu}(\eta) \\
&\quad = \langle g\xi \mid g\eta \rangle_o - \langle \xi \mid \eta \rangle_o + \sigma(g, \xi) + \sigma(g, \eta) + O(1)
\end{aligned}$$

In view of (2.9) the latter is uniformly bounded over $g \in \Gamma$ and $(\xi, \eta) \in \partial^2 M$. We can now invoke the following general fact.

Lemma 3.4.
Let $\Gamma \curvearrowright X$ be a measurable action on a Borel space, and $\rho : \Gamma \times X \to \mathbb{R}$ be a Borel cocycle. Assume that ρ is pointwise bounded in Γ, i.e.

$$|\rho(g, x)| \leq C(x) < +\infty$$

Then ρ is a coboundary: $\rho(g, x) = \phi(gx) - \phi(x)$ of some Borel function $\phi : X \to \mathbb{R}$ satisfying $|\phi(x)| \leq 2C(x)$.

In particular, a uniformly bounded Borel cocycle, $\rho(g, x) = O(1)$, is a coboundary of a bounded Borel function.

Proof.
Applying $-\sup_h$ to the cocycle equation $\rho(g, x) = \rho(hg, x) - \rho(h, gx)$, and the function

$$\phi(x) = -\sup\{\rho(h, x) \mid h \in \Gamma\}$$

gives the a.e. identity $\rho(g, x) = \phi(gx) - \phi(x)$. □

Finally we set

$$F(\xi,\eta) := \delta_\Gamma \cdot \langle \xi \mid \eta \rangle_o + \phi(\xi,\eta)$$

where the function $\phi \in L^\infty(\partial^2 M, \nu \times \nu)$ is obtained from Lemma 3.4 applied to the logarithmic Radon–Nikodym cocycle

$$\log \frac{\mathrm{d}g_*^{-1} m_o}{\mathrm{d}m_o}(\xi,\eta)$$

over the Γ-action on $(\partial^2 M, [\nu^{\mathrm{PS}} \times \nu^{\mathrm{PS}}])$. This completes the proof of Proposition 3.3. $\square$

Now let us define a measurable function$\rho : \Gamma \times \partial M \to \mathbb{R}$ by

$$\rho(g,\xi) := \delta_\Gamma \cdot \log \frac{\mathrm{d}g_*^{-1}\nu}{\mathrm{d}\nu}(\xi) \tag{3.1}$$

and observe the following properties:

- ρ is a measurable cocycle: for $[\nu^{\mathrm{PS}}]$-a.e. $\xi \in \partial\Gamma$
$$\rho(gh,\xi) = \rho(g,h\xi) + \rho(h,\xi) \qquad (g,h \in \Gamma).$$
- ρ is bounded distance from σ:
$$\rho(g,\xi) = \sigma(g,\xi) + O(1)$$
- The square of ρ is a coboundary of a measurable F:
$$\rho(g,\xi) + \rho(g,\eta) = \nabla_g F(\xi,\eta)$$
where $\nabla_g F := F \circ g - F$.

The last identity gives the formula

$$\begin{aligned} \tau(g,(\xi,\eta)) &:= \frac{\rho(g,\eta) - \rho(g,\xi)}{2} \\ &= \rho(g,\eta) - \nabla_g F(\xi,\eta) \\ &= -\rho(g,\xi) + \nabla_g F(\xi,\eta). \end{aligned} \tag{3.2}$$

This is a measurable **cocycle** $\Gamma \times \partial^2\Gamma \to \mathbb{R}$, i.e. we have a.e. identity

$$\tau(gh,(\xi,\eta)) = \tau(g,(h.\xi,h.\eta)) + \tau(h,(\xi,\eta)) \qquad (g,h \in \Gamma).$$

We can use it to define a measurable Γ-action on the $\mathbb{R}$-extension $\partial^2\Gamma \times \mathbb{R}$ of the Γ-action on $(\partial^2\Gamma, m^{\mathrm{BMS}})$ by the formula

$$g \cdot (\xi,\eta,t) := (g\xi, g\eta, t + \tau(g,(\xi,\eta))). \tag{3.3}$$

Since we used an actual cocycle τ (rather than an almost cocycle), we obtain a Γ-**action**, namely we have an identity:

$$gh \cdot (\xi,\eta,t) = g \cdot (h \cdot (\xi,\eta,t)) \qquad (g,h \in \Gamma).$$

The key properties of this action are summarized in the following Proposition. We denote by $\mathcal{L}$ the Lebesgue measure on $\mathbb{R}$.

Proposition 3.5.
The above measurable Γ-action on $(\partial^2\Gamma \times \mathbb{R}, m^{\mathrm{BMS}} \times \mathcal{L})$ has the following properties:

(a) *It preserves the infinite measure $m^{\mathrm{BMS}} \times \mathcal{L}$.*
(b) *It commutes with the $\Phi^{\mathbb{R}}$-action $\Phi^s : (\xi,\eta,t) \mapsto (\xi,\eta,t+s)$.*
(c) *It commutes with the flip: $(\xi,\eta,t) \mapsto (\eta,\xi,-t)$.*

(d) *It is at essentially bounded distance from the* Γ*-almost-action (2.11); more precisely there exists a function* $s \in L^\infty(\partial^2\Gamma, m^{\mathrm{BMS}})$ *so that*

$$g \cdot (\xi, \eta, t) = \Phi^{s(\xi,\eta)}\left(g * (\xi, \eta, t)\right) \qquad (g \in \Gamma).$$

(e) *There is a measurable precompact subset* $\hat{X} \subset \partial^2\Gamma \times \mathbb{R}$ *that meets a.e.* Γ*-orbit once. There is a bound on the size of* Γ*-stabilizer of* m^{BMS}*-a.e. point in* $\partial^2\Gamma$*.*

(f) *The quotient space* $X = (\partial^2\Gamma \times \mathbb{R})/\Gamma$ *is endowed with a probability measure* μ^{BM} *and measure-preserving flow* $\phi^{\mathbb{R}}$*, satisfying* $\phi^t \circ \mathrm{pr} = \mathrm{pr} \circ \Phi^t$ *a.e., and a flip* $F : X \to X$ *so that* $F \circ \phi^t = \phi^{-t} \circ F$*.*

Remark 3.6. There is a natural identification (modulo null sets) of (X, μ^{BM}) with $\hat{X}$, equipped with the normalized restriction of the measure $m^{\mathrm{BMS}} \times \mathcal{L}$ to $\hat{X}$. In fact, hereafter we choose the scaling for m^{BMS} to ensure that

$$(m^{\mathrm{BMS}} \times \mathcal{L})(\hat{X}) = 1.$$

Remark 3.7. Let $\tau' : \Gamma \times (\partial^2\Gamma, m^{\mathrm{BMS}}) \to \mathbb{R}$ be a cocycle measurably cohomologous to τ, i.e. assume that

$$\tau'(g, (\xi, \eta)) - \tau(g, (\xi, \eta)) = \nabla_g H(\xi, \eta)$$

for some measurable $H : \partial^2\Gamma \to \mathbb{R}$. Then τ' can be used to define a Γ-action on $\partial^2\Gamma \times \mathbb{R}$, which is measurably isomorphic to (3.3) and still commutes with $\Phi^{\mathbb{R}}$ (but not necessarily with the flip). In particular, one could use (as in [10]) the measurable cocycle

$$\tau_1(g, (\xi, \eta)) := \rho(g, \xi), \qquad \text{or} \qquad \tau_2(g, (\xi, \eta)) := -\rho(g, \eta).$$

This leads to the same normalization of m^{BMS} and the same (i.e. measure-theoretically isomorphic) action of $\Gamma \times \Phi^{\mathbb{R}}$ on $(\partial^2\Gamma \times \mathbb{R}, m^{\mathrm{BMS}} \times \mathcal{L})$.

Remark 3.8. We will show below (Lemma 4.2) that the Γ-action on $(\partial^2\Gamma, m^{\mathrm{BMS}})$ is essentially free. (Note that if Γ is known to be torsion-free, then this fact follows from the finiteness of stabilizers). Essential freeness implies that $\hat{X}$ is a measurable fundamental domain for the Γ-action on $(\partial^2\Gamma \times \mathbb{R}, m^{\mathrm{BMS}} \times \mathcal{L})$, and one obtains a measurable cocycle

$$c_{\hat{X}} : \mathbb{R} \times X \to \Gamma, \qquad \text{defined by} \qquad c_{\hat{X}}(t, x) = g \quad \text{if} \quad \Phi^t(\hat{x}) \in g^{-1}\hat{X},$$

where $\hat{x} \in \hat{X}$ corresponds to $x \in X$. Different choices of Γ-fundamental domains $\hat{X}$ correspond to measurably conjugate cocycles into Γ.

Remark 3.9. In the geometric Example 1.2.(a), X represents the unit tangent bundle T^1N, μ^{BM} is the Bowen–Margulis measure on T^1N, and $\phi^{\mathbb{R}}$ is the geodesic flow. The cocycle $c : \mathbb{R} \times T^1N \to \Gamma = \pi_1(N, o)$ can be defined as follows. Fix a base point $o \in N$, for each $p \in N$ choose a path γ_p in N connecting o to p and make this choice in a Borel measurable way. Then for $t \in \mathbb{R}$ and $x \in T^1N$ let $c(t, x) \in \pi_1(N, o)$ be the homotopy class of the path obtained by connecting o to the base point of x, followed by the geodesic flow for time t, and then using the chosen path to connect back to o.

Proof of Proposition 3.5. Statements (a), (b), (c) follow from the definition of the Γ-action. The Γ-action clearly preserves the $m^{\mathrm{BMS}} \times \mathcal{L}$ measure, and satisfies (d) because $|\rho - \sigma|$ is uniformly bounded.

It follows from (b) and (d) that for a fixed $p \in M$ and large enough R the preimage $A = \pi^{-1}(\mathrm{B}(p, R))$ is a measurable subset of $\partial^2\Gamma \times \mathbb{R}$ of finite $m^{\mathrm{BMS}} \times \mathcal{L}$-measure with the property that for a.e. $x \in \partial^2\Gamma \times \mathbb{R}$ the set $\{g \in \Gamma \mid gx \in A\}$ is non-empty and finite. In particular, the stabilizer of a.e. point is finite; in fact uniformly bounded. Therefore there is a measurable choice of one representative from each such orbit; it forms a measurable subset $\hat{X} \subset A$ as in (e).

Statement (f) follows from (d) and the fact that the actions of Γ and $\Phi^{\mathbb{R}}$ commute. ☐

Let us record the following diagram that generalizes the well known construction in the context of geodesic flow on negatively curved manifolds:

(3.4)
$$\begin{array}{ccc} & (\partial^2\Gamma \times \mathbb{R}, m^{\mathrm{BMS}} \times \mathcal{L}) & \\ \swarrow & & \searrow \\ (X, \mu^{\mathrm{BM}}) & & (\partial^2\Gamma, m^{\mathrm{BMS}}) \end{array}$$

Let us fix a precompact measurable subset $\hat{X} \subset \partial^2\Gamma \times \mathbb{R}$ as in Proposition 3.5, and for a.e $x \in X$ and $t \in \mathbb{R}$ choose (in a measurable way) an element $\gamma_{t,x} \in \Gamma$ so that

$$\Phi^t(\hat{x}) \in \gamma_{t,x}^{-1}\hat{X}$$

where $\hat{x} \in \hat{X}$ corresponds to $x \in X$. We shall also denote by (x_-, x_+) the components of the projection of $\hat{x} \in \hat{X} \subset \partial^2\Gamma \times \mathbb{R}$ to $\partial^2\Gamma$. We observe that by the construction for a.e. $x \in X$, $t \in \mathbb{R}$

$$d\left(\gamma_{t,x}^{-1}o, \pi(x_-, x_+, t)\right) = O(1). \tag{3.5}$$

Thus, up to a uniformly bounded error, element $\gamma_{t,x}^{-1}$ pushes point o distance t along the almost geodesic line from x_- towards x_+.

Note that once the essential freeness of the Γ-action on $\partial^2\Gamma$ is established (in Lemma 4.2), we can write $\gamma_{t,x} = c_{\hat{X}}(t, x)$.

4. Stronger Ergodicity via Lebesgue differentiation

The main goal of this section is to prove the stronger ergodic properties of $\Gamma \curvearrowright (\partial^2\Gamma, [\nu^{\mathrm{PS}} \times \nu^{\mathrm{PS}}])$, namely Theorem 1.6, using Lebesgue differentiation.

Below we shall need to estimate the value of

$$\frac{\mathrm{d}\gamma_{t,x}^{-1}\nu}{\mathrm{d}\nu}(\zeta) = e^{\delta_\Gamma \sigma(\gamma_{t,x}, \zeta) + O(1)}$$

as a function of ζ. For $n \gg 1$ consider the following partition of $\partial\Gamma$

$$\partial\Gamma = B^- \sqcup S_1 \sqcup \cdots \sqcup S_{n-1} \sqcup B_n^+, \tag{4.1}$$

where

$$\begin{aligned} S_k &:= \mathrm{B}\left(x_+, \alpha^k\right) \setminus \mathrm{B}\left(x_+, \alpha^{k+1}\right) = \{\zeta \in \partial\Gamma \mid k \le \langle x_+ \mid \zeta \rangle_o < k+1\}, \\ B_n^+ &:= \mathrm{B}\left(x_+, \alpha^n\right) = \{\zeta \in \partial\Gamma \mid n \le \langle x_+ \mid \zeta \rangle_o\}, \end{aligned}$$

and B^- is the rest. This partition roughly corresponds to the location of the "projection" of $\zeta \in \partial\Gamma$ to the almost-geodesic line from x_- to x_+, as follows. If we choose $t(\zeta) \in \mathbb{R}$ so that

$$\pi(x_-, x_+, t(\zeta)) \in \mathrm{QM}(x_-, x_+, \zeta)$$

then, up to a bounded error, B^- consists of those $\zeta \in \partial\Gamma$ for which $t(\zeta) \le 0$, S_k of those $\zeta \in \partial\Gamma$ for which $t(\zeta) \in [k, k+1]$, and B_n^+ comprises $\zeta \in \partial\Gamma$ for which $t(\zeta) \ge n$. We deduce

$$\sigma(\gamma_{n,x}, \zeta) = \begin{cases} -n + O(1) & \zeta \in B^- \\ 2k - n + O(1) & \zeta \in S_k,\ k = 1, \dots, n \\ n + O(1) & \zeta \in B_n^+. \end{cases} \tag{4.2}$$

Let us also note that

$$\nu(S_k) \le \nu\left(\mathrm{B}\left(x_+, \alpha^k\right)\right) = e^{-\delta_\Gamma k + O(1)}, \qquad \nu(B_n^+) = e^{-\delta_\Gamma n + O(1)}. \tag{4.3}$$

For the proof of Theorem 1.6 we will use a version of Lebesgue differentiation (Lemma 4.1) combined with Poincaré recurrence for the $\phi^{\mathbb{R}}$-flow on the probability space (X, μ^{BM}).

Lemma 4.1.
Given $f \in L^1(\partial\Gamma, \nu)$, for m-a.e. $x \in X$ one has

$$\lim_{n\to\infty} \int_{\partial\Gamma} \left| f(\gamma_{n,x}^{-1}\zeta) - f(x_+) \right| \, \mathrm{d}\nu(\zeta) = 0$$

and therefore

$$\lim_{n\to\infty} \int_{\partial\Gamma} f(\gamma_{n,x}^{-1}\zeta)\, \mathrm{d}\nu(\zeta) = f(x_+).$$

In particular, for any measurable $E \subset \partial\Gamma$, for m-a.e. $x \in X$ one has

$$\lim_{n\to\infty} \nu\left(\gamma_{n,x} E\right) = 1_E(x_+).$$

Proof. Let $L_f \subset \partial\Gamma$ be the set of all $\xi \in \partial\Gamma$ for which

$$\lim_{\epsilon\to 0} \frac{1}{\nu\left(\mathrm{B}\left(\xi, \epsilon\right)\right)} \int_{\mathrm{B}(\xi,\epsilon)} |f(\zeta) - f(\xi)| \, \mathrm{d}\nu(\zeta) = 0. \tag{4.4}$$

By the Lebesgue differentiation theorem $\nu(L_f) = 1$. Let $X_0 \subset X$ denote the set of $x \in X$ for which $x_+ \in L_f$ and (3.5) holds for $t \in \mathbb{N}$. This is a full measure set. Fix $x \in X_0$, denote

$$h(\zeta) = |f(\zeta) - f(x_+)|$$

and consider the partition of $\partial\Gamma$ as in (4.1). Using

$$\frac{\mathrm{d}\gamma_{n,x}^{-1}\nu}{\mathrm{d}\nu}(\zeta) = e^{\delta_\Gamma \rho(\gamma_{n,x}, \zeta)} = e^{\delta_\Gamma \sigma(\gamma_{n,x}, \zeta) + O(1)}$$

and (4.2) we estimate

$$\int_{\partial\Gamma} h \circ \gamma_{n,x}^{-1} \, \mathrm{d}\nu = \int_{\partial\Gamma} h \cdot \frac{\mathrm{d}\gamma_{n,x}^{-1}\nu}{\mathrm{d}\nu} \, \mathrm{d}\nu = \left(\int_{B^-} + \int_{S_1} + \cdots + \int_{S_{n-1}} + \int_{B_n^+} \right) h \cdot \frac{\mathrm{d}\gamma_{n,x}^{-1}\nu}{\mathrm{d}\nu} \, \mathrm{d}\nu$$

as follows. For the first term we have the bound

$$\int_{B^-} h \cdot \frac{\mathrm{d}\gamma_{n,x}^{-1}\nu}{\mathrm{d}\nu}\,\mathrm{d}\nu \le e^{-\delta_\Gamma n + O(1)} \cdot \int_{B^-} h\,\mathrm{d}\nu$$

that tends to 0 as $n \to \infty$. For the last term, we use (4.3) to write

$$\int_{B_n^+} h \cdot \frac{\mathrm{d}g_n^{-1}\nu}{\mathrm{d}\nu}\,\mathrm{d}\nu \le e^{\delta_\Gamma n + O(1)} \cdot \int_{B_n^+} h\,\mathrm{d}\nu = O(1)\frac{1}{\nu(B_n^+)}\int_{B_n^+} h\,\mathrm{d}\nu.$$

The latter converges to 0 by (4.4) because $x_+ \in L_f$. For the S_k-term we have

$$\begin{aligned}\int_{S_k} h \cdot \frac{\mathrm{d}\gamma_{n,x}^{-1}\nu}{\mathrm{d}\nu}\,\mathrm{d}\nu &\le e^{\delta_\Gamma(2k-n)+O(1)} \cdot \int_{\mathrm{B}\left(x_+,\alpha^k\right)} h\,\mathrm{d}\nu \\ &= e^{\delta_\Gamma(k-n)+O(1)} \cdot \left(\frac{1}{\nu\left(\mathrm{B}\left(x_+,\alpha^k\right)\right)}\int_{\mathrm{B}\left(x_+,\alpha^k\right)} h\,\mathrm{d}\nu\right).\end{aligned}$$

By (4.4) the average θ_k of h over $\mathrm{B}\left(x_+,\alpha^k\right)$

$$\theta_k := \frac{1}{\nu\left(\mathrm{B}\left(x_+,\alpha^k\right)\right)}\int_{\mathrm{B}\left(x_+,\alpha^k\right)} h\,\mathrm{d}\nu$$

tends to 0 as $k \to \infty$, while the terms

$$w_{n,k} = e^{\delta_\Gamma(k-n)}$$

form longer and longer geometric progressions with bounded sums and diminishing terms:

$$\sum_{k=1}^{n} w_{n,k} = O(1), \qquad \lim_{n\to\infty} w_{n,k} = 0 \quad (k = 1,\ldots,n).$$

Therefore $\sum_{k=1}^{n} w_{n,k} \cdot \theta_k \to 0$ as $n \to \infty$, and

$$\lim_{n\to\infty}\sum_{k=1}^{n}\int_{S_k} h \cdot \frac{\mathrm{d}\gamma_{n,x}^{-1}\nu}{\mathrm{d}\nu}\,\mathrm{d}\nu = 0.$$

This proves that for every $x \in X_0$

$$\lim_{n\to\infty}\int_{\partial\Gamma} |f(\gamma_{n,x}^{-1}\zeta) - f(x_+)|\,\mathrm{d}\nu(\zeta) = 0.$$

In particular, for μ^{BM}-a.e. $x \in X$

$$\lim_{n\to\infty}\int_{\partial\Gamma} f(\gamma_{n,x}^{-1}\zeta)\,\mathrm{d}\nu(\zeta) = f(x_+).$$

This completes the proof of the lemma. □

Proof of Theorem 1.6.

Denote $\mathrm{pr}_-, \mathrm{pr}_+ : \partial^2\Gamma \to \partial\Gamma$ the projections to the first and second components, and for a set $A \subset \partial^2\Gamma$ and $\xi, \eta \in \partial\Gamma$ denote the slices

$$A_\xi^+ := \{\eta \in \partial\Gamma \mid (\xi,\eta) \in A\}, \qquad A_\eta^- := \{\xi \in \partial\Gamma \mid (\xi,\eta) \in A\}.$$

To prove relative (SAT) we shall show that given $A \subset \partial^2\Gamma$ with $\nu \times \nu(A) > 0$ and $\epsilon > 0$, there is $g \in \Gamma$ and a positive measure subset $B \subset \mathrm{pr}_-(A) \cap g\left(\mathrm{pr}_-(A)\right)$ so that for all $\xi \in B$

$$\nu\left(g^{-1}A_\xi^+\right) > 1 - \epsilon.$$

Observe that it suffices to show this claim for any positive measure subset of A, or any fixed Γ-translate g_0A of A. Since Γ-translates of X cover $\partial^2\Gamma \times \mathbb{R}$, up to a null set, upon replacing the given A by a subset of some translate g_0A, we may assume that $X_A := \{x \in X \mid (x_-, x_+) \in A\}$ has $\mu^{\mathrm{BM}}(X_A) > 0$. By Lemma 4.1 for a full measure subset of $x \in X_A$ we have

$$\lim_{n\to\infty} \nu(\gamma_{n,x}^{-1} A^+_{x_+}) = 1.$$

Hence there exists N, so that the set

$$X_{A,N} := \left\{x \in X_A \mid \forall n \geq N : \quad \nu\left(\gamma_{n,x}^{-1} A^+_{x_+}\right) > 1 - \epsilon\right\}$$

has $\mu^{\mathrm{BM}}(X_{A,N}) > 0$. We can now apply Poincaré recurrence theorem on $(X, \mu^{\mathrm{BM}}, \phi^{\mathbb{R}})$ to deduce that there exists $n > N$ for which

$$\mu^{\mathrm{BM}}(\phi^{-n} X_{A,N} \cap X_{A,N}) > 0.$$

Denote $Y = \phi^{-n} X_{A,N} \cap X_{A,N}$, and consider its partition according to the Γ-value of $\gamma_{n,x}$:

$$Y = \bigsqcup_{g\in\Gamma} Y_g, \qquad Y_g = \left\{x \in Y \mid \gamma_{n,x}^{-1} = g\right\}.$$

Choose $g \in \Gamma$ so that $\nu \times \nu(Y_g) > 0$, set $B = \mathrm{pr}_-(Y_g)$, and observe that:

- $\nu(B) > 0$, because $\nu \times \nu(Y_g) > 0$.
- $B \subset \mathrm{pr}_-(A) \cap g^{-1}\left(\mathrm{pr}_-(A)\right)$, because $B \subset \mathrm{pr}_-(X_{A,N}) \subset \mathrm{pr}_-(A)$ and
$$g^{-1}B \subset \mathrm{pr}_-(g^{-1}C_g) \subset \mathrm{pr}_-(X_{A,N}) \subset \mathrm{pr}_-(A).$$
- For $\xi \in B$ there is $x \in Y_g$ so that $\xi = x_-$ and, since $\gamma_{n,x} = g$, we have
$$\nu\left(gA^+_\xi\right) = \nu\left(\gamma_{n,x}^{-1} A^+_{x_+}\right) > 1 - \epsilon$$
as required.

This completes the proof that $\mathrm{pr}_-, \mathrm{pr}_+ : \partial^2\Gamma \to \partial\Gamma$ are relatively SAT. By Equation (2.2) it follows that these maps are also relatively metrically ergodic. The last statement of the Theorem follows from [1, Remark 2.4(1)]. □

We can finally prove

Lemma 4.2. *The* Γ*-actions on* $(\partial\Gamma, [\nu^{\mathrm{PS}}])$*, on* $(\partial^2\Gamma, m^{\mathrm{BMS}})$*, and on* $(\partial^2\Gamma \times \mathbb{R}, m^{\mathrm{BMS}} \times \mathcal{L})$ *are essentially free.*

Proof. Denote by FSub_Γ the space of finite subgroups of Γ. This is a countable set, on which Γ acts by conjugation. By Proposition 3.5 pointwise stabilizers of a.e. point in $\partial^2\Gamma \times \mathbb{R}$ are finite. This gives a measurable Γ-equivariant map

$$\mathrm{Stab} : \partial^2\Gamma \times \mathbb{R} \to \mathrm{FSub}_\Gamma$$

that descends to a Γ-map $\partial^2\Gamma \to \mathrm{FSub}_\Gamma$, because Stab is $\Phi^{\mathbb{R}}$-invariant. Ergodicity of the Γ-action on $(\partial^2\Gamma, m^{\mathrm{BMS}})$ implies that Stab takes values in a Γ-fixed point of FSub_Γ, i.e. a finite normal subgroup $N \triangleleft \Gamma$. Such a finite normal subgroup acts trivially on $\partial\Gamma$, and therefore is trivial by assumption in our framework 1.1. This proves that the Γ-stabilizer of m^{BMS}-a.e. point in $\partial^2\Gamma$ is trivial. Note that this implies essential freeness of the Γ-action on $(\partial\Gamma, [\nu^{\mathrm{PS}}])$. Indeed, if $g \in \Gamma$ pointwise fixes a set $A \subset \partial\Gamma$ of positive ν^{PS}-measure, then it also fixes pointwise the set $A \times A \subset \partial^2\Gamma$ of positive m^{BMS}-measure. Thus $g = 1$. □

We can now justified (see Remark 3.8) the construction of a measurable cocycle $c_{\hat{X}} : \mathbb{R} \times X \to \Gamma$ without assuming that Γ is torsion-free.

Proof of Corollary 1.7.
Combine Theorem 1.6 with Lemma 2.2. □

Proof of Corollary 1.8.
Let (Ω, ω, Γ) be an ergodic p.m.p. system and $(X \times \Omega, \mu^{\mathrm{BM}} \times \omega, \phi_c^{\mathbb{R}})$ the induced flow. To show ergodicity of the latter flow, consider a measurable $\phi_c^{\mathbb{R}}$-invariant function

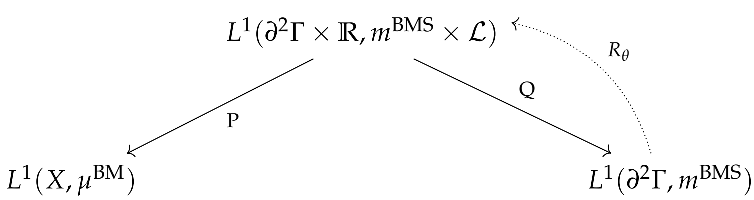
$$f : X \times \Omega \to \mathbb{R}.$$

It lifts to a $\Gamma \times \Phi_c^{\mathbb{R}}$-invariant function $F : \partial^2\Gamma \times \mathbb{R} \times \Omega \to \mathbb{R}$, and the latter descends to a Γ-invariant function $f_1 : \partial^2\Gamma \times \Omega \to \mathbb{R}$. This function f_1 has to be essentially constant by Corollary 1.7. Hence f was essentially constant. □

5. Double Ergodicity via Hopf argument

In this section we will prove Theorem 1.5 and deduce the ergodicity of the diagonal Γ-action on $(\partial^2\Gamma, m^{\mathrm{BMS}})$ and of the geodesic flow $(X, \mu^{\mathrm{BM}}, \phi^{\mathbb{R}})$, via an analogue of Hopf's argument. This argument is independent of Section 4.

Recall the quotient maps (3.4) where X is viewed both as the quotient by Γ and as a Γ-fundamental domain $\hat{X} \subset \partial^2\Gamma \times \mathbb{R}$. We consider the following diagram



where $P : L^1(\partial^2\Gamma \times \mathbb{R}, m^{\mathrm{BMS}} \times \mathcal{L}) \longrightarrow L^1(X, \mu^{\mathrm{BM}})$ is the operator of summation over the Γ-orbits

$$PF := \sum_{g \in \Gamma} F \circ g,$$

the operator $Q : L^1(\partial^2\Gamma \times \mathbb{R}, m^{\mathrm{BMS}} \times \mathcal{L}) \longrightarrow L^1(\partial^2\Gamma, m^{\mathrm{BMS}})$ is the integration over the $\Phi^{\mathbb{R}}$-orbits

$$QF(\xi, \eta) := \int_{\mathbb{R}} F(\xi, \eta, t)\, \mathrm{d}t.$$

Given a positive kernel $\theta \in L^1(\mathbb{R}, \mathcal{L})$, namely given a non-negative function $\theta \geq 0$ with integral one

$$\theta(t) \geq 0, \qquad \int_{\mathbb{R}} \theta(t)\, \mathrm{d}t = 1,$$

we define an operator R_θ that acts by

$$R_\theta f(\xi, \eta, t) = f(\xi, \eta) \cdot \theta(t).$$

The operators P, Q, R_θ are positive and have operator norm one. For each kernel θ one has

$$Q \circ R_\theta = \mathrm{id}_{L^1(\partial^2\Gamma, m^{\mathrm{BMS}})}.$$

But we will be interested in the positive contractions

$$L^1(\partial^2\Gamma, m^{\mathrm{BMS}}) \xrightarrow{R_\theta} L^1(\partial^2\Gamma \times \mathbb{R}) \xrightarrow{P} L^1(X, \mu^{\mathrm{BM}}), \qquad f \mapsto \bar{f}_\theta := P \circ R_\theta(f).$$

For example, if $\theta = (b-a)^{-1} \cdot 1_{[a,b]}$, one has

$$\bar{f}_\theta(\xi, \eta) = \frac{1}{b-a} \cdot \sum_{\{g \in \Gamma \mid a \le \tau(g,x) \le b\}} f(gx_-, gx_+).$$

These operators are positive (i.e. $f \ge 0 \implies \bar{f}_\theta \ge 0$) and preserve integrals:

$$\int_{\partial^2\Gamma} f \, \mathrm{d}m^{\mathrm{BMS}} = \int_X \bar{f}_\theta \, \mathrm{d}\mu^{\mathrm{BM}}. \tag{5.1}$$

For $f \in L^1(\partial^2\Gamma, m^{\mathrm{BMS}})$, with a slight abuse of notation, we write

$$\bar{f}_{[0,1]} := P \circ R_{1_{[0,1]}}(f) \in L^1(X, \mu^{\mathrm{BM}}).$$

With these preliminary observations we proceed to the proofs of the claimed theorems.

Proof of Theorem 1.5. Given $f \in L^1(\partial^2\Gamma, m^{\mathrm{BMS}})$ define $\bar{f}_{[0,1]} \in L^1(X, \mu^{\mathrm{BM}})$ as above, and consider its average over the $\phi^{\mathbb{R}}$-flow over time interval $[a,b]$. One easily calculates

$$\frac{1}{b-a} \int_a^b \bar{f}_{[0,1]}(\phi^t x) \, \mathrm{d}t = \bar{f}_{\theta_a^b}(x) = \sum_{g \in \Gamma} \theta_a^b(\tau(g,x)) \cdot f(gx_-, gx_+)$$

where θ_a^b is the convolution

$$\theta_a^b = \frac{1}{b-a} \cdot 1_{[a,b]} * 1_{[0,1]}$$

which is a linear interpolation between the value $1/(b-a)$ on $[a+1, b]$ and 0 on $(-\infty, a] \cup [b+1, \infty)$. By Birkhoff's ergodic theorem, applied to $\bar{f}_{[0,1]}$, we have a.e. convergence

$$\lim_{T\to\infty} \sum_{g \in \Gamma} \theta_0^T(\tau(g,x)) \cdot f(gx_-, gx_+) = \bar{f}_\infty(x) \tag{5.2}$$

where $\bar{f}_\infty = \mathbb{E}(\bar{f}_{[0,1]} \mid \mathcal{F}^\phi)$ is the conditional expectation of $\bar{f}_{[0,1]} \in L^1(X, \mu^{\mathrm{BM}})$ with respect to the σ-algebra $\mathcal{F}^\phi$ of $\phi^{\mathbb{R}}$-invariant measurable sets in (X, μ^{BM}). In fact, viewing X as a subset of $\partial^2\Gamma \times \mathbb{R}$, $\phi^{\mathbb{R}}$-invariant functions depend only on $\partial^2\Gamma$-coordinate, and therefore $\bar{f}_\infty$ may be written as $\bar{f}_\infty(x_-, x_+)$. So for m^{BMS}-a.e. (ξ, η) in $\mathrm{pr}(X)$, one has

$$\bar{f}_\infty(\xi, \eta) = \lim_{T\to\infty} \sum_{g \in \Gamma} \theta_0^T(\tau(g, \xi, \eta)) \cdot f(g\xi, g\eta) \tag{5.3}$$

where $\tau(g, \xi, \eta)$ is as in (3.2). It will be convenient to use the following notation

$$I_a^b(f)(\xi, \eta) := \frac{1}{b-a} \cdot \sum_{\{g \in \Gamma \ \mid \ \tau(g,\xi,\eta) \in [a,b]\}} f(g\xi, g\eta) \tag{5.4}$$

and to focus on non-negative integrable functions $f \in L^1_+(\partial^2\Gamma, m^{\mathrm{BMS}})$. For such f the inequalities

$$I_0^T f(\xi,\eta) - \frac{1}{T} I_0^1 f(\xi,\eta) \le \sum_{g\in\Gamma} \theta_0^T(\tau(g,\xi,\eta)) \cdot f(g\xi, g\eta) \le \frac{T+1}{T} \cdot I_0^{T+1} f(\xi,\eta)$$

prove a.e. convergence $I_0^T f(\xi,\eta) \to \bar{f}_\infty(\xi,\eta)$ as $T \to \infty$. It also follows that for any fixed $a, b \in \mathbb{R}$

$$\lim_{T\to\infty} I_a^{b+T} f(\xi,\eta) = \bar{f}_\infty(\xi,\eta). \tag{5.5}$$

Formally, these statements apply to (ξ,η) that were $\partial^2\Gamma$-projections of a typical point $x \in X \subset \partial^2\Gamma \times \mathbb{R}$, but for a g_0-translate of such a point, we write

$$\tau(g, g_0\xi, g_0\eta) = \tau(gg_0, \xi, \eta) - \tau_0 \qquad \text{with} \qquad \tau_0 = \tau(g_0, \xi, \eta)$$

so the same limit holds. We also note that the equalities extend from $L^1_+(\partial^2\Gamma, m^{\mathrm{BMS}})$ to all $L^1(\partial^2\Gamma, m^{\mathrm{BMS}})$ by linearity.

We want to show that $\bar{f}_\infty$ is a.e. constant. In view of (5.1) this constant has to be

$$\int_{\partial^2\Gamma} f \, \mathrm{d}m^{\mathrm{BMS}}.$$

First we observe that $\hat{f}_\infty$ is flip-invariant:

$$\bar{f}_\infty(\xi,\eta) = \bar{f}_\infty(\eta,\xi) \tag{5.6}$$

This follows from the fact that the flip $(\xi,\eta,t) \mapsto (\eta,\xi,-t)$ commutes with $\Phi^{\mathbb{R}}$ and Γ on $\partial^2\Gamma \times \mathbb{R}$, and so the σ-algebra of $\phi^{\mathbb{R}}$-invariant sets is also flip-invariant. One can also deduce this from the above Birkhoff's ergodic theorem argument with reversed time: $T \to -\infty$.

Since the operator

$$L^1(\partial^2\Gamma, m^{\mathrm{BMS}}) \longrightarrow L^1(X, \mu^{\mathrm{BM}}), \qquad f \mapsto \hat{f}_\infty$$

has norm ≤ 1, it suffices to prove that $\hat{f}_\infty$ is a.e. constant for a dense in L^1 family of functions f. We shall focus on continuous functions with compact support

$$f \in C_c(\partial^2\Gamma) \subset L^1(\partial^2\Gamma, m^{\mathrm{BMS}}).$$

Fix such an f and $\epsilon > 0$ and an arbitrary compact set $K \subset \partial^2\Gamma$ containing $\mathrm{supp}(f)$ in its interior. Using uniform continuity of f and the geometric Lemma 2.7, we can find $a > 0$ so that for $\xi, \eta, \eta' \in \partial\Gamma$ with $(\xi,\eta), (\xi,\eta') \in K$ and any $g \in \Gamma$ one has

$$\tau(g,\xi,\eta) > a \qquad \Longrightarrow \qquad |f(g\xi, g\eta) - f(g\xi, g\eta')| < \epsilon.$$

Indeed, for $g \in \Gamma$ for which $(g\xi, g\eta) \in K$ by Lemma 2.7 the points $(g\xi, g\eta)$ and $(g\xi, g\eta')$ are close and uniform continuity of f can be used. For those $g \in \Gamma$ with $(g\xi, g\eta) \notin K$, we have $f(g\xi, g\eta) = 0$ and claim that $f(g\xi, g\eta') = 0$ also. Otherwise, $(g\xi, g\eta') \in \mathrm{supp}(f) \subset \mathrm{Int}(K)$ and Lemma 2.7 would apply that $(g\xi, g\eta)$ is close enough to lie in K, provided $a \gg 0$.

We conclude that for ν^{PS}-a.e. $\xi \in \partial\Gamma$ for ν^{PS}-a.e. $\eta, \eta' \in \partial\Gamma$ with $(\xi,\eta), (\xi,\eta') \in K$ one has

$$|\hat{f}_\infty(\xi,\eta) - \hat{f}_\infty(\xi,\eta')| \le \epsilon.$$

Taking $K \to \partial^2\Gamma$ and $\epsilon \to 0$ we deduce that for ν^{PS}-a.e. ξ the function $\hat{f}_\infty(\xi,-)$ is essentially constant. But since $\hat{f}_\infty$ is also flip invariant (5.6), it follows that $\hat{f}_\infty$

is a.e. constant as claimed. Putting together the previous remarks we proved that for any $f \in L^1(\partial^2\Gamma, m^{\mathrm{BMS}})$ there is m^{BMS}-a.e. convergence

$$I_0^T f(\xi,\eta) \longrightarrow \int_{\partial^2\Gamma} f \,\mathrm{d}m^{\mathrm{BMS}}.$$

The claim in Theorem 1.5 concerns a.e. convergence of averages, defined in a way similar to $I_0^T f$, but with τ replaced by

$$\widetilde{\sigma}(g,\xi,\eta) = \frac{\sigma(g,\eta)-\sigma(g,\xi)}{2}, \qquad \sigma(g,\eta), \qquad \text{or} \qquad \sigma(g,\xi).$$

First consider averages $J_0^T f$ defined by $\widetilde{\sigma}$ instead of τ that were used in $I_0^T f$. Since $\tau - \sigma'$ is uniformly bounded, say by C, for any $f \in L^1_+$, we have pointwise comparison

$$\frac{T-2C}{T}\cdot I_C^{T-C} f \le J_0^T f \le \frac{T+2C}{T}\cdot I_{-C}^{T+C} f.$$

Convergence (5.5) implies a.e. convergence

$$\lim_{T\to\infty} J_0^T f(\xi,\eta) = \int_{\partial^2\Gamma} f \,\mathrm{d}m^{\mathrm{BMS}}.$$

By linearity, this applies to all $f \in L^1$.

Next consider averages $J_0^T f$ defined similarly to $I_0^T f$ but using $\sigma(g,\eta)$ instead of $\tau(g,\xi,\eta)$. Fix an exhaustion of $\partial^2\Gamma$ by a nested sequence of compact sets $K_1 \subset K_2 \subset \dots$. There exist constants C_n (e.g. $C_n = 2\|1_{K_n}\cdot F\|_\infty$ in (3.2)) so that for $(\xi,\eta)\in K_n$ and $g \in \Gamma$ with $(g\xi, g\eta) \in K_n$ one has

$$|\tau(g,\xi,\eta) - \sigma(g,\eta)| \le C_n.$$

For each $n \in \mathbb{N}$, for $(\xi,\eta) \in K_n$ and function $f_n = 1_{K_n}\cdot f$, one has an estimate of $J_0^T f_n$ in terms of $I_{C_n}^{T_n - C_n} f_n$ and $I_{-C_n}^{T_n+C_n} f_n$ as above, and conclude

$$\lim_{T\to\infty} J_0^T f_n(\xi,\eta) = \int_{\partial^2\Gamma} f_n \,\mathrm{d}m^{\mathrm{BMS}} = \int_{K_n} f \,\mathrm{d}m^{\mathrm{BMS}}$$

Taking $n \to \infty$ and using monotonicity, we deduce $J_0^T f(\xi,\eta) \to \int f \,\mathrm{d}m^{\mathrm{BMS}}$ a.e. on $\partial^2\Gamma$. By linearity this result extends to all $f \in L^1$. The case of $\sigma(g,\xi)$ is essentially identical to the last one (but with time reversed). This completes the proof of Theorem 1.5. □

Proof of Theorem 1.4. Let us prove that any measurable Γ-invariant set $E \subset \partial^2\Gamma$ of positive measure is conull, i.e. $m^{\mathrm{BMS}}(E^c) = 0$. Choose a compact subset $K \subset \partial^2\Gamma$ so that $m^{\mathrm{BMS}}(K \cap E) > 0$. Then the indicator function $f = 1_{K\cap E}$ is integrable and

$$\int f \,\mathrm{d}m^{\mathrm{BMS}} > 0.$$

For $(\xi,\eta) \in E^c$ we have $f(g\xi, g\eta) = 0$ for all $g \in \Gamma$. Thus Theorem 1.5 implies that $m^{\mathrm{BMS}}(E^c) = 0$ as claimed.

This proves ergodicity of the Γ-action on $(\partial^2\Gamma, m^{\mathrm{BMS}})$, which implies ergodicity of the other two actions in the diagram (3.4). Indeed, any non-trivial (mod μ^{BM}) measurable partition $X = A_1 \sqcup A_2$ into $\phi^{\mathbb{R}}$-invariant sets lifts to a non-trivial $\Gamma \times \Phi^{\mathbb{R}}$-invariant partition $\Gamma A_1 \sqcup \Gamma A_2 = \partial^2\Gamma \times \mathbb{R}$. In turn, it has to be of the form $\Gamma A_i = E_i \times \mathbb{R}$ for a non-trivial (m^{BMS}) partition $\partial^2\Gamma = E_1 \sqcup E_2$ into Γ-invariant sets, which leads to a contradiction. This completes the proof of Theorem 1.4. □

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