

# INTEGRABLE MEASURE EQUIVALENCE AND RIGIDITY OF HYPERBOLIC LATTICES

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ABSTRACT. We study rigidity properties of lattices in  $\text{Isom}(\mathbf{H}^n) \simeq \text{SO}_{n,1}(\mathbb{R})$ ,  $n \geq 3$ , and of surface groups in  $\text{Isom}(\mathbf{H}^2) \simeq \text{SL}_2(\mathbb{R})$  in the context of *integrable measure equivalence*. The results for lattices in  $\text{Isom}(\mathbf{H}^n)$ ,  $n \geq 3$ , are generalizations of Mostow rigidity; they include a *cocycle* version of strong rigidity and an integrable measure equivalence classification. For surface groups integrable measure equivalence rigidity is obtained via a cocycle version of the Milnor-Wood inequality. The integrability condition appears in certain (co)homological tools pertaining to bounded cohomology. Some of these homological tools are developed in a companion paper [2].

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## 1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

**1.1. Introduction.** *Measure equivalence* is an equivalence relation on groups, introduced by Gromov [25] as a measure-theoretic counterpart to quasi-isometry of finitely generated groups. It is intimately related to *orbit equivalence* in ergodic theory, to the theory of von Neumann algebras, and to questions in descriptive set theory. The study of rigidity in measure equivalence or orbit equivalence goes back to Zimmer’s paper [60], which extended Margulis’ superrigidity of higher rank lattices [39] to the context of *measurable cocycles* and applied it to problems of orbit equivalence. The study of measure equivalence and related problems has recently experienced a rapid growth, with [14, 15, 21, 22, 26, 28, 29, 33, 34, 42, 45–48, 50] being only a partial list of important advances. We refer to [17, 49, 55] for surveys and further references.

One particularly fruitful direction of research in this area has been in obtaining the complete description of groups that are measure equivalent to a given one from a well understood class of groups. This has been achieved for lattices in simple Lie groups of higher rank [15], products of hyperbolic-like groups [42], mapping class groups [33–35], and certain amalgams of groups as above [36]. In all these results, the measure equivalence class of one of such groups turns out to be small and to consist of a list of “obvious” examples obtained by simple modifications of the original group. This phenomenon is referred to as *measure equivalence rigidity*. On the other hand, the class of groups measure equivalent to lattices in  $\mathrm{SL}_2(\mathbb{R})$  is very rich: it is uncountable, includes wide classes of groups and does not seem to have an explicit description (cf. [3, 23]).

In the present paper we obtain measure equivalence rigidity results for lattices in the least rigid family of simple Lie groups  $\mathrm{Isom}(\mathbf{H}^n) \simeq \mathrm{SO}_{n,1}(\mathbb{R})$  for  $n \geq 2$ , including surface groups, albeit within a more restricted category of *integrable measure equivalence*, hereafter also called  *$L^1$ -measure equivalence* or just  *$L^1$ -ME*. Let us briefly state the classification result, before giving the precise definitions and stating more detailed results.

**Theorem A.** *Let  $\Gamma$  be a lattice in  $G = \mathrm{Isom}(\mathbf{H}^n)$ ,  $n \geq 2$ ; in the case  $n = 2$  assume that  $\Gamma$  is cocompact. Then the class of all finitely generated groups that are  $L^1$ -measure equivalent to  $\Gamma$  consists of those  $\Lambda$ , which admit a short exact sequence  $\{1\} \rightarrow F \rightarrow \Lambda \rightarrow \bar{\Lambda} \rightarrow \{1\}$  where  $F$  is finite and  $\bar{\Lambda}$  is a lattice in  $G$ ; in the case  $n = 2$ ,  $\bar{\Lambda}$  is cocompact in  $G = \mathrm{Isom}(\mathbf{H}^2)$ .*

The integrability assumption is necessary for the validity of the rigidity results for cocompact lattices in  $\mathrm{Isom}(\mathbf{H}^2) \cong \mathrm{PGL}_2(\mathbb{R})$ . It remains possible, however, that the  $L^1$ -integrability assumption is superfluous for lattices in  $\mathrm{Isom}(\mathbf{H}^n)$ ,  $n \geq 3$ . We also note that a result of Fisher and Hitchman [12] can be used to obtain  $L^2$ -ME rigidity results similar to Theorem A for the family of rank one Lie groups  $\mathrm{Isom}(\mathbf{H}_{\mathbb{H}}^n) \simeq \mathrm{Sp}_{n,1}(\mathbb{R})$  and  $\mathrm{Isom}(\mathbf{H}_{\mathbb{O}}^2) \simeq F_{4(-20)}^1$ ; here we do not know either whether the  $L^2$ -integrability assumption is necessary.

The proof of Theorem A for the case  $n \geq 3$  proceeds through a cocycle version of Mostow’s strong rigidity theorem stated in Theorems B and 1.8. This cocycle version relates to the original Mostow’s strong rigidity theorem in the same way in which Zimmer’s cocycle superrigidity theorem relates to the original Margulis’

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<sup>1</sup>Here  $\simeq$  means *locally isomorphic*.

superrigidity for higher rank lattices. Our proof of the cocycle version of Mostow rigidity, which is inspired by Gromov-Thurston's proof of Mostow rigidity using simplicial volume [58] and Burger-Iozzi's proof for dimension 3 [5], heavily uses bounded cohomology and other homological methods. A major part of the relevant homological technique (like *Sobolev homology*), which applies to general Gromov hyperbolic groups, is developed in the companion paper [2]; *in fact, Theorem 3.2 taken from [2] is the only place in this paper where we require the integrability assumption.*

Theorem A and the more detailed Theorem D are deduced from the strong rigidity for integrable cocycles (Theorem B) using a general method described in Theorem 2.1. The latter extends and streamlines the approach developed in [15], and further used in [42] and in [34].

The proof of Theorem A for surfaces uses a cocycle version of the fact that an abstract isomorphism between uniform lattices in  $\mathrm{PGL}_2(\mathbb{R})$  is realized by conjugation in  $\mathrm{Homeo}(S^1)$ . The proof of this generalization uses homological methods mentioned above and a cocycle version of the Milnor-Wood-Ghys phenomenon (Theorem C), in which an integrable ME-cocycle between surface groups is conjugate to the identity map in  $\mathrm{Homeo}(S^1)$ . In the case of surfaces in Theorem A, this result is used together with Theorem 2.1 to construct a representation  $\rho : \Lambda \rightarrow \mathrm{Homeo}(S^1)$ . Additional arguments (Lemma 3.14 and Theorem 4.1) are then needed to deduce that  $\rho(\Lambda)$  is a uniform lattice in  $\mathrm{PGL}_2(\mathbb{R})$ .

Let us now make precise definitions and describe in more detail the main results.

## 1.2. Basic notions.

1.2.1. **Measure equivalence of locally compact groups.** We recall the central notion of *measure equivalence* which was suggested by Gromov [25, 0.5.E]. It will be convenient to work with general unimodular, locally compact second countable (lcsc) groups rather than just countable ones.

**Definition 1.1.** Let  $G, H$  be unimodular lcsc groups with Haar measures  $m_G$  and  $m_H$ . A  $(G, H)$ -coupling is a Lebesgue measure space  $(\Omega, m)$  with a measurable, measure-preserving action of  $G \times H$  such that there exist finite measure spaces  $(X, \mu), (Y, \nu)$  and equivariant measure space isomorphisms

$$(1.1) \quad \begin{aligned} i : (G, m_G) \times (Y, \nu) &\xrightarrow{\cong} (\Omega, m) & \text{so that } g : i(g', y) &\mapsto i(gg', y), \\ j : (H, m_H) \times (X, \mu) &\xrightarrow{\cong} (\Omega, m) & \text{so that } h : j(h', x) &\mapsto j(hh', x), \end{aligned}$$

for  $g, g' \in G$  and  $h, h' \in H$ . Groups which admit such a coupling are said to be *measure equivalent* (abbreviated ME).

In the case where  $G$  and  $H$  are countable groups, the condition on the commuting actions  $G \curvearrowright (\Omega, m)$  and  $H \curvearrowright (\Omega, m)$  is that they admit finite  $m$ -measure Borel fundamental domains  $X, Y \subset \Omega$  with  $\mu = m|_X$  and  $\nu = m|_Y$  being the restrictions.

As the name suggest, measure equivalence is an equivalence relation between unimodular lcsc groups. For reflexivity, consider the  $G \times G$ -action on  $(G, m_G)$ ,  $(g_1, g_2) : g \mapsto g_1 g g_2^{-1}$ . We refer to this as the *tautological self coupling* of  $G$ . The symmetry of the equivalence relation is obvious. For transitivity and more details we refer to Appendix A.1.

**Example 1.2.** Let  $\Gamma_1, \Gamma_2$  be lattices in a lcsc group  $G^2$ . Then  $\Gamma_1$  and  $\Gamma_2$  are measure equivalent, with  $(G, m_G)$  serving as a natural  $(\Gamma_1, \Gamma_2)$ -coupling when equipped with the action  $(\gamma_1, \gamma_2) : g \mapsto \gamma_1 g \gamma_2^{-1}$  for  $\gamma_i \in \Gamma_i$ . In fact, any lattice  $\Gamma < G$  is measure equivalent to  $G$ , with  $(G, m_G)$  serving as a natural  $(G, \Gamma)$ -coupling when equipped with the action  $(g, \gamma) : g' \mapsto gg'\gamma^{-1}$ .

1.2.2. **Taut groups.** We now introduce the following key notion of taut couplings and taut groups.

**Definition 1.3** (Taut couplings, taut groups). A  $(G, G)$ -coupling  $(\Omega, m)$  is *taut* if it has the tautological coupling as a factor uniquely; in other words if it admits a up to null sets unique measurable map  $\Phi : \Omega \rightarrow G$  so that for  $m$ -a.e.  $\omega \in \Omega$  and all  $g_1, g_2 \in G^3$

$$\Phi((g_1, g_2)\omega) = g_1 \Phi(\omega) g_2^{-1}.$$

Such a  $G \times G$ -equivariant map  $\Omega \rightarrow G$  will be called a *tautening map*. A unimodular lcsc group  $G$  is *taut* if every  $(G, G)$ -coupling is taut.

The requirement of uniqueness for tautening maps in the definition of taut groups is equivalent to the property that the group in question is *strongly ICC* (see Definition 2.2). This property is rather common; in particular it is satisfied by all center free semi-simple Lie groups and all ICC countable groups, i.e. countable groups with infinite conjugacy classes. On the other hand the existence of tautening maps for  $(G, G)$ -coupling is hard to obtain; in particular taut groups necessarily satisfy Mostow's strong rigidity property.

**Lemma 1.4** (Taut groups satisfy Mostow rigidity). *Let  $G$  be a taut unimodular lcsc group. If  $\tau : \Gamma_1 \xrightarrow{\cong} \Gamma_2$  is an isomorphism of two lattices  $\Gamma_1$  and  $\Gamma_2$  in  $G$ , then there exists a unique  $g \in G$  so that  $\Gamma_2 = g^{-1}\Gamma_1g$  and  $\tau(\gamma_1) = g^{-1}\gamma_1g$  for  $\gamma_1 \in \Gamma_1$ .*

The lemma follows from considering the tautness of the measure equivalence  $(G, G)$ -coupling given by the  $G \times G$ -homogeneous space  $G \times G/\Delta_\tau$ , where  $\Delta_\tau$  is the graph of the isomorphism  $\tau : \Gamma_1 \rightarrow \Gamma_2$ ; see Lemma A.3 for details.

The phenomenon, that any isomorphism between lattices in  $G$  is realized by an inner conjugation in  $G$ , known as *strong rigidity* or *Mostow rigidity*, holds for all simple Lie groups<sup>4</sup>  $G \not\cong \mathrm{SL}_2(\mathbb{R})$ . More precisely, if  $X$  is an irreducible non-compact, non-Euclidean symmetric space with the exception of the hyperbolic plane  $\mathbf{H}^2$ , then  $G = \mathrm{Isom}(X)$  is Mostow rigid. Mostow proved this remarkable rigidity property for uniform lattices [44]. It was then extended to the non-uniform cases by Prasad [51] ( $\mathrm{rk}(X) = 1$ ) and by Margulis [38] ( $\mathrm{rk}(X) \geq 2$ ).

In the higher rank case, more precisely, if  $X$  is a symmetric space without compact and Euclidean factors with  $\mathrm{rk}(X) \geq 2$ , Margulis proved a stronger rigidity property, which became known as *superrigidity* [39]. Margulis' superrigidity for lattices in higher rank, was extended by Zimmer in the cocycle superrigidity theorem [60]. Zimmer's cocycle superrigidity was used in [15] to show that higher rank simple Lie groups  $G$  are *taut* (the use of term tautness in this context is new). In [42] Monod and Shalom proved another case of cocycle superrigidity and proved a

<sup>2</sup>Any lcsc group containing a lattice is necessarily unimodular.

<sup>3</sup>If one only requires equivariance for almost all  $g_1, g_2 \in G$  one can always modify  $\Phi$  on a null set to get an everywhere equivariant map [61, Appendix B].

<sup>4</sup>For the formulation of Mostow rigidity above we have to assume that  $G$  has trivial center.

version of tautness property for certain products  $G = \Gamma_1 \times \cdots \times \Gamma_n$  with  $n \geq 2$ . In [33, 34] Kida proved that mapping class groups are taut. Kida's result may be viewed as a cocycle generalization of Ivanov's theorem [31].

**1.2.3. Measurable cocycles.** Let us elaborate on this connection between tautness and rigidity of measurable cocycles. Recall that a cocycle over a group action  $G \curvearrowright X$  to another group  $H$  is a map  $c : G \times X \rightarrow H$  such that for all  $g_1, g_2 \in G$

$$c(g_2 g_1, x) = c(g_2, g_1 x) \cdot c(g_1, x).$$

Cocycles that are independent of the space variable are precisely homomorphisms  $G \rightarrow H$ . One can conjugate a cocycle  $c : G \times X \rightarrow H$  by a map  $f : X \rightarrow H$  to produce a new cocycle  $c^f : G \times X \rightarrow H$  given by

$$c^f(g, x) = f(g.x)^{-1} c(g, x) f(x).$$

In our context,  $G$  is a lcsc group,  $H$  is lcsc or, more generally, a Polish group, and  $G \curvearrowright (X, \mu)$  is a measurable measure-preserving action on a Lebesgue finite measure space. In this context all maps, including the cocycle  $c$ , are assumed to be  $\mu$ -measurable, and all equations should hold  $\mu$ -a.e.; we then say that  $c$  is a *measurable cocycle*.

Let  $(\Omega, m)$  be a  $(G, H)$ -coupling and  $H \times X \xrightarrow{j} \Omega \xrightarrow{i^{-1}} G \times Y$  be as in (1.1). Since the actions  $G \curvearrowright \Omega$  and  $H \curvearrowright \Omega$  commute,  $G$  acts on the space of  $H$ -orbits in  $\Omega$ , which is naturally identified with  $X$ . This  $G$ -action preserves the finite measure  $\mu$ . Similarly, we get the measure preserving  $H$ -action on  $(Y, \nu)$ . These actions will be denoted by a dot,  $g : x \mapsto g.x$ ,  $h : y \mapsto h.y$ , to distinguish them from the  $G \times H$  action on  $\Omega$ . Observe that in  $\Omega$  one has

$$g : j(h, x) \mapsto j(hh_1^{-1}, g.x)$$

where  $h_1 \in H$  depends only on  $g \in G$  and  $x \in X$ , and therefore may be denoted by  $\alpha(g, x)$ . One easily checks that the map

$$\alpha : G \times X \rightarrow H$$

that was just defined, is a measurable cocycle. Similarly, one obtains a measurable cocycle  $\beta : H \times Y \rightarrow G$ . These cocycles depend on the choice of the measure isomorphisms in (1.1), but different measure isomorphisms produce conjugate cocycles. Identifying  $(\Omega, m)$  with  $(H, m_H) \times (X, \mu)$ , the action  $G \times H$  takes the form

$$(1.2) \quad (g, h) : j(h', x) \mapsto j(hh'\alpha(g, x)^{-1}, g.x).$$

Similarly, cocycle  $\beta : H \times Y \rightarrow G$  describes the  $G \times H$ -action on  $(\Omega, m)$  when identified with  $(G, m_G) \times (Y, \nu)$ . In general, we call a measurable cocycle  $G \times X \rightarrow H$  that arises from a  $(G, H)$ -coupling as above an *ME-cocycle*.

The connection between tautness and cocycle rigidity is in the observation (see Lemma A.4) that a  $(G, G)$ -coupling  $(\Omega, m)$  is taut iff the ME-cocycle  $\alpha : G \times X \rightarrow G$  is conjugate to the identity isomorphism

$$\alpha(g, x) = f(g.x)^{-1} g f(x)$$

by a unique measurable  $f : X \rightarrow G$ . Hence one might say that  $G$  is *taut* iff it satisfies a *cocycle version* of Mostow rigidity.

1.2.4. **Integrability conditions.** Our first main result – Theorem B below – shows that  $G = \text{Isom}(\mathbf{H}^n)$ ,  $n \geq 3$ , are 1-*taut* groups, i.e. all *integrable*  $(G, G)$ -couplings are taut. We shall now define these terms more precisely.

A *norm* on a group  $G$  is a map  $|\cdot| : G \rightarrow [0, \infty)$  so that  $|gh| \leq |g| + |h|$  and  $|g^{-1}| = |g|$  for all  $g, h \in G$ . A norm on a lsc group is *proper* if it is measurable and the balls with respect to this norm are pre-compact. Two norms  $|\cdot|$  and  $|\cdot|'$  are equivalent if there are  $a, b > 0$  such that  $|g|' \leq a \cdot |g| + b$  and  $|g| \leq a \cdot |g|' + b$  for every  $g \in G$ . On a compactly generated group<sup>5</sup>  $G$  with compact generating symmetric set  $K$  the function  $|g|_K = \min\{n \in \mathbb{N} \mid g \in K^n\}$  defines a proper norm, whose equivalence class does not depend on the chosen  $K$ . Unless stated otherwise, we mean a norm in this equivalence class when referring to a proper norm on a compactly generated group.

**Definition 1.5** (Integrability of cocycles). Let  $H$  be a compactly generated group with a proper norm  $|\cdot|$  and  $G$  be a lsc group. Let  $p \in [1, \infty]$ . A measurable cocycle  $c : G \times X \rightarrow H$  is  $L^p$ -*integrable* if for a.e.  $g \in G$

$$\int_X |c(g, x)|^p d\mu(x) < \infty.$$

For  $p = 0$  we require that the essential supremum of  $|c(g, -)|$  is finite for a.e.  $g \in G$ . If  $p = 1$ , we also say that  $c$  is *integrable*. If  $p = 0$ , we say that  $c$  is *bounded*.

The integrability condition is independent of the choice of a norm within a class of equivalent norms.  $L^p$ -integrability implies  $L^q$ -integrability whenever  $1 \leq q \leq p$ . In the Appendix A.2 we show that, if  $G$  is also compactly generated, the  $L^p$ -integrability of  $c$  implies that the above integral is uniformly bounded on compact subsets of  $G$ .

**Definition 1.6** (Integrability of couplings). A  $(G, H)$ -coupling  $(\Omega, m)$  of compactly generated, unimodular, lsc groups is  $L^p$ -*integrable*, if there exist measure isomorphisms as in (1.1) so that the corresponding ME-cocycles  $G \times X \rightarrow H$  and  $H \times Y \rightarrow G$  are  $L^p$ -integrable. If  $p = 1$  we just say that  $(\Omega, m)$  is *integrable*. Groups  $G$  and  $H$  that admit an  $L^p$ -integrable  $(G, H)$ -coupling are said to be  $L^p$ -*measure equivalent*.

For each  $p \in [1, \infty]$ , being  $L^p$ -measure equivalent is an equivalence relation on compactly generated, unimodular, lsc groups (see Lemma A.1). Furthermore,  $L^p$ -measure equivalence implies  $L^q$ -measure equivalence if  $1 \leq q \leq p$ . So among the  $L^p$ -measure equivalence relations,  $L^\infty$ -measure equivalence is the strongest and  $L^1$ -measure equivalence is the weakest one; all being subrelations of the (unrestricted) measure equivalence.

**Definition 1.7** ( $p$ -taut groups). A lsc group  $G$  is  $p$ -*taut* if every  $L^p$ -integrable  $(G, G)$ -coupling is taut.

The definition of  $L^p$ -integrability for couplings is motivated by the older definition of  $L^p$ -*integrability for lattices*, which e.g. plays an important role in [54] and which we recall here. Let  $\Gamma < G$  be a lattice, where  $\Gamma$  is a finitely generated and  $G$  is a compactly generated group. Then  $\Gamma$  is  $L^p$ -*integrable* if  $(G, m_G)$  is an  $L^p$ -coupling. Equivalently, if there exists a Borel cross-section  $s : G/\Gamma \rightarrow G$  of the projection, so that the cocycle  $c : G \times G/\Gamma \rightarrow \Gamma$ ,  $c(g, x) = s(g.x)^{-1}gs(x)$  is  $L^p$ -integrable. Note also that  $L^\infty$ -integrable lattices are precisely the uniform ones.

<sup>5</sup>Every connected lsc group is compactly generated [56, Corollary 6.12 on p. 58].

### 1.3. Statement of the main results.

**Theorem B.** *The groups  $G = \text{Isom}(\mathbf{H}^n)$ ,  $n \geq 3$ , are 1-taut.*

This result has an equivalent formulation in terms of cocycles.

**Theorem 1.8** (Integrable cocycle strong rigidity). *Let  $G = \text{Isom}(\mathbf{H}^n)$ ,  $n \geq 3$ ,  $G \curvearrowright (X, \mu)$  be a probability measure preserving action, and  $c : G \times X \rightarrow G$  be an integrable ME-cocycle. Then there is a measurable map  $f : X \rightarrow G$ , which is unique up to null sets, such that for  $\mu$ -a.e.  $x \in X$  and every  $g \in G$  we have*

$$c(g, x) = f(g.x)^{-1} g f(x).$$

Note that this result generalizes Mostow-Prasad rigidity for lattices in these groups. This follows from the fact that any 1-taut group satisfies Mostow rigidity for  $L^1$ -integrable lattices, and the fact, due to Shalom, that all lattices in groups  $G = \text{Isom}(\mathbf{H}^n)$ ,  $n \geq 3$ , are  $L^1$ -integrable.

**Theorem 1.9** ([54, Theorem 3.6]). *All lattices in simple Lie groups not locally isomorphic to  $\text{Isom}(\mathbf{H}^2) \simeq \text{PSL}_2(\mathbb{R})$ ,  $\text{Isom}(\mathbf{H}^3) \simeq \text{PSL}_2(\mathbb{C})$ , are  $L^2$ -integrable, hence also  $L^1$ -integrable. Further, lattices in  $\text{Isom}(\mathbf{H}^3)$  are  $L^1$ -integrable.*

The second assertion is not stated in this form in [54, Theorem 3.6] but the proof therein shows exactly that. In fact, for lattices in  $\text{Isom}(\mathbf{H}^n)$  Shalom shows  $L^{n-1-\epsilon}$ -integrability.

Lattices in  $G = \text{Isom}(\mathbf{H}^2) \cong \text{PGL}_2(\mathbb{R})$ , such as surface groups, admit a rich space of deformations – the Teichmüller space. In particular, these groups do not satisfy Mostow rigidity, and therefore are not taut (they are not even  $\infty$ -taut). However, it is well known viewing  $G = \text{Isom}(\mathbf{H}^2) \cong \text{PGL}_2(\mathbb{R})$  as acting on the circle  $S^1 \cong \partial\mathbf{H}^2 \cong \mathbb{R}P^1$ , any abstract isomorphism  $\tau : \Gamma \rightarrow \Gamma'$  between cocompact lattices  $\Gamma, \Gamma' < G$  can be realized by a conjugation in  $\text{Homeo}(S^1)$ , that is,

$$\exists f \in \text{Homeo}(S^1) \forall \gamma \in \Gamma : \pi \circ \tau(\gamma) = f^{-1} \circ \pi(\gamma) \circ f,$$

where  $\pi : G \rightarrow \text{Homeo}(S^1)$  is the imbedding as above. Motivated by this observation we generalize the notion of tautness as follows.

**Definition 1.10.** Let  $G$  be a unimodular lsc group,  $\mathcal{G}$  a Polish group,  $\pi : G \rightarrow \mathcal{G}$  a continuous homomorphism. A  $(G, G)$ -coupling is *taut relative to  $\pi : G \rightarrow \mathcal{G}$*  if there exists a up to null sets unique measurable map  $\Phi : \Omega \rightarrow \mathcal{G}$  such that for  $m$ -a.e.  $\omega \in \Omega$  and all  $g_1, g_2 \in G$

$$\Phi((g_1, g_2)\omega) = \pi(g_1)\Phi(\omega)\pi(g_2)^{-1}.$$

We say that  $G$  is *taut* (resp. *p-taut*) *relative to  $\pi : G \rightarrow \mathcal{G}$*  if all (resp. all  $L^p$ -integrable)  $(G, G)$ -couplings are taut relative to  $\pi : G \rightarrow \mathcal{G}$ .

Observe that  $G$  is taut iff it is taut relative to itself. Note also that if  $\Gamma < G$  is a lattice, then  $G$  is taut iff  $\Gamma$  is taut relative to the inclusion  $\Gamma < G$ ; and  $G$  is taut relative to  $\pi : G \rightarrow \mathcal{G}$  iff  $\Gamma$  is taut relative to  $\pi|_{\Gamma} : \Gamma \rightarrow \mathcal{G}$  (Proposition 2.8). If  $\Gamma < G$  is  $L^p$ -integrable, then these equivalences apply to  $p$ -tautness.

**Theorem C.** *The group  $G = \text{Isom}(\mathbf{H}^2) \cong \text{PGL}_2(\mathbb{R})$  is 1-taut relative to the natural embedding  $G < \text{Homeo}(S^1)$ . Cocompact lattices  $\Gamma < G$  are 1-taut relative to the embedding  $\Gamma < G < \text{Homeo}(S^1)$ .*

We skip the obvious equivalent cocycle reformulation of this result.

- Remarks 1.11.** (1) The  $L^1$ -assumption cannot be dropped from Theorem C. Indeed, the free group  $\mathbf{F}_2$  can be realized as a lattice in  $\mathrm{PSL}_2(\mathbb{R})$ , but most automorphisms of  $\mathbf{F}_2$  cannot be realized by homeomorphisms of the circle.
- (2) Realizing isomorphisms between surface groups in  $\mathrm{Homeo}(S^1)$ , one obtains somewhat regular maps: they are Hölder continuous and quasi-symmetric. We do not know (and do not expect) Theorem C to hold with  $\mathrm{Homeo}(S^1)$  being replaced by the corresponding subgroups.

We now state the  $L^1$ -ME rigidity result which is deduced from Theorem B, focusing on the case of countable, finitely generated groups.

**Theorem D** ( $L^1$ -Measure equivalence rigidity). *Let  $G = \mathrm{Isom}(\mathbf{H}^n)$  with  $n \geq 3$ , and  $\Gamma < G$  be a lattice. Let  $\Lambda$  be a finitely generated group, and let  $(\Omega, m)$  be an integrable  $(\Gamma, \Lambda)$ -coupling. Then*

- (1) *there exists a short exact sequence*

$$1 \rightarrow F \rightarrow \Lambda \rightarrow \bar{\Lambda} \rightarrow 1$$

*where  $F$  is finite and  $\bar{\Lambda}$  is a lattice in  $G$ ,*

- (2) *and a measurable map  $\Phi : \Omega \rightarrow G$  so that for  $m$ -a.e.  $\omega \in \Omega$  and every  $\gamma \in \Gamma$  and every  $\lambda \in \Lambda$*

$$\Phi((\gamma, \lambda)\omega) = \gamma\Phi(\omega)\bar{\lambda}^{-1}.$$

*Moreover, if  $\Gamma \times \Lambda \curvearrowright (\Omega, m)$  is ergodic, then*

- (2a) *either the push-forward measure  $\Phi_*m$  is a positive multiple of the Haar measure  $m_G$  or  $m_{G^0}$ ;*
- (3a) *or, one may assume that  $\Gamma$  and  $\bar{\Lambda}$  share a subgroup of finite index and  $\Phi_*m$  is a positive multiple of the counting measure on the double coset  $\Gamma\bar{\Lambda} \subset G$ .*

This result is completely analogous to the higher rank case considered in [15], except for the  $L^1$ -assumption. We do not know whether Theorem D remains valid in the broader ME category, that is, without the  $L^1$ -condition, but should point out that if the  $L^1$  condition can be removed from Theorem B then it can also be removed from Theorem D.

Theorem D can also be stated in the broader context of unimodular lcsc groups, in which case the  $L^1$ -measure equivalence rigidity states that a compactly generated unimodular lcsc group  $H$  that is  $L^1$ -measure equivalent to  $G = \mathrm{Isom}(\mathbf{H}^n)$ ,  $n \geq 3$ , admits a short exact sequence  $1 \rightarrow K \rightarrow H \rightarrow \bar{H} \rightarrow 1$  where  $K$  is compact and  $\bar{H}$  is either  $G$ , or its index two subgroup  $G^0$ , or is a lattice in  $G$ .

Measure equivalence rigidity results have natural consequences for (*stable, or weak*) *orbit equivalence* of essentially free probability measure-preserving group actions (cf. [14, 35, 42, 48]). Two probability measure preserving actions  $\Gamma \curvearrowright (X, \mu)$ ,  $\Lambda \curvearrowright (Y, \nu)$  are *weakly, or stably, orbit equivalent* if there exist measurable maps  $p : X \rightarrow Y$ ,  $q : Y \rightarrow X$  with  $p_*\mu \ll \nu$ ,  $q_*\nu \ll \mu$  so that a.e.

$$p(\Gamma.x) \subset \Lambda.p(x), \quad q(\Lambda.y) \subset \Gamma.q(y), \quad q \circ p(x) \in \Gamma.x, \quad p \circ q(y) \in \Lambda.y.$$

(see [14, §2] for other equivalent definitions). If  $\Gamma_1, \Gamma_2$  are lattices in some lcsc group  $G$ , then  $\Gamma_1 \curvearrowright G/\Gamma_2$  and  $\Gamma_2 \curvearrowright G/\Gamma_1$  are stably orbit equivalent via  $p(x) = s_1(x)^{-1}$ ,  $q(y) = s_2(y)^{-1}$ , where  $s_i : G/\Gamma_i \rightarrow G$  are measurable cross-sections. Moreover, given any (essentially) free, ergodic, probability measure preserving (p.m.p.) action  $\Gamma_1 \curvearrowright (X_1, \mu_1)$  and  $\Gamma_1$ -equivariant quotient map  $\pi_1 : X_1 \rightarrow G/\Gamma_2$ , there exists a

canonically defined free, ergodic p.m.p. action  $\Gamma_2 \curvearrowright (X_2, \mu_2)$  with equivariant quotient  $\pi_2 : X_2 \rightarrow G/\Gamma_1$  so that  $\Gamma_i \curvearrowright (X_i, \mu_i)$  are stably orbit equivalent in a way compatible to  $\pi_i : X_i \rightarrow G/\Gamma_{3-i}$  [14, Theorem C].

We shall now introduce integrability conditions on weak orbit equivalence, assuming  $\Gamma$  and  $\Lambda$  are finitely generated groups. Let  $|\cdot|_\Gamma, |\cdot|_\Lambda$  denote some word metrics on  $\Gamma, \Lambda$  respectively, and let  $\Gamma \curvearrowright (X, \mu)$  be an essentially free action. Define an extended metric  $d_\Gamma : X \times X \rightarrow [0, \infty]$  on  $X$  by setting  $d_\Gamma(x_1, x_2) = |\gamma|_\Gamma$  if  $\gamma.x_1 = x_2$  and set  $d_\Gamma(x_1, x_2) = \infty$  otherwise. Let  $d_\Lambda$  denote the extended metric on  $Y$ , defined in a similar fashion. We say that  $\Gamma \curvearrowright (X, \mu)$  and  $\Lambda \curvearrowright (Y, \nu)$  are  $L^p$ -weakly/stably orbit equivalent, if there exists maps  $p : X \rightarrow Y, q : Y \rightarrow X$  as above, and such that for every  $\gamma \in \Gamma, \lambda \in \Lambda$

$$(x \mapsto d_\Lambda(p(\gamma.x), p(x))) \in L^p(X, \mu), \quad (x \mapsto d_\Gamma(q(\lambda.y), q(y))) \in L^p(Y, \nu).$$

Note that the last condition is independent of the choice of word metrics.

The following result<sup>6</sup> is deduced from Theorem D in essentially the same way Theorems A and C in [14] are deduced from the corresponding measure equivalence rigidity theorem in [15]. The only additional observation is that the constructions respect the integrability conditions.

**Theorem E** ( $L^1$ -Orbit equivalence rigidity). *Let  $G = \text{Isom}(\mathbf{H}^n)$  where  $n \geq 3$ , and  $\Gamma < G$  be a lattice. Assume that there is a finitely generated group  $\Lambda$  and essentially free, ergodic, p.m.p actions  $\Gamma \curvearrowright (X, \mu)$  and  $\Lambda \curvearrowright (Y, \nu)$ , which admit a stable  $L^1$ -orbit equivalence  $p : X \rightarrow Y, q : Y \rightarrow X$  as above. Then either one the following two cases occurs:*

**Virtual isomorphism:** *There exists a short exact sequence  $1 \rightarrow F \rightarrow \Lambda \rightarrow \bar{\Lambda} \rightarrow 1$ , where  $F$  is a finite group and  $\bar{\Lambda} < G$  is a lattice with  $\Delta = \Gamma \cap \bar{\Lambda}$  having finite index in both  $\Gamma$  and  $\bar{\Lambda}$ , and an essentially free ergodic p.m.p action  $\Delta \curvearrowright (Z, \zeta)$  so that  $\Gamma \curvearrowright (X, \mu)$  is isomorphic to the induced action  $\Gamma \curvearrowright \Gamma \times_\Delta (Z, \zeta)$ , and the quotient action  $\bar{\Lambda} \curvearrowright (\bar{Y}, \bar{\nu}) = (Y, \nu)/F$  is isomorphic to the induced action  $\bar{\Lambda} \curvearrowright \bar{\Lambda} \times_\Delta (Z, \zeta)$ , or*

**Standard quotients:** *There exists a short exact sequence  $1 \rightarrow F \rightarrow \Lambda \rightarrow \bar{\Lambda} \rightarrow 1$ , where  $F$  is a finite group and  $\bar{\Lambda} < G$  is a lattice, and for  $G' = G$  or  $G' = G^0$  (only if  $\bar{\Lambda}, \Gamma < G^0$ ), and equivariant measure space quotient maps*

$$\pi : (X, \mu) \rightarrow (G'/\bar{\Lambda}, m_{G'/\bar{\Lambda}}), \quad \sigma : (Y, \nu) \rightarrow (G'/\Gamma, m_{G'/\Gamma})$$

*with  $\pi(\gamma.x) = \gamma.\pi(x), \sigma(\lambda.y) = \bar{\lambda}.\sigma(y)$ . Moreover, the action  $\bar{\Lambda} \curvearrowright (\bar{Y}, \bar{\nu}) = (Y, \nu)/F$  is isomorphic to the canonical action associated to  $\Gamma \curvearrowright (X, \mu)$  and the quotient map  $\pi : X \rightarrow G'/\bar{\Lambda}$ .*

The family of rank one simple real Lie groups  $\text{Isom}(\mathbf{H}^n)$  is the least rigid one among simple Lie groups. As higher rank simple Lie groups are rigid with respect to measure equivalence, one wonders about the remaining families of simple real Lie groups:  $\text{Isom}(\mathbf{H}_\mathbb{C}^n) \simeq \text{SU}_{n,1}(\mathbb{R})$ ,  $\text{Isom}(\mathbf{H}_\mathbb{H}^n) \simeq \text{Sp}_{n,1}(\mathbb{R})$ , and the exceptional group  $\text{Isom}(\mathbf{H}_\mathbb{O}^2) \simeq F_{4(-20)}$ . The question of measure equivalence rigidity (or  $L^p$ -measure equivalence rigidity) for the former family remains open, but the latter groups are rigid with regard to  $L^2$ -measure equivalence. Indeed, recently, using harmonic maps techniques (after Corlette [9] and Corlette-Zimmer [10]), Fisher and Hitchman [12] proved an  $L^2$ -cocycle superrigidity result for isometries of quaternionic hyperbolic

<sup>6</sup>The formulation of the virtual isomorphism case in terms of induced actions is due to Kida [35].

space  $\mathbf{H}_{\mathbb{H}}^n$  and the Cayley plane  $\mathbf{H}_{\mathbb{O}}^2$ . This theorem can be used to deduce the following.

**Theorem 1.12** (Corollary of [12]). *The rank one Lie groups  $\text{Isom}(\mathbf{H}_{\mathbb{H}}^n) \simeq \text{Sp}_{n,1}(\mathbb{R})$  and  $\text{Isom}(\mathbf{H}_{\mathbb{O}}^2) \simeq F_{4(-20)}$  are 2-taut.*

Using this result as an input to the general machinery described above one obtains:

**Corollary 1.13.** *The conclusions of Theorems D and E hold for all lattices in  $\text{Isom}(\mathbf{H}_{\mathbb{H}}^n)$  and  $\text{Isom}(\mathbf{H}_{\mathbb{O}}^2)$  provided the  $L^1$ -conditions are replaced by  $L^2$ -ones.*

**1.4. Organization of the paper.** The rigidity properties of general taut groups, including Theorem 2.1, are proved in Section 2. The generalizations of Mostow rigidity, Theorem B, and cocycle version of Milnor-Wood-Ghys phenomenon, Theorem C, are proved in Section 3 using the homological methods. In Section 4 the remaining results stated in the introduction are deduced, including the  $L^1$ -measure equivalence rigidity of surface groups (Theorem A case  $n = 2$ ). General facts about measure equivalence which are used throughout the paper are collected in the Appendix A.

**1.5. Acknowledgements.** Uri Bader and Alex Furman were supported in part by the BSF grant 2008267. Uri Bader was also supported in part by the ISF grant number 704/08 and the GIF grant 2191-1826.6/2007. Alex Furman was also supported in part by the NSF grants DMS 0604611 and 0905977. Roman Sauer acknowledges support from the *Deutsche Forschungsgemeinschaft*, made through grant SA 1661/1-2.

## 2. MEASURE EQUIVALENCE RIGIDITY FOR TAUT GROUPS

This section contains general tools related to the notion of taut couplings and taut groups. The results of this section apply to general unimodular lsc groups, including countable groups, and are not specific to  $\text{Isom}(\mathbf{H}^n)$  or semi-simple Lie groups. Whenever we refer to  $L^p$ -integrability conditions, we assume that the groups are also compactly generated. We shall rely on some basic facts about measure equivalence which are collected in Appendix A.

The basic tool is the following:

**Theorem 2.1.** *Let  $G$  be a unimodular lsc group that is taut (resp.  $p$ -taut). Any unimodular lsc group  $H$  that is measure equivalent (resp.  $L^p$ -measure equivalent) to  $G$  admits a short exact sequence with continuous homomorphisms*

$$1 \rightarrow K \rightarrow H \rightarrow \bar{H} \rightarrow 1,$$

where  $K$  is compact and  $\bar{H}$  is a closed subgroup in  $G$  such that  $G/\bar{H}$  carries a  $G$ -invariant Borel probability measure.

Theorem 2.5 below contains a more general statement.

**2.1. The strong ICC property and strongly proximal actions.** We need to introduce a notion of *strongly ICC* group  $G$  and, more generally, the notion of a group  $\mathcal{G}$  being strongly ICC relative to a subgroup  $\mathcal{G}_0 < \mathcal{G}$ .

**Definition 2.2.** A Polish group  $\mathcal{G}$  is *strongly ICC relative* to  $\mathcal{G}_0 < \mathcal{G}$  if  $\mathcal{G} \setminus \{e\}$  does not support any Borel probability measure that is invariant under the conjugation action of  $\mathcal{G}_0$  on  $\mathcal{G} \setminus \{e\}$ . A Polish group  $\mathcal{G}$  is *strongly ICC* if it is strongly ICC relative to itself.

The key properties of this notion are discussed in the appendix [A.4](#).

**Example 2.3.** The basic examples of the strong ICC property include the following:

- (1) A countable group  $\Gamma$  is strongly ICC iff it is an ICC<sup>7</sup> group.
- (2) Center-free semi-simple Lie groups  $G$  without compact factors are examples of non-discrete strongly ICC groups. In fact, such groups are strongly ICC relative to any unbounded Zariski dense subgroup (cf. [[16](#), Proof of Theorem 2.3]).
- (3) The Polish group  $\text{Homeo}(S^1)$  is strongly ICC relative to  $\text{PGL}_2(\mathbb{R})$  (see Lemma [2.4](#)).

Let  $M$  be a compact metrizable space. Recall that a continuous action  $G \curvearrowright M$  is *minimal and strongly proximal* if the following equivalent conditions hold:

- (1) for every Borel probability measure  $\nu \in \text{Prob}(M)$  and every non-empty open subset  $V \subset M$  one has

$$\sup_{g \in G} g_*\nu(V) = 1.$$

- (2) for every  $\nu \in \text{Prob}(M)$  the convex hull of the  $G$ -orbit  $g_*\nu$  is dense in  $\text{Prob}(M)$  in the weak-\* topology.

Recall that this condition is satisfied by the standard action of  $G = \text{PGL}_2(\mathbb{R})$  and its lattices on the circle. More generally, any connected semi-simple center free group  $G$  without compact factors  $G$  acts on  $M = G/Q$  where  $Q$  is a parabolic in a minimal and strongly proximal fashion [[40](#), Theorem 3.7 on p. 205].

**Lemma 2.4.** *Let  $M$  be a compact metrizable space and  $G < \text{Homeo}(M)$  be a subgroup which acts minimally and strongly proximally on  $M$ . Then  $\text{Homeo}(M)$  is strongly ICC relative to  $G$ .*

*Proof.* Let  $\mu$  be a probability measure on  $\text{Homeo}(M)$ . The set of  $\mu$ -stationary probability measures on  $M$

$$\text{Prob}_\mu(M) = \left\{ \nu \in \text{Prob}(M) \mid \nu = \mu * \nu = \int f_*\nu d\mu(f) \right\}$$

is a non-empty convex closed (hence compact) subset of  $\text{Prob}(M)$ , with respect to the weak-\* topology. Suppose  $\mu$  is invariant under conjugations by  $g \in G$ . Since

$$g_*(\mu * \nu) = \mu^g * (g_*\nu) = \mu * (g_*\nu)$$

it follows that  $\text{Prob}_\mu(M)$  is a  $G$ -invariant set. Minimality and strong proximality of the  $G$ -action implies that  $\text{Prob}_\mu(M) = \text{Prob}(M)$ . In particular, every Dirac measure  $\nu_x$  is  $\mu$ -stationary; hence  $\mu\{f \mid f(x) = x\} = 1$ . It follows that  $\mu = \delta_e$ .  $\square$

<sup>7</sup> has Infinite Conjugacy Classes.

## 2.2. Tautness and the passage to self couplings.

**Theorem 2.5.** *Let  $G, H$  be unimodular lcsc groups. Let  $(\Omega, m)$  be a  $(G, H)$ -coupling. Let  $\mathcal{G}$  be a Polish group and  $\pi : G \rightarrow \mathcal{G}$  a continuous homomorphism. Assume that  $\mathcal{G}$  is strongly ICC relative to  $\pi(G)$  and the  $(G, G)$ -coupling  $\Omega \times_H \Omega^*$  is taut relative to  $\pi$ .*

*Then there exists a continuous homomorphism  $\rho : H \rightarrow \mathcal{G}$  and a measurable map  $\Psi : \Omega \rightarrow \mathcal{G}$  so that a.e.:*

$$\Psi((g, h)\omega) = \pi(g)\Psi(\omega)\rho(h)^{-1} \quad (g \in G, h \in H)$$

*and the unique tautening map  $\Phi : \Omega \times_H \Omega^* \rightarrow \mathcal{G}$  is given by*

$$\Phi([\omega_1, \omega_2]) = \Psi(\omega_1) \cdot \Psi(\omega_2)^{-1}.$$

*The pair  $(\Psi, \rho)$  is unique up to conjugations  $(\Psi^x, \rho^x)$  by  $x \in \mathcal{G}$ , where*

$$\Psi^x(\omega) = \Psi(\omega)x^{-1}, \quad \rho^x(h) = x\rho(h)x^{-1}.$$

*If, in addition,  $\pi : G \rightarrow \mathcal{G}$  has compact kernel and closed image  $\bar{G} = \pi(G)$ , then the same applies to  $\rho : H \rightarrow \mathcal{G}$ , and there exists a Borel measure  $\bar{m}$  on  $\mathcal{G}$ , which is invariant under*

$$(g, h) : x \mapsto \pi(g)x\rho(h)$$

*and descends to finite measures on  $\pi(G)\backslash\mathcal{G}$  and  $\mathcal{G}/\rho(H)$ . In other words,  $(\mathcal{G}, \bar{m})$  is a  $(\pi(G), \rho(H))$ -coupling which is a quotient of  $(\text{Ker}(\pi) \times \text{Ker}(\rho))\backslash(\Omega, m)$ .*

*Proof.* We shall first construct a homomorphism  $\rho : H \rightarrow \mathcal{G}$  and the  $G \times H$ -equivariant map  $\Psi : \Omega \rightarrow \mathcal{G}$ . Consider the space  $\Omega^3 = \Omega \times \Omega \times \Omega$  and the three maps  $p_{1,2}, p_{2,3}, p_{1,3}$ , where

$$p_{i,j} : \Omega^3 \longrightarrow \Omega^2 \longrightarrow \Omega \times_H \Omega^*$$

is the projection to the  $i$ -th and  $j$ -th factor followed by the natural projection. Consider the  $G^3 \times H$ -action on  $\Omega^3$ :

$$(g_1, g_2, g_3, h) : (\omega_1, \omega_2, \omega_3) \mapsto ((g_1, h)\omega_1, (g_2, h)\omega_2, (g_3, h)\omega_3).$$

For  $i \in \{1, 2, 3\}$  denote by  $G_i$  the corresponding  $G$ -factor in  $G^3$ . For  $i, j \in \{1, 2, 3\}$  with  $i \neq j$  the group  $G_i \times G_j < G_1 \times G_2 \times G_3$  acts on  $\Omega \times_H \Omega^*$  and on  $\mathcal{G}$  by

$$(g_i, g_j) : [\omega_1, \omega_2] \mapsto [g_i\omega_1, g_j\omega_2], \quad (g_i, g_j) : x \mapsto \pi(g_i)x\pi(g_j)^{-1} \quad (x \in \mathcal{G})$$

respectively. Let  $\{i, j, k\} = \{1, 2, 3\}$ . The map  $p_{i,j} : \Omega^3 \rightarrow \Omega \times_H \Omega^*$  is  $G_k \times H$ -invariant and  $G_i \times G_j$ -equivariant. This is also true of the maps

$$F_{i,j} = \Phi \circ p_{i,j} : \Omega^3 \xrightarrow{p_{i,j}} \Omega \times_H \Omega^* \xrightarrow{\Phi} \mathcal{G},$$

where  $\Phi : \Omega \times_H \Omega^* \rightarrow \mathcal{G}$  is the tautening map. For  $\{i, j, k\} = \{1, 2, 3\}$ , the three maps  $F_{i,j}, F_{j,i}^{-1}$  and  $F_{i,k} \cdot F_{k,j}$  are all  $G_k \times H$ -invariant, hence factor through the natural map

$$\Omega^3 \rightarrow \Sigma_k = (G_k \times H)\backslash\Omega^3.$$

By an obvious variation on the argument in Appendix A.1.3 one verifies that  $\Sigma_k$  is a  $(G_i, G_j)$ -coupling. The three maps  $F_{i,j}, F_{j,i}^{-1}$  and  $F_{i,k} \cdot F_{k,j}$  are  $G_i \times G_j$ -equivariant. Since  $\mathcal{G}$  is strongly ICC relative to  $\pi(G)$ , there is at most one  $G_i \times G_j$ -equivariant measurable map  $\Sigma_k \rightarrow \mathcal{G}$  according to Lemma A.6. Therefore, we get  $m^3$ -a.e. identities

$$(2.1) \quad F_{i,j} = F_{j,i}^{-1} = F_{i,k} \cdot F_{k,j}.$$

Denote by  $\bar{\Phi} : \Omega^2 \rightarrow \mathcal{G}$  the composition  $\Omega^2 \rightarrow \Omega \times_H \Omega^* \xrightarrow{\Phi} \mathcal{G}$ . By Fubini's theorem, (2.1) implies that for  $m$ -a.e.  $\omega_2 \in \Omega$ , for  $m \times m$ -a.e.  $(\omega_1, \omega_3)$

$$\bar{\Phi}(\omega_1, \omega_3) = \bar{\Phi}(\omega_1, \omega_2) \cdot \bar{\Phi}(\omega_2, \omega_3) = \bar{\Phi}(\omega_1, \omega_2) \cdot \bar{\Phi}(\omega_3, \omega_2)^{-1}.$$

Fix such a generic  $\omega_2 \in \Omega$  and define  $\Psi : \Omega \rightarrow \mathcal{G}$  by  $\Psi(\omega) = \bar{\Phi}(\omega, \omega_2)$ . Then for a.e.  $[\omega, \omega'] \in \Omega \times_H \Omega$

$$(2.2) \quad \Phi([\omega, \omega']) = \bar{\Phi}(\omega, \omega') = \Psi(\omega) \cdot \Psi(\omega')^{-1}.$$

We proceed to construct a representation  $\rho : H \rightarrow \mathcal{G}$ . Equation (2.2) implies that for every  $h \in H$  and for a.e.  $\omega, \omega' \in \Omega$ :

$$\Psi(h\omega)\Psi(h\omega')^{-1} = \bar{\Phi}(h\omega, h\omega') = \bar{\Phi}(\omega, \omega') = \Psi(\omega)\Psi(\omega')^{-1},$$

and in particular, we get

$$\Psi(h\omega)^{-1}\Psi(\omega) = \Psi(h\omega')^{-1}\Psi(\omega').$$

Observe that the left hand side is independent of  $\omega' \in \Omega$ , while the right hand side is independent of  $\omega \in \Omega$ . Hence both are  $m$ -a.e. constant, and we denote by  $\rho(h) \in \mathcal{G}$  the constant value. Being coboundaries the above expressions are cocycles; being independent of the space variable they give a homomorphism  $\rho : H \rightarrow \mathcal{G}$ . To see this explicitly, for  $h, h' \in H$  we compute using  $m$ -a.e.  $\omega \in \Omega$ :

$$\begin{aligned} \rho(hh') &= \Psi(hh'\omega)^{-1}\Psi(\omega) \\ &= \Psi(hh'\omega)^{-1}\Psi(h'\omega)\Psi(h'\omega)^{-1}\Psi(\omega) \\ &= \rho(h)\rho(h'). \end{aligned}$$

Since the homomorphism  $\rho$  is measurable, it is also continuous [61, Theorem B.3 on p. 198]. By definition of  $\rho$  we have for  $h \in H$  and  $m$ -a.e.  $\omega \in \Omega$ :

$$(2.3) \quad \Psi(h\omega) = \Psi(\omega)\rho(h)^{-1}.$$

Since  $\Psi(\omega) = \bar{\Phi}(\omega, \omega_2)$ , it also follows that for  $g \in G$  and  $m$ -a.e.  $\omega \in \Omega$

$$(2.4) \quad \Psi(g\omega) = \pi(g)\Psi(\omega).$$

Consider the collection of all pairs  $(\Psi, \rho)$  satisfying (2.3) and (2.4). Clearly,  $\mathcal{G}$  acts on this set by  $x : (\Psi, \rho) \mapsto (\Psi^x, \rho^x) = (\Psi \cdot x, x^{-1}\rho x)$ ; and we claim that this action is transitive. Let  $(\Psi_i, \rho_i)$ ,  $i = 1, 2$ , be two such pairs in the above set. Then

$$\tilde{\Phi}_i(\omega, \omega') = \Psi_i(\omega)\Psi_i(\omega')^{-1} \quad (i = 1, 2)$$

are  $G \times G$ -equivariant measurable maps  $\Omega \times \Omega \rightarrow \mathcal{G}$ , which are invariant under  $H$ . Hence they descend to  $G \times G$ -equivariant maps  $\Phi_i : \Omega \times_H \Omega^* \rightarrow \mathcal{G}$ . The assumption that  $\mathcal{G}$  is strongly ICC relative to  $\pi(G)$ , implies a.e. identities  $\Phi_1 = \Phi_2$ ,  $\tilde{\Phi}_1 = \tilde{\Phi}_2$ . Hence for a.e.  $\omega, \omega'$

$$\Psi_1(\omega)^{-1}\Psi_2(\omega) = \Psi_1(\omega')^{-1}\Psi_2(\omega').$$

Since the left hand side depends only on  $\omega$ , while the right hand side only on  $\omega'$ , it follows that both sides are a.e. constant  $x \in \mathcal{G}$ . This gives  $\Psi_1 = \Psi_2^x$ . The a.e. identity

$$\Psi_1(\omega)\rho_1(h) = \Psi_1(h^{-1}\omega) = \Psi_2(h^{-1}\omega)x = \Psi_2(\omega)\rho_2(h)x = \Psi_1(\omega)x^{-1}\rho_2(h)x$$

implies  $\rho_1 = \rho_2^x$ . This completes the proof of the first part of the theorem.

Next, we assume that  $\text{Ker}(\pi)$  is compact and  $\pi(G)$  is closed in  $\mathcal{G}$ , and will show that the kernel  $K = \text{Ker}(\rho)$  is compact, the image  $\bar{H} = \rho(H)$  is closed in  $\mathcal{G}$ , and that  $\mathcal{G}/\bar{H}$ ,  $\pi(G) \backslash \mathcal{G}$  carry finite measures. These properties will be deduced

from the assumption on  $\pi$  and the existence of the measurable map  $\Psi : \Omega \rightarrow \mathcal{G}$  satisfying (2.3) and (2.4). We need the next lemma, which says that  $\Omega$  has measure space isomorphisms as in (1.1) with special properties.

**Lemma 2.6.** *Let  $\rho : H \rightarrow \mathcal{G}$  and  $\Psi : \Omega \rightarrow \mathcal{G}$  be as above. Then there exist measure space isomorphisms  $i : G \times Y \cong \Omega$  and  $j : H \times X \cong \Omega$  as in (1.1) that satisfy in addition*

$$\Psi(i(g, y)) = \pi(g), \quad \Psi(j(h, x)) = \rho(h).$$

*Proof.* We start from some measure space isomorphisms  $i_0 : G \times Y \cong \Omega$  and  $j_0 : H \times X \cong \Omega$  as in (1.1) and will replace them by

$$i(g, y) = i_0(gg_y, y), \quad j(h, x) = j_0(hh_x, x)$$

for some appropriately chosen measurable maps  $Y \rightarrow G$ ,  $y \mapsto g_y$  and  $X \rightarrow H$ ,  $x \mapsto h_x$ . The conditions (1.1) remain valid after any such alteration.

Let us construct  $y \mapsto g_y$  with the required property; the map  $x \mapsto h_x$  can be constructed in a similar manner. By (2.4) for  $m_G \times \nu$ -a.e.  $(g_1, y) \in G \times Y$  the value

$$\pi(g)^{-1}\Psi \circ i_0(gg_1, y)$$

is  $m_G$ -a.e. independent of  $g$ ; denote it by  $f(g_1, y) \in \mathcal{G}$ . Fix  $g_1 \in G$  for which

$$\Psi \circ i_0(gg_1, y) = \pi(g)f(g_1, y)$$

holds for  $m_G$ -a.e.  $g \in G$  and  $\nu$ -a.e.  $y \in Y$ . There exists a Borel cross section  $\mathcal{G} \rightarrow G$  to  $\pi : G \rightarrow \mathcal{G}$ . Using such, we get a measurable choice for  $g_y$  so that

$$\pi(g_y) = f(g_1, y)^{-1}\pi(g_1).$$

Setting  $i(g, y) = i_0(gg_y, y)$ , we get  $m_G \times \nu$ -a.e. that  $\Psi \circ i(g, y) = \pi(g)$ .  $\square$

**Lemma 2.7.** *Given a neighborhood of the identity  $V \subset H$  and a compact subset  $Q \subset \mathcal{G}$ , the set  $\rho^{-1}(Q)$  can be covered by finitely many translates of  $V$ :*

$$\rho^{-1}(Q) \subset h_1V \cup \dots \cup h_NV.$$

*Proof.* Since  $\pi : G \rightarrow \mathcal{G}$  is assumed to be continuous, having closed image and compact kernel, for any compact  $Q \subset \mathcal{G}$  the preimage  $\pi^{-1}(Q) \subset G$  is also compact. Let  $W \subset H$  be an open neighborhood of identity so that  $W \cdot W^{-1} \subset V$ ; we may assume  $W$  has compact closure in  $H$ . Then  $\pi^{-1}(Q) \cdot W$  is precompact. Hence there is an open precompact set  $U \subset G$  with  $\pi^{-1}(Q) \cdot W \subset U$ . Consider subsets  $A = j(W \times X)$ , and  $B = i(U \times Y)$  of  $\Omega$ , where  $i$  and  $j$  are as in the previous lemma. Then

$$m(A) = m_H(W) \cdot \nu(Y) > 0, \quad m(B) = m_G(U) \cdot \mu(X) < \infty.$$

Let  $\{h_1, \dots, h_n\} \subset \rho^{-1}(Q)$  be such that  $h_kW \cap h_lW = \emptyset$  for  $k \neq l \in \{1, \dots, n\}$ . Then the sets  $h_kA = j(h_kW \times X)$  are also pairwise disjoint and have  $m(h_kA) = m(A) > 0$  for  $1 \leq k \leq n$ . Since

$$\Psi(h_kA) = \rho(h_kW) = \rho(h_k)\rho(W) \subset Q \cdot \rho(W) \subset \rho(U),$$

it follows that  $h_kA \subset B$  for every  $1 \leq k \leq n$ . Hence  $n \leq m(B)/m(A)$ . Choosing a maximal such set  $\{h_1, \dots, h_N\}$ , we obtain the desired cover.  $\square$

*Continuation of the proof of Theorem 2.5.* Lemma 2.7 implies that the closed subgroup  $K = \text{Ker}(\rho)$  is compact. More generally, it implies that the continuous homomorphism  $\rho : H \rightarrow \mathcal{G}$  is proper, that is, preimages of compact sets are compact. Therefore  $\bar{H} = \rho(H)$  is closed in  $\mathcal{G}$ .

We push forward the measure  $m$  to a measure  $\bar{m}$  on  $\mathcal{G}$  via the map  $\Psi : \Omega \rightarrow \mathcal{G}$ . The measure  $\bar{m}$  is invariant under the action  $x \mapsto \pi(g)x\rho(h)$ . Since  $\bar{H} = \rho(H) \cong H/\text{ker}(\rho)$  is closed in  $\mathcal{G}$ , the space  $\mathcal{G}/\bar{H}$  is Hausdorff. As  $\text{Ker}(\rho)$  and  $\text{Ker}(\pi)$  are compact normal subgroups in  $H$  and  $G$ , respectively, the map  $\Psi : \Omega \rightarrow \mathcal{G}$  factors through

$$(\Omega, m) \longrightarrow (\Omega', m') = (\text{Ker}(\pi) \times \text{Ker}(\rho)) \backslash (\Omega, m) \xrightarrow{\Psi'} \mathcal{G}.$$

Let  $\bar{G} = G/\text{Ker}(\pi)$ . Starting from measure isomorphisms as in Lemma 2.6, we obtain equivariant measure isomorphisms  $(\Omega', m') \cong (\bar{H} \times X, m_{\bar{H}} \times \mu)$  and  $(\Omega', m') \cong (\bar{G} \times Y, m_{\bar{G}} \times \nu)$ . In particular,  $(\Omega', m')$  is a  $(\bar{G}, \bar{H})$ -coupling. The measure  $\bar{m}$  on  $\mathcal{G}$  descends to the  $\bar{G}$ -invariant finite measure on  $\mathcal{G}/\bar{H}$  obtained by pushing forward  $\mu$ . Similarly,  $\bar{m}$  descends to the  $\bar{H}$ -invariant finite measure on  $\bar{G} \backslash \mathcal{G}$  obtained by pushing forward  $\nu$ . This completes the proof of Theorem 2.5.  $\square$

*Proof of Theorem 2.1.* Theorem 2.1 immediately follows from Theorem 2.5. In case of  $L^p$ -conditions, one observes that if  $(\Omega, m)$  is an  $L^p$ -integrable  $(G, H)$ -coupling, then  $\Omega \times_H \Omega^*$  is an  $L^p$ -integrable  $(G, G)$ -coupling (Lemma A.2); so it is taut under the assumption that  $G$  is  $p$ -taut.  $\square$

### 2.3. Lattices in taut groups.

**Proposition 2.8** (Taut groups and lattices). *Let  $G$  be a unimodular lcsc group,  $\mathcal{G}$  a Polish group,  $\pi : G \rightarrow \mathcal{G}$  a continuous homomorphism. Assume that  $\mathcal{G}$  is strongly ICC relative to  $\pi(G)$ .*

*Then  $G$  is taut (resp.  $p$ -taut) relative to  $\pi : G \rightarrow \mathcal{G}$  iff  $\Gamma$  is taut (resp.  $p$ -taut) relative to  $\pi|_{\Gamma} : \Gamma \rightarrow \mathcal{G}$ .*

*In particular,  $G$  is taut iff any/all lattices in  $G$  are taut relative to the inclusion  $\Gamma < G$ .*

For the proof of this proposition we shall need the following.

**Lemma 2.9** (Induction). *Let  $G$  be a unimodular lcsc group,  $\mathcal{G}$  a Polish group,  $\pi : G \rightarrow \mathcal{G}$  a continuous homomorphism, and  $\Gamma_1, \Gamma_2 < G$  lattices. Let  $(\Omega, m)$  be a  $(\Gamma_1, \Gamma_2)$ -coupling, and assume that the  $(G, G)$ -coupling  $\bar{\Omega} = G \times_{\Gamma_1} \Omega \times_{\Gamma_2} G$  is taut relative to  $\pi : G \rightarrow \mathcal{G}$ . Then there exists a  $\Gamma_1 \times \Gamma_2$ -equivariant map  $\Omega \rightarrow \mathcal{G}$ .*

*Proof.* It is convenient to have a concrete model for  $\bar{\Omega}$ . Choose Borel cross-sections  $\sigma_i$  from  $X_i = G/\Gamma_i$  to  $G$ , and form the cocycles  $c_i : G \times X_i \rightarrow \Gamma_i$  by

$$c_i(g, x) = \sigma_i(g.x)^{-1}g\sigma_i(x), \quad (i = 1, 2).$$

Then, suppressing the obvious measure from the notations,  $\bar{\Omega}$  identifies with  $X_1 \times X_2 \times \Omega$ , while the  $G \times G$ -action is given by

$$(g_1, g_2) : (x_1, x_2, \omega) \mapsto (g_1.x_1, g_2.x_2, (\gamma_1, \gamma_2)\omega) \quad \text{where} \quad \gamma_i = c_i(g_i, x_i).$$

By the assumption there exists a measurable map  $\bar{\Phi} : \bar{\Omega} \rightarrow \mathcal{G}$  so that

$$\bar{\Phi}((g_1, g_2)(x_1, x_2, \omega)) = \pi(g_1) \cdot \bar{\Phi}(x_1, x_2, \omega) \cdot \pi(g_2)^{-1} \quad (g_1, g_2 \in G)$$

for a.e.  $(x_1, x_2, \omega) \in \bar{\Omega}$ . Fix a generic pair  $(x_1, x_2) \in X_1 \times X_2$ , denote  $h_i = \sigma_i(x_i)$  and consider  $g_i = \gamma_i^{h_i}$  ( $= h_i \gamma_i h_i^{-1}$ ), where  $\gamma_i \in \Gamma_i$  for  $i \in \{1, 2\}$ . Then  $g_i.x_i = x_i$ ,

$c_i(g_i, x_i) = \gamma_i$  and the map  $\Phi' : \Omega \rightarrow \mathcal{G}$  defined by  $\Phi'(\omega) = \bar{\Phi}(x_1, x_2, \omega)$  satisfies  $m$ -a.e.

$$\begin{aligned} \Phi'((\gamma_1, \gamma_2)\omega) &= \bar{\Phi}((g_1, g_2)(x_1, x_2, \omega)) = \pi(g_1) \cdot \Phi'(\omega) \cdot \pi(g_2)^{-1} \\ &= \pi(\gamma_1^{h_1}) \cdot \Phi'(\omega) \cdot \pi(\gamma_2^{h_2})^{-1}. \end{aligned}$$

Thus  $\Phi(\omega) = \pi(h_1)^{-1}\Phi'(\omega)\pi(h_2)$  is a  $\Gamma_1 \times \Gamma_2$ -equivariant measurable map  $\Omega \rightarrow \mathcal{G}$ , as required.  $\square$

*Proof of Proposition 2.8.* Assuming that  $G$  is taut relative to  $\pi : G \rightarrow \mathcal{G}$  and  $\Gamma < G$  is a lattice, we shall show that  $\Gamma$  is taut relative to  $\pi|_{\Gamma} : \Gamma \rightarrow \mathcal{G}$ .

Let  $(\Omega, m)$  be a  $(\Gamma, \Gamma)$ -coupling. Then the  $(G, G)$ -coupling  $\bar{\Omega} = G \times_{\Gamma} \Omega \times_{\Gamma} G$  is taut relative to  $\mathcal{G}$ , and by Lemma 2.9,  $\bar{\Omega}$  admits a  $\Gamma \times \Gamma$ -tautening map  $\bar{\Phi} : \bar{\Omega} \rightarrow \mathcal{G}$ . Since  $\mathcal{G}$  is strongly ICC relative to  $\pi(G) < \mathcal{G}$ , the map  $\bar{\Phi} : \bar{\Omega} \rightarrow \mathcal{G}$  is unique as a  $\Gamma \times \Gamma$ -equivariant map (Lemma A.6.(3)). This shows that  $\Gamma$  is taut relative to  $\mathcal{G}$ .

Observe, that if  $G$  is assumed to be only  $p$ -taut, while  $\Gamma < G$  to be  $L^p$ -integrable, then the preceding argument for the existence of  $\Gamma \times \Gamma$ -tautening map for a  $L^p$ -integrable  $(\Gamma, \Gamma)$ -coupling  $\bar{\Omega}$  still applies. Indeed, the composed coupling  $\bar{\Omega} = G \times_{\Gamma} \Omega \times_{\Gamma} G$  is then  $L^p$ -integrable and therefore admits a  $G \times G$ -tautening map  $\bar{\Phi} : \bar{\Omega} \rightarrow \mathcal{G}$ , leading to a  $\Gamma \times \Gamma$ -tautening map  $\bar{\Phi} : \bar{\Omega} \rightarrow \mathcal{G}$ .

Next assume that  $\Gamma < G$  is a lattice and  $\Gamma$  is taut (resp.  $p$ -taut) relative to  $\pi|_{\Gamma} : \Gamma \rightarrow \mathcal{G}$ . Let  $(\Omega, m)$  be a  $(G, G)$ -coupling (resp. a  $L^p$ -integrable one). Then  $(\Omega, m)$  is also a  $(\Gamma, \Gamma)$ -coupling (resp. a  $L^p$ -integrable one). Since  $\Gamma$  is assumed to be taut (resp.  $p$ -taut) there is a  $\Gamma \times \Gamma$ -equivariant map  $\bar{\Phi} : \bar{\Omega} \rightarrow \mathcal{G}$ . As  $\mathcal{G}$  is strongly ICC relative to  $\pi(G)$  it follows from (4) in Lemma A.6 that  $\bar{\Phi} : \bar{\Omega} \rightarrow \mathcal{G}$  is automatically  $G \times G$ -equivariant. The uniqueness of tautening maps follows from the strong ICC assumption.  $\square$

**Remark 2.10.** The explicit assumption that  $\mathcal{G}$  is strongly ICC relative to  $\pi(G)$  is superfluous. If no integrability assumptions are imposed, the strong ICC follows from the tautness assumption by Lemma A.5. However, if one assumes merely  $p$ -tautness, the above lemma yields strong ICC property for a restricted class of measures; and the argument that this is sufficient becomes unjustifiably technical in this case.

### 3. THE ISOMETRY GROUP OF HYPERBOLIC SPACE IS 1-TAUT

**3.1. Proofs of Theorems B and C.** We prove Theorems B and C relying on the results of Subsections 3.2–3.5. Throughout, let  $G = \text{Isom}(\mathbf{H}^n)$  be the isometry group of hyperbolic  $n$ -space. We assume that  $n \geq 2$ . Let

$$\mathcal{G} = \begin{cases} G & \text{if } n \geq 3, \\ \text{Homeo}(S^1) & \text{if } n = 2. \end{cases}$$

Further, we define

$$G \xrightarrow{\pi} \mathcal{G} = \begin{cases} G \xrightarrow{\text{id}} G & \text{if } n \geq 3, \\ \text{standard action of } G \text{ on } \partial\mathbf{H}^2 \cong S^1 & \text{if } n = 2. \end{cases}$$

Theorems B and C state that  $G$  is 1-taut relative to  $\pi : G \rightarrow \mathcal{G}$ .

**3.1.1. Reduction to cocycles of lattices and ergodicity.** Let  $\Gamma < G^0$  be a torsion-free uniform lattice in the connected component  $G^0 \subset G$  of the identity. By Proposition 2.8 it suffices to show that  $\Gamma$  is 1-taut relative to  $\pi = \pi|_{\Gamma}: \Gamma \rightarrow \mathcal{G}$ . Let  $(\Omega, m)$  be an integrable  $(\Gamma, \Gamma)$ -coupling. It is sometimes convenient to denote the left copy of  $\Gamma$  by  $\Gamma_l$  and the right copy by  $\Gamma_r$ .

By Lemma A.4 the  $(\Gamma, \Gamma)$ -coupling  $\Omega$  is taut relative to  $\pi$  if and only if there is an essentially unique measurable map  $f: X \rightarrow \mathcal{G}$  such that a.e.

$$(3.1) \quad \pi \circ \alpha(\gamma, x) = f(\gamma.x)\pi(\gamma)f(x)^{-1},$$

that is, the cocycle  $\pi \circ \alpha$  is conjugate to the constant cocycle  $\pi$ . By [13, Corollary 3.6]<sup>8</sup> it is sufficient to prove (3.1) on a.e. ergodic component of  $X$ , each of which corresponds to an ergodic component in the ergodic decomposition  $(\Omega, m_t)$  of the coupling  $\Omega$  [15, Lemma 2.2] where  $m = \int m_t d\eta(t)$  and  $\eta$  some probability measure. Let  $\alpha: \Gamma_r \times X \rightarrow \Gamma_l$  be an integrable ME-cocycle associated to a  $\Gamma_l$ -fundamental domain  $X \subset \Omega$ . Let  $|\cdot|: \Gamma \rightarrow \mathbb{N}$  be the length function associated to some word-metric on  $\Gamma$ . Then the integrability of  $(\Omega, m)$  means that for every  $\gamma \in \Gamma$

$$\int \int_X |\alpha(\gamma, x)| dm_t(x) d\eta(t) = \int_X |\alpha(\gamma, x)| dm(x) < \infty,$$

which yields that  $\int_X |\alpha(\gamma, x)| dm_t(x) < \infty$  for  $\eta$ -a.e.  $t$ . Hence  $(\Omega, m_t)$  is integrable for  $\eta$ -a.e.  $t$ . Thus we may assume for the rest of the proof that  $(\Omega, m)$  is an ergodic and integrable  $(\Gamma, \Gamma)$ -coupling. By [2, Corollary 1.11], the coupling index of  $\Omega$  is 1.

**3.1.2. Volume cocycle.** We identify the boundary at infinity  $\partial\mathbf{H}^n$  with  $B = G/P$  and endow it with the measure class of the push-forward of the Haar measure on  $G$ . In the functorial theory of bounded cohomology as developed by Burger-Monod [6, 41], the measurable map

$$\text{dvol}_b: B^{n+1} \rightarrow \mathbb{R}$$

that assigns to  $(b_0, \dots, b_n)$  the oriented volume of the geodesic, ideal simplex with vertices  $b_0, \dots, b_n$  is a  $\Gamma$ -invariant (even  $G^0$ -invariant) cocycle and defines an element  $\text{dvol}_b \in H_b^n(\Gamma, \mathbb{R})$  (Theorem B.4). The forgetful map (comparison map) from bounded cohomology to ordinary cohomology is denoted by

$$\text{comp}^\bullet: H_b^\bullet(\Gamma, \mathbb{R}) \rightarrow H^\bullet(\Gamma, \mathbb{R}).$$

By Theorem 3.9 the bounded cocycle  $\text{dvol}_b$  is a lift of the volume cocycle  $\text{dvol} \in H^n(\Gamma, \mathbb{R}) \cong H^n(\Gamma \backslash \mathbf{H}^n, \mathbb{R})$  of the  $n$ -dimensional closed manifold  $\Gamma \backslash \mathbf{H}^n$ , that is,

$$\text{comp}^n(\text{dvol}_b) = \text{dvol}.$$

**3.1.3. A higher-dimensional Milnor-Wood inequality.** To show the existence of  $f$  as in (3.1), we consider the induction homomorphism

$$H_b^\bullet(\Omega): H_b^\bullet(\Gamma_l, L^\infty(\Gamma_r \backslash \Omega)) \rightarrow H_b^\bullet(\Gamma_r, L^\infty(\Gamma_l \backslash \Omega))$$

in bounded cohomology associated to  $\Omega$  (see Subsection 3.2). Let

$$H_b^\bullet(j^\bullet): H_b^\bullet(\Gamma_l, \mathbb{R}) \rightarrow H_b^\bullet(\Gamma_l, L^\infty(\Gamma_r \backslash \Omega))$$

$$H_b^\bullet(\Gamma^\bullet): H_b^\bullet(\Gamma_r, L^\infty(\Gamma_l \backslash \Omega)) \rightarrow H_b^\bullet(\Gamma_r, \mathbb{R})$$

<sup>8</sup>The target  $\mathcal{G}$  is assumed to be locally compact in this reference but the proof therein works the same for a Polish group  $\mathcal{G}$ .

be the homomorphisms induced by inclusion of constant functions in the coefficients and by integration in the coefficients, respectively. Inspired by the classical Euler number of a surface representation we define:

**Definition 3.1** (Higher-dimensional Euler number). Let  $[\Gamma] \in H_n(\Gamma, \mathbb{R})$  be the homological fundamental class of the manifold  $\Gamma \backslash \mathbf{H}^n$ . The *Euler number*  $\text{eu}(\Omega)$  of  $\Omega$  is the evaluation of the cohomology class  $\text{comp}^n \circ H_b^n(\mathbf{I}^\bullet) \circ H_b^n(\Omega) \circ H_b^n(j^\bullet)(\text{dvol}_b)$  against the fundamental class  $[\Gamma]$

$$\text{eu}(\Omega) = \langle \text{comp}^n \circ H_b^n(\mathbf{I}^\bullet) \circ H_b^n(\Omega) \circ H_b^n(j^\bullet)(\text{dvol}_b), [\Gamma] \rangle.$$

For any  $(\Gamma, \Gamma)$ -coupling  $\Omega$  (without assuming integrability) we prove the following *higher-dimensional Milnor-Wood inequality* in Theorem 3.10:

$$(3.2) \quad |\text{eu}(\Omega)| \leq \text{vol}(\Gamma \backslash \mathbf{H}^n).$$

**3.1.4. Maximality of the Euler number provided  $\Omega$  is integrable.** Next we appeal to the following general result from our companion paper [2], which relies on the integrability of the coupling. In fact, this is the only place in the proof where we use the integrability.

**Theorem 3.2** ([2, Theorem 5.12]). *Let  $M$  and  $N$  be closed, oriented, negatively curved manifolds of dimension  $n$ . Let  $(\Omega, \mu)$  be an ergodic, integrable ME-coupling  $(\Omega, \mu)$  of the fundamental groups  $\Gamma = \pi_1(M)$  and  $\Lambda = \pi_1(N)$  with coupling index  $c = \frac{\mu(\Lambda \backslash \Omega)}{\mu(\Gamma \backslash \Omega)}$ . Suppose that  $x_\Gamma^b \in H_b^n(\Gamma, \mathbb{R})$  is an element that maps to the cohomological fundamental class  $x_G \in H^n(\Gamma, \mathbb{R}) \cong H^n(M, \mathbb{R})$  of  $M$  under the comparison map. Define  $x_\Lambda \in H^n(\Lambda, \mathbb{R})$  analogously. Then the composition*

$$(3.3) \quad H_b^n(\Gamma, \mathbb{R}) \xrightarrow{H_b^n(j^\bullet)} H_b^n(\Gamma, L^\infty(\Lambda \backslash \Omega, \mathbb{R})) \xrightarrow{H_b^n(\Omega)} H_b^n(\Lambda, L^\infty(\Gamma \backslash \Omega, \mathbb{R})) \\ \xrightarrow{H_b^n(\mathbf{I}^\bullet)} H_b^n(\Lambda, \mathbb{R}) \xrightarrow{\text{comp}^n} H^n(\Lambda, \mathbb{R})$$

sends  $x_\Gamma^b$  to  $\pm c \cdot x_\Lambda$ .

We apply this theorem to  $M = N = \Gamma \backslash \mathbf{H}^n$  and  $\Lambda = \Gamma$ . In our case the coupling index is 1. Therefore the bounded class  $\text{dvol}_b$  is mapped to  $\pm \text{dvol}$  under (3.3). In other words, the Euler class of  $\Omega$  is maximal:

$$(3.4) \quad \text{eu}(\Omega) = \pm \text{vol}(\Gamma \backslash \mathbf{H}^n).$$

**3.1.5. Boundary maps.** Next we want to express (3.4) in terms of the *boundary map* associated to the cocycle  $\alpha$ . Boundary theory, in the sense of Furstenberg [18] (see [7, Corollary 3.2], or [43, Proposition 3.3] for a detailed argument applying to our situation), yields the existence of an essentially unique measurable map, called *boundary map* or *Furstenberg map*,

$$(3.5) \quad \phi : X \times B \rightarrow B \quad \text{satisfying} \quad \phi(\gamma x, \gamma b) = \alpha(\gamma, x)\phi(x, b)$$

for every  $\gamma \in \Gamma$  and a.e.  $(x, b) \in X \times B$ .

To deal with some measurability issues we need the following construction. For a standard Borel probability space  $(S, \nu)$  and a Polish space  $W$  we consider the set  $F(S, W)$  of measurable functions  $S \rightarrow W$ , where two functions are identified if they agree on  $\nu$ -conull set. One can endow  $F(S, W)$  with the topology of convergence in measure. The Borel algebra of this topology turns  $F(S, W)$  into a standard Borel

space [13, Section 2A]. Two different Polish topologies on  $W$  with the same Borel algebra give rise to the same standard Borel space  $F(S, W)$  [13, Remark 2.5].

In our situation, the map  $\phi$  gives rise to a measurable map  $f : X \rightarrow F(B, B)$  defined for almost every  $x \in X$  by  $f(x) = \phi(x, -)$  [13, Corollary 2.9]. We also write  $f(x) = \phi_x$ .

Theorem 3.10 below, allows us to express the Euler class  $\text{eu}(\Omega)$  in terms of the boundary map  $\phi$ , and interpret the equality in (3.4) as

$$\int_X \int_{G^0/\Gamma} \text{vol}(\phi_x(gz_0), \dots, \phi_x(gz_n)) dg d\mu(x) = \pm v_{\max}.$$

where  $v_{\max}$  is the volume of a positively oriented ideal maximal simplex  $(z_0, \dots, z_n)$  in  $B^{n+1}$  and the quotient  $G^0/\Gamma$  carries the normalized Haar measure. Since the integrand is at most  $v_{\max}$  for a.e.  $x \in X$ , we conclude: Either the ideal simplex  $(\phi_x(gb_0), \dots, \phi_x(gb_n))$  is non-degenerate and positively oriented for a.e.  $g \in G$  and a.e.  $x \in X$ , or  $(\phi_x(gb_0), \dots, \phi_x(gb_n))$  is non-degenerate and negatively oriented for a.e.  $g \in G$  and a.e.  $x \in X$ .

For  $n = 2$  any non-degenerate ideal triangle has oriented volume  $\pm v_{\max}$ ; hence either for a.e.  $x \in X$  the map  $\phi_x : S^1 \rightarrow S^1$  preserves the cyclic order of a.e. triple of points on the circle  $B = \partial\mathbf{H}^2 = S^1$ , or for a.e.  $x \in X$  the map  $\phi_x$  reverses the orientation of a.e. triple on the circle.

In the  $n \geq 3$  case it follows that  $(\phi_x(gb_0), \dots, \phi_x(gb_n))$  is a maximal, hence regular, ideal simplex for a.e.  $g \in G$  and a.e.  $x \in X$ .

**3.1.6. Conclusion.** Firstly consider the case  $n \geq 3$ . By a general fact about standard Borel spaces the measurable injection  $j : G \rightarrow F(B, B)$  given by the action of  $G$  on  $B$  by Moebius transformations is a Borel isomorphism onto its image and the image is measurable in  $F(B, B)$  [32, Corollary 15.2 on p. 89]. Lemma 3.16 yields that the image of  $f : X \rightarrow F(B, B)$  is contained in  $j(G)$ , thus  $f$  can be regarded as a measurable map  $X \rightarrow G$ . Equation (3.5) for  $\phi$  implies that  $f$  satisfies equation (3.1), which concludes the proof of Theorem B.

Next let  $n = 2$ . Here  $B = S^1$ . By *loc. cit.* the measurable injective map  $j : \text{Homeo}(S^1) \rightarrow F(S^1, S^1)$  is a Borel isomorphism of  $\text{Homeo}(S^1)$  onto its measurable image. By Proposition 3.13 the image of  $f : X \rightarrow F(S^1, S^1)$  is contained in the image of  $j$ . Thus  $f$  can be regarded as a measurable map to  $\text{Homeo}(S^1)$ . Again, we conclude that  $f$  satisfies (3.1), which finishes the proof of Theorem C.

**3.2. The cohomological induction map.** The following *cohomological induction map* associated to an ME-coupling was introduced by Monod and Shalom in [42].

**Proposition 3.3** (Monod-Shalom). *Let  $(\Omega, m)$  be a  $(\Gamma, \Lambda)$ -coupling. Let  $Y \subset B$  be a measurable fundamental domain for the  $\Gamma$ -action. Let  $\chi : \Omega \rightarrow \Gamma$  be the measurable  $\Gamma$ -equivariant map uniquely defined by  $\chi(\omega)^{-1}\omega \in Y$  for  $\omega \in \Omega$ . The maps*

$$\begin{aligned} C_b^\bullet(\chi) : C_b^\bullet(\Gamma, L^\infty(\Omega)) &\rightarrow C_b^\bullet(\Lambda, L^\infty(\Omega)) \\ C_b^k(\chi)(f)(\lambda_0, \dots, \lambda_k)(y) &= f(\chi(\lambda_0^{-1}y), \dots, \chi(\lambda_k^{-1}y))(y) \end{aligned}$$

*defines a  $\Gamma \times \Lambda$ -equivariant chain morphism with regard to the following actions: The  $\Gamma \times \Lambda$ -action on  $C_b^\bullet(\Gamma, L^\infty(\Omega)) \cong L^\infty(\Gamma^{\bullet+1} \times \Omega)$  is induced by  $\Gamma$  acting diagonally on  $\Gamma^{\bullet+1} \times \Omega$  and by  $\Lambda$  acting only on  $\Omega$ . The  $\Gamma \times \Lambda$ -action on  $C_b^\bullet(\Lambda, L^\infty(\Omega)) \cong L^\infty(\Lambda^{\bullet+1} \times \Omega)$  is induced by  $\Lambda$  acting diagonally on  $\Lambda^{\bullet+1} \times \Omega$  and by  $\Gamma$  acting only on  $\Omega$ .*

The chain map  $C_b^\bullet(\chi)$  induces, after taking  $\Gamma \times \Lambda$ -invariants and identifying  $L^\infty(\Gamma \backslash \Omega)$  with  $L^\infty(\Omega)^\Gamma$  and similarly for  $\Lambda$ , an isometric isomorphism

$$H_b^\bullet(\chi): H_b^\bullet(\Gamma, L^\infty(\Lambda \backslash \Omega)) \xrightarrow{\cong} H_b^\bullet(\Lambda, L^\infty(\Gamma \backslash \Omega)).$$

in cohomology. This map does not depend on the choice of  $Y$ , or equivalently  $\chi$ , and will be denoted by  $H_b^\bullet(\Omega)$ . We call  $H_b^\bullet(\Omega)$  the cohomological induction map associated to  $\Omega$ .

*Proof.* Apart from the fact that the isomorphism is isometric, this is exactly Proposition 4.6 in [42] (with  $S = \Omega$  and  $E = \mathbb{R}$ ). The proof therein relies on [41, Theorem 7.5.3 in §7], which also yields the isometry statement.  $\square$

**Proposition 3.4.** *Retain the setting of the previous proposition. Let  $\alpha: \Lambda \times Y \rightarrow \Gamma$  be the corresponding ME-cocycle. Let  $B_\Gamma$  and  $B_\Lambda$  be standard Borel spaces endowed with probability Borel measures and measure-class preserving Borel actions of  $\Gamma$  and  $\Lambda$ , respectively. Assume the action on  $B_\Lambda$  is amenable in the sense of Zimmer. Let  $\phi: B_\Lambda \times \Gamma \backslash \Omega \rightarrow B_\Gamma$  be a measurable  $\alpha$ -equivariant map (upon identifying  $Y$  with  $\Gamma \backslash \Omega$ ). Then the chain morphism*

$$\begin{aligned} C_b^\bullet(\phi): \mathcal{B}^\infty(B_\Gamma^{\bullet+1}, \mathbb{R}) &\rightarrow L_{w^*}^\infty(B_\Lambda^{\bullet+1}, L^\infty(\Omega)) \\ C_b^k(\phi)(f)(\dots, b_i, \dots)(\omega) &= f(\dots, \chi(\omega)\phi(b_i, [\omega]), \dots). \end{aligned}$$

is  $\Gamma \times \Lambda$ -equivariant with regard to the following actions: The action on  $\mathcal{B}^\infty(B_\Gamma^{\bullet+1}, \mathbb{R})$  is induced from  $\Gamma$  acting diagonally on  $B_\Gamma^{\bullet+1}$  and  $\Lambda$  acting trivially. The action on  $L_{w^*}^\infty(B_\Lambda^{\bullet+1}, L^\infty(\Omega)) \cong L^\infty(B_\Lambda^{\bullet+1} \times \Omega)$  is induced from  $\Lambda$  acting diagonally on  $B_\Lambda^{\bullet+1} \times \Omega$  and from  $\Gamma$  acting only on  $\Omega$ .

Further, every  $\Gamma \times \Lambda$ -chain morphism from  $\mathcal{B}^\infty(B_\Gamma^{\bullet+1}, \mathbb{R})$  to  $L_{w^*}^\infty(B_\Lambda^{\bullet+1}, L^\infty(\Omega))$  that induces the same homomorphism on  $H_b^0$  as  $C_b^\bullet(\phi)$  is equivariantly chain homotopic to  $C_b^\bullet(\phi)$ .

*Proof.* Firstly, we show equivariance of  $C_b^\bullet(\phi)$ . By definition we have

$$C_b^\bullet(\phi)((\gamma, \lambda)f)(\dots, b_i, \dots)(\omega) = f(\dots, \gamma^{-1}\chi(\omega)\phi(b_i, [\omega]), \dots).$$

By definition,  $\Gamma$ -equivariance of  $\chi$ , and  $\alpha$ -equivariance of  $\phi$  we have

$$C_b^\bullet(\phi)(f)(\dots, \lambda^{-1}b_i, \dots)(\gamma^{-1}\lambda^{-1}\omega) = f(\dots, \gamma^{-1}\chi(\lambda^{-1}\omega)\alpha(\lambda^{-1}, [\omega])\phi(b_i, [\omega]), \dots).$$

It remains to check that

$$\chi(\lambda^{-1}\omega)\alpha(\lambda^{-1}, [\omega]) = \chi(\omega).$$

Since both sides are  $\Gamma$ -equivariant, we may assume that  $\omega \in Y$ , i.e.,  $\chi(\omega) = 1$ . In this case it follows from the defining properties of  $\chi$  and  $\alpha$ .

Next we prove the uniqueness up to equivariant chain homotopy. By Proposition B.3 the complex  $\mathcal{B}^\infty(B_\Gamma^{\bullet+1}, \mathbb{R})$  is a strong resolution of the trivial  $\Gamma \times \Lambda$ -module  $\mathbb{R}$ . The  $\Gamma \times \Lambda$ -action on  $B_\Lambda^{\bullet+1} \times \Omega$  is amenable if the  $\Lambda$ -action on  $B_\Lambda^{\bullet+1} \times \Gamma \backslash \Omega$  is amenable [1, Corollary C]. The latter action is amenable since the  $\Lambda$ -action on  $B_\Lambda$  is amenable and because of [61, Proposition 4.3.4 on p. 79]. By Theorem B.4  $L_{w^*}^\infty(B_\Lambda^{\bullet+1}, L^\infty(\Omega)) \cong L^\infty(B_\Lambda^{\bullet+1} \times \Omega)$  is a relatively injective, strong resolution of the trivial  $\Gamma \times \Lambda$ -module, and Theorem B.1 yields uniqueness up to equivariant homotopy.  $\square$

**Remark 3.5.** The map  $C_b^\bullet(\phi)$  cannot be defined on  $L^\infty(B_\Gamma^{\bullet+1}, \mathbb{R})$  since we do not assume that  $\phi$  preserves the measure class. The idea to work with the complex  $\mathcal{B}^\infty(B_\Gamma^{\bullet+1}, \mathbb{R})$  to circumvent this problem in the context of boundary maps is due to Burger and Iozzi [4].

**3.3. The Euler number in terms of boundary maps.** In the Burger-Monod approach to bounded cohomology one can realize bounded cocycles in the bounded cohomology of  $\Gamma$  as cocycles on the boundary  $B$ . However, it is not immediately clear how the evaluation of a bounded  $n$ -cocycle realized on  $B$  at the fundamental class of  $\Gamma \backslash \mathbf{H}^n$  can be explicitly computed since the fundamental class is not defined in terms of the boundary. Lemma 3.8 below achieves just that. The two important ingredients that go into its proof are Thurston's measure homology and the cohomological Poisson transform  $PT^\bullet : L^\infty(B^{\bullet+1}, \mathbb{R}) \rightarrow C_b^\bullet(\Gamma, \mathbb{R})$  (see Definition B.5).

**Definition 3.6.** For  $z \in \mathbf{H}^n$  let  $\nu_z$  be the *visual measure* at  $z$  on the boundary  $B = \partial \mathbf{H}^n$  at infinity, that is,  $\nu_z$  is the push-forward of the Lebesgue measure on the unit tangent sphere  $T_z^1 \mathbf{H}^n$  under the homeomorphism  $T_z^1 \mathbf{H}^n \rightarrow \partial \mathbf{H}^n$  given by the exponential map. For a  $(k+1)$ -tuple  $\sigma = (z_0, \dots, z_k)$  of points in  $\mathbf{H}^n$  we denote the product of the  $\nu_{z_i}$  on  $B^{k+1}$  by  $\nu_\sigma$ .

**Remark 3.7.** The measure  $\nu_z$  is the unique Borel probability measure on  $B$  that is invariant with respect to the stabilizer of  $z$ . All visual measures are in the same measure class. Moreover, we have  $\nu_{gz} = g_* \nu_z = \nu_z(g^{-1} \cdot)$  for every  $g \in G$ .

**Lemma 3.8.** *Let  $\Gamma \subset G^0$  be a torsion-free and uniform lattice. Let  $\sigma_0 = (z_0, \dots, z_n)$  be a positively oriented geodesic simplex in  $\mathbf{H}^n$ . Let  $f \in L^\infty(B^{n+1}, \mathbb{R})^\Gamma$  be an alternating cocycle. Then*

$$\langle \text{comp}^n \circ H_b^n(PT^\bullet)(f), [\Gamma] \rangle = \frac{\text{vol}(\Gamma \backslash \mathbf{H}^n)}{\text{vol}(\sigma_0)} \int_{\Gamma \backslash G^0} \int_{B^{n+1}} f(gb_0, \dots, gb_n) d\nu_{\sigma_0} dg.$$

*Proof.* We need Thurston's description of singular homology by *measure cycles* [58]: Let  $M$  be a topological space. We equip the space  $\mathcal{S}_k(M) = \text{Map}(\Delta^k, M)$  of continuous maps from the standard  $k$ -simplex to  $M$  with the compact-open topology. The group  $C_k^m(M)$  is the vector space of all signed, compactly supported Borel measures on  $\mathcal{S}_k(M)$  with finite total variation. The usual face maps  $\partial_i : \mathcal{S}_k(M) \rightarrow \mathcal{S}_{k-1}(M)$  are measurable, and the maps  $C_k^m(M) \rightarrow C_{k-1}^m(M)$  that send  $\mu$  to  $\sum_{i=0}^k (-1)^i (\partial_i)_* \mu$  turn  $C_\bullet^m(M)$  into a chain complex. The map

$$D_\bullet : C_\bullet(M) \rightarrow C_\bullet^m(M), \quad \sigma \mapsto \delta_\sigma$$

that maps a singular simplex  $\sigma$  to the point measure concentrated at  $\sigma$  is a chain map that induces an (isometric) homology isomorphism provided  $M$  is homeomorphic to a CW-complex [37, 59]. We will consider the case  $M = \Gamma \backslash \mathbf{H}^n$  next.

Fix a basepoint  $x \in \mathbf{H}^n$ . Consider the  $\Gamma$ -equivariant chain homomorphism  $j_k : C_k(\Gamma) \rightarrow C_k(\mathbf{H}^n)$  that maps  $(\gamma_0, \dots, \gamma_k)$  to the geodesic simplex with vertices  $(\gamma_0 x, \dots, \gamma_k x)$ . Let  $\mathcal{B}^\infty(\mathcal{S}_\bullet(\mathbf{H}^n), \mathbb{R}) \subset C^\bullet(\mathbf{H}^n)$  be the subcomplex of bounded measurable singular cochains on  $\mathbf{H}^n$ . The Poisson transform<sup>9</sup> factorizes as

$$L^\infty(B^{\bullet+1}, \mathbb{R}) \xrightarrow{P^\bullet} \mathcal{B}^\infty(\mathcal{S}_\bullet(\mathbf{H}^n), \mathbb{R}) \xrightarrow{R^\bullet} C_b^\bullet(\Gamma)$$

<sup>9</sup>Here the Poisson transform is defined in terms of  $\nu_x$ . Since the visual measures are all in the same measure class, the Poisson transform in cohomology does not depend on the choice of  $x$  (see the remark after Definition B.5).

where  $P^k(l)(\sigma) = \int_{B^{k+1}} l(b_0, \dots, b_k) d\nu_\sigma$  and  $R^k(f) = f \circ j_k$ . For every  $k \geq 0$  there is a Borel section  $s_k : \mathcal{S}_k(\Gamma \backslash \mathbf{H}^n) \rightarrow \mathcal{S}_k(\mathbf{H}^n)$  of the projection [37, Theorem 4.1]. The following pairing is independent of the choice of  $s_k$  and descends to cohomology:

$$\begin{aligned} \langle -, - \rangle_m : \mathcal{B}^\infty(\mathcal{S}_\bullet(\mathbf{H}^n), \mathbb{R})^\Gamma \otimes \mathbb{C}_\bullet^m(\Gamma \backslash \mathbf{H}^n) &\rightarrow \mathbb{R} \\ \langle l, \mu \rangle_m &= \int_{\mathcal{S}_\bullet(\Gamma \backslash \mathbf{H}^n)} l(s_\bullet(\sigma)) d\mu(\sigma) \end{aligned}$$

One sees directly from the definitions that for every  $x \in H_n(\Gamma)$

$$(3.6) \quad \begin{aligned} \langle \text{comp}^n \circ H^n(\text{PT}^\bullet)(f), x \rangle &= \langle \text{comp}^n \circ H^n(\mathbf{R}^\bullet) \circ H^n(\mathbf{P}^\bullet)(f), x \rangle \\ &= \langle H^n(\mathbf{P}^\bullet)(f), H_n(\mathbf{D}_\bullet \circ j_\bullet)(x) \rangle_m. \end{aligned}$$

For any positively oriented geodesic  $n$ -simplex  $\sigma$ , let  $\text{sm}(\sigma)$  denote the push-forward of the normalized Haar measure under the measurable map

$$\Gamma \backslash G^0 \rightarrow \text{Map}(\Delta^n, \Gamma \backslash \mathbf{H}^n), \quad g \mapsto \text{pr}(g\sigma).$$

Let  $\rho \in G$  be the orientation reversing isometry that maps  $(z_0, z_1, \dots, z_n)$  to  $(z_1, z_0, \dots, z_n)$ . By [52, Theorem 11.5.4 on p. 551] the image  $H_n(\mathbf{D}_\bullet \circ j_\bullet)([\Gamma])$  of the fundamental class  $[\Gamma] \in H_n(\Gamma) \cong H_n(M)$  of  $M$  is represented by the measure<sup>10</sup>

$$\frac{\text{vol}(\Gamma \backslash \mathbf{H}^n)}{2 \text{vol}(\sigma_0)} (\text{sm}(\sigma_0) - \text{sm}(\rho \circ \sigma_0)).$$

In combination with (3.6) and the fact that  $f$  is alternating, this yields the assertion.  $\square$

The next theorem is well known to experts, and we only prove it for the lack of a good reference. Although it can be seen as a special case of Theorem 3.10 we separate the proofs. The proofs of Theorems 3.9 and 3.10 are given at the end of the subsection.

**Theorem 3.9.** *Let  $\Gamma \subset G^0$  be a torsion-free and uniform lattice. Then*

$$\langle \text{comp}^n(\text{dvol}_b), [\Gamma] \rangle = \text{vol}(\Gamma \backslash \mathbf{H}^n).$$

*Equivalently, this means that  $\text{comp}^n(\text{dvol}_b) = \text{dvol}$ .*

We view the following theorem as a higher-dimensional cocycle analog of the Milnor-Wood inequality for homomorphisms of a surface group into  $\text{Homeo}_+(S^1)$ .

**Theorem 3.10** (Higher-dimensional Milnor-Wood inequality). *Let  $(\Omega, m)$  be a  $(\Gamma, \Gamma)$ -coupling of a torsion-free and uniform lattice  $\Gamma \subset G^0$ . Let  $\phi : X \times B \rightarrow B$  be the  $\alpha$ -equivariant boundary map from (3.5), where  $\alpha : \Gamma_r \times X \rightarrow \Gamma_l$  is a ME-cocycle for  $\Omega$ . If  $\sigma = (z_0, \dots, z_n)$  with  $z_i \in B$  is a positively oriented ideal regular simplex, then*

$$\text{eu}(\Omega) = \frac{\text{vol}(\Gamma \backslash \mathbf{H}^n)}{v_{\max}} \int_X \int_{\Gamma \backslash G^0} \text{vol}(\phi_x(gz_0), \dots, \phi_x(gz_n)) dg d\mu(x).$$

*In particular, we have the inequality  $|\text{eu}(\Omega)| \leq \text{vol}(\Gamma \backslash \mathbf{H}^n)$ .*

We shall need the upcoming, auxiliary Lemmas 3.11 and 3.12 before we conclude the proof of the preceding theorem at the end of this subsection. We retain the setting of Theorem 3.10 for the rest of this subsection.

<sup>10</sup>The reader should note that in *loc. cit.* the Haar measure is normalized by  $\text{vol}(\Gamma \backslash \mathbf{H}^n)$  whereas we normalize it by 1.

**Lemma 3.11.** *If  $\sigma = (z_0, \dots, z_n)$  with  $z_i \in \mathbf{H}^n$  is a positively oriented geodesic simplex, then*

$$\text{eu}(\Omega) = \frac{\text{vol}(\Gamma \backslash \mathbf{H}^n)}{\text{vol}(\sigma)} \int_X \int_{G^0/\Gamma} \int_{B^{n+1}} \text{vol}(\phi_x(gb_0), \dots, \phi_x(gb_n)) d\nu_\sigma dg d\mu(x).$$

*Proof.* Consider the diagram below. The unlabelled maps are the obvious ones, sending a function to its equivalence class up to null sets and inclusion of constant functions. For better readability, we denote the copy of  $B$  on which  $\Gamma_l$  acts by  $B_l$ ; similarly for  $B_r$ . All the maps are  $\Gamma_l \times \Gamma_r$ -equivariant chain morphisms as explained now. On  $L_{w*}^\infty(B_l^{\bullet+1}, \mathbb{R})$  and  $C_b^\bullet(\Gamma_l, \mathbb{R})$  we have the usual  $\Gamma_l$ -actions and the trivial  $\Gamma_r$ -actions. The lower Poisson transform is then clearly  $\Gamma_l \times \Gamma_r$ -equivariant. The actions on the domain and target of the maps  $C_b^\bullet(\chi)$  and  $C_b^\bullet(\phi)$  are defined in Propositions 3.3 and 3.4, and is proven there that these maps are  $\Gamma_l \times \Gamma_r$ -equivariant. The Poisson transform in the upper row, which is  $\Gamma_r$ -equivariant, is also  $\Gamma_l$ -equivariant, since  $\Gamma_l$  acts only by its natural action on  $\Omega$ .

$$\begin{array}{ccccc} \mathcal{B}^\infty(B_l^{\bullet+1}, \mathbb{R}) & \xrightarrow{C_b^\bullet(\phi)} & L_{w*}^\infty(B_r^{\bullet+1}, L^\infty(\Omega)) & \xrightarrow{\text{PT}^\bullet} & C_b^\bullet(\Gamma_r, L^\infty(\Omega)) \\ \downarrow & & & & \uparrow C_b^\bullet(\chi) \\ L_{w*}^\infty(B_l^{\bullet+1}, \mathbb{R}) & \xrightarrow{\text{PT}^\bullet} & C_b^\bullet(\Gamma_l, \mathbb{R}) & \longrightarrow & C_b^\bullet(\Gamma_l, L^\infty(\Omega)) \end{array}$$

Using Proposition 3.4 again, one sees that the diagram commutes up to equivariant chain homotopy.

The volume cocycle  $d\text{vol}_b$ , which we defined as a cocycle in  $L_{w*}^\infty(B^{n+1}, \mathbb{R})$ , is everywhere defined and everywhere  $\Gamma$ -invariant and strictly satisfies the cocycle condition; hence it lifts to a cocycle in  $\mathcal{B}^\infty(B^{n+1}, \mathbb{R})$  which we denote by  $d\text{vol}_{\text{strict}}$ . The commutativity of the diagram up to equivariant chain homotopy yields that

$$(3.7) \quad \text{eu}(\Omega) = \langle \text{comp}^n \circ H_b^n(\Gamma^\bullet) \circ H_b^n(\text{PT}^\bullet) \circ H_b^n(\phi)(d\text{vol}_{\text{strict}}), [\Gamma] \rangle.$$

The Poisson transform in the upper row after taking  $\Gamma_l$ -invariants followed by integration in the coefficients

$$L_{w*}^\infty(B_r^{\bullet+1}, L^\infty(\Gamma_l \backslash \Omega)) \xrightarrow{\text{PT}^\bullet} C_b^\bullet(\Gamma_r, L^\infty(\Gamma_l \backslash \Omega)) \rightarrow C_b^\bullet(\Gamma_r, \mathbb{R})$$

is the same as first integrating the coefficients followed by the Poisson transform with trivial coefficients. With this fact and (3.7) in mind, we invoke Lemma 3.8 to conclude the proof.  $\square$

**Lemma 3.12.** *Fix points  $o \in \mathbf{H}^n$  and  $b_0 \in \partial \mathbf{H}^n$ . Denote by  $d = d_o$  the visual metric on  $\partial \mathbf{H}^n$  associated with  $o$ . Let  $\{z^{(k)}\}_{k=1}^\infty$  be a sequence in  $\mathbf{H}^n$  converging radially to  $b_0$ . Let  $\phi: B \rightarrow B$  be a measurable map. For every  $\epsilon > 0$  and for a.e.  $g \in G$  we have*

$$\lim_{k \rightarrow \infty} \nu_{z^{(k)}} \{b \in B \mid d(\phi(gb), \phi(gb_0)) > \epsilon\} = 0.$$

*Proof.* For the domain of  $\phi$ , it is convenient to represent  $\partial \mathbf{H}^n$  as the boundary  $\hat{\mathbb{R}}^n = \{(x_1, \dots, x_n, 0) \mid x_i \in \mathbb{R}\} \cup \{\infty\}$  of the upper half space model

$$\mathbf{H}^n = \{(x_1, \dots, x_{n+1}) \mid x_{n+1} > 0\} \subset \mathbb{R}^{n+1}.$$

We may assume that  $o = (0, \dots, 0, 1)$  and  $b_0 = 0 \in \mathbb{R}^n \subset \hat{\mathbb{R}}^n$ . The points  $z^{(k)}$  lie on the line  $l$  between  $o$  and  $b_0$ . The subgroup of  $G$  consisting of reflections along

hyperplanes containing  $l$  and perpendicular to  $\{x_{n+1} = 0\}$  leaves the measures  $\nu_{z^{(k)}}$  invariant, i.e. each  $\nu_{z^{(k)}}$  is  $O(n)$ -invariant. Since the probability measure  $\nu_{z^{(k)}}$  is in the Lebesgue measure class, the Radon-Nikodym theorem, combined with the  $O(n)$ -invariance, yields the existence of a measurable functions  $h_k : [0, \infty) \rightarrow [0, \infty)$  such that for any bounded measurable function  $l$

$$\int l d\nu_{z^{(k)}} = \int_0^\infty \left( \frac{1}{\text{vol}(B(0, r))} \int_{B(0, r)} l(y) dy \right) h_k(r) dr$$

holds<sup>11</sup> and

$$\int_0^\infty h_k(r) dr = 1.$$

Since the  $\nu_{z^{(k)}}$  weakly converge to the Dirac measure at  $0 \in \mathbb{R}^n$ , we have for every  $r_0 > 0$

$$(3.8) \quad \lim_{k \rightarrow \infty} \int_{r_0}^\infty h_k(r) dr = 0.$$

For the target of  $\phi$ , we represent  $B = \partial \mathbf{H}^n$  as the boundary  $S^{n-1} \subset \mathbb{R}^n$  of the Poincare disk model. The visual metric is then just the standard metric of the unit sphere. Considering coordinates in the target, it suffices to prove that every measurable function  $f : \mathbb{R}^n \rightarrow [-1, 1]$  satisfies

$$\lim_{k \rightarrow \infty} \int_{\hat{\mathbb{R}}^n} |f(gx) - f(g0)| d\nu_{z^{(k)}}(x) = 0.$$

for a.e.  $g \in G$ . By the Lebesgue differentiation theorem the set  $L_f$  of points  $x \in \mathbb{R}^n$  with the property

$$(3.9) \quad \lim_{r \rightarrow 0} \frac{1}{\text{vol}(B(0, r))} \int_{B(x, r)} |f(y) - f(x)| dy = 0$$

is conull in  $\mathbb{R}^n$ . The subset of elements  $g \in G$  such that  $g0 \in L_f$  and  $g0 \neq \infty$  is conull with respect to the Haar measure. From now on we fix such an element  $g \in G$ . By compactness there is  $L > 0$  such that the diffeomorphism of  $\hat{\mathbb{R}}^n$  given by  $g$  has Lipschitz constant at most  $L$  and its Jacobian satisfies  $|\text{Jac}(g)| > 1/L$  everywhere on  $\mathbb{R}^n \subset \hat{\mathbb{R}}^n$ . Let  $\epsilon > 0$ . According to (3.9) choose  $r_0 > 0$  such that for all  $r < r_0$

$$(3.10) \quad \frac{L}{\text{vol}(B(0, r))} \int_{B(g0, Lr)} |f(y) - f(g0)| dy < \frac{\epsilon}{2}.$$

According to (3.8) choose  $k_0 \in \mathbb{N}$  such that

$$\int_{r_0}^\infty h_k(r) dr < \frac{\epsilon}{4}$$

for every  $k > k_0$ . So we obtain that

$$\begin{aligned} \int_{\hat{\mathbb{R}}^n} |f(gx) - f(g0)| d\nu_{z^{(k)}} &< \int_0^{r_0} \frac{1}{\text{vol}(B(0, r))} \int_{B(0, r)} |f(gx) - f(g0)| dx h_k(r) dr + \frac{\epsilon}{2} \\ &\leq \int_0^{r_0} \frac{L}{\text{vol}(B(0, r))} \int_{gB(0, r)} |f(y) - f(g0)| dy h_k(r) dr + \frac{\epsilon}{2} \end{aligned}$$

<sup>11</sup> $\text{vol}(B(0, r))$  is here the Lebesgue measure of the Euclidean ball of radius  $r$  around  $0 \in \mathbb{R}^n$ .

for  $k > k_0$ . Because of  $gB(0, r) \subset B(g0, Lr)$  and (3.10) we obtain that for  $k > k_0$

$$\int_{\hat{\mathbb{R}}^n} |f(gx) - f(g0)| d\nu_{z^{(k)}} < \epsilon. \quad \square$$

*Proofs of Theorems 3.9 and 3.10.* We start with the proof of Theorem 3.10. For every  $i \in \{0, \dots, n\}$  we pick a sequence  $(z_i^{(k)})_{k \in \mathbb{N}}$  on the geodesic ray from a base-point  $o$  to  $z_i$  converging to  $z_i$ . Let  $\sigma_k$  be the geodesic simplex spanned by the vertices  $z_0^{(k)}, \dots, z_n^{(k)}$ . By Lemma 3.11,

$$\text{eu}(\Omega) = \frac{\text{vol}(\Gamma \backslash \mathbf{H}^n)}{\text{vol}(\sigma_k)} \int_X \int_{\Gamma \backslash G^0} \int_{B^{n+1}} \text{vol}(\phi_x(gb_0), \dots, \phi_x(gb_n)) d\nu_{\sigma_k} dg d\mu.$$

We now let  $k$  go to  $\infty$ . Note that the left hand side does not depend on  $k$ . First of all, the volumes  $\text{vol}(\sigma_k)$  converge to  $\text{vol}(\sigma) = v_{\max}$ . By Lemma 3.12,

$$\lim_{k \rightarrow \infty} \nu_{\sigma_k} \{ (b_0, \dots, b_n) \mid d(\phi_x(gz_i), \phi_x(gb_i)) < \epsilon \} = 1$$

for every  $\epsilon > 0$  and a.e.  $(x, g) \in X \times G$ . Since the volume is continuous on  $B^{n+1}$  for  $n \geq 3$  [52, Theorem 11.4.2 on p. 541] and constant on non-degenerate ideal simplices for  $n = 2$ , this implies that

$$\lim_{k \rightarrow \infty} \int_{B^{n+1}} \text{vol}(\phi_x(gb_0), \dots, \phi_x(gb_n)) d\nu_{\sigma_k} dg = \text{vol}(\phi_x(gz_0), \dots, \phi_x(gz_n)),$$

for a.e.  $(x, g) \in X \times G$ , which finally yields Theorem 3.10 by the dominated convergence theorem. The proof of Theorem 3.9 is even easier since it does not require Lemma 3.12. One obtains from Lemma 3.8 that

$$\langle \text{comp}^n(\text{dvol}_b), [\Gamma] \rangle = \frac{\text{vol}(\Gamma \backslash \mathbf{H}^n)}{\text{vol}(\sigma_k)} \int_X \int_{\Gamma \backslash G^0} \int_{B^{n+1}} \text{vol}(gb_0, \dots, gb_n) d\nu_{\sigma_k} dg d\mu$$

which converges for  $k \rightarrow \infty$  to  $\text{vol}(\Gamma \backslash \mathbf{H}^n)$  by continuity of  $\text{vol}: B^{n+1} \rightarrow \mathbb{R}$  and the weak convergence of  $\nu_{z_i^{(k)}}$  to the point measure at  $z_i$  for every  $i \in \{0, \dots, n\}$ .  $\square$

**3.4. Order-preserving measurable self maps of the circle.** Consider the function  $c$ , called the *orientation cocycle*, which is defined on triples of points on the circle  $S^1 = \partial \mathbf{H}^2$  by

$$c(b_0, b_1, b_2) = v_{\max}^{-1} \cdot \text{vol}(b_0, b_1, b_2).$$

It takes values in  $\{-1, 0, 1\}$  with  $c(b_0, b_1, b_2) = 1$  if the triple  $(b_0, b_1, b_2)$  consists of distinct points in the positive orientation/cyclic order,  $c = -1$  if the cyclic order is reversed, and  $c = 0$  if the triple is degenerate. Let  $\nu$  denote a probability measure in the Lebesgue class, and suppose that  $\phi: (S^1, \nu) \rightarrow S^1$  is a measurable map so that for  $\nu^3$ -a.e.  $(b_0, b_1, b_2)$ :

$$c(\phi(b_0), \phi(b_1), \phi(b_2)) = c(b_0, b_1, b_2).$$

It follows from [30, Proposition 5.5] that the following conditions on such measurable orientation preserving  $\phi: (S^1, \nu) \rightarrow S^1$  are equivalent:

- (1) The push-forward measure  $\phi_*\nu$  has full support;
- (2)  $\phi$  agrees a.e. with a homeomorphism  $f \in \text{Homeo}(S^1)$ .

Let  $\alpha: \Gamma \times X \rightarrow \Gamma$  be the ME-cocycle associated with an ergodic  $(\Gamma, \Gamma)$ -coupling  $(\Omega, m)$  and an identification  $i: \Gamma \times X \rightarrow \Omega$ . Let  $\phi_x: (S^1, \nu) \rightarrow S^1$ ,  $x \in X$ , be the boundary map associated to  $\alpha$  as in Subsection 3.1.5.

**Proposition 3.13.** *If the orientation cocycle is preserved by  $\phi_x$  a.e., that is,*

$$c(\phi_x(b_0), \phi_x(b_1), \phi_x(b_2)) = c(b_0, b_1, b_2) \quad \nu^3\text{-a.e.}$$

*for a.e.  $x \in X$ , then the map  $\phi_x$  agree a.e. with a homeomorphism  $f_x \in \text{Homeo}(S^1)$  for a.e.  $x \in X$ .*

*Proof.* We have to prove that the measurable family of open sets  $U_x = S^1 \setminus \text{supp}(\phi_x \nu)$  satisfies a.e.  $U_x = \emptyset$ . The fact that  $\nu$  is  $\Gamma$ -quasi-invariant and the identity

$$\phi_{\gamma.x}(\gamma b) = \alpha(\gamma, x)\phi_x(b)$$

imply the following *a priori* equivariance of  $\{U_x \mid x \in X\}$

$$(3.11) \quad U_{\gamma.x} = \alpha(\gamma, x)U_x.$$

Since  $U_x \neq S^1$  for every  $x \in X$ , the proposition is implied by the following general lemma and the fact that the action of  $G = \text{PSL}_2(\mathbb{R})$  and of its lattices on the circle  $S^1$  is minimal and strongly proximal [19, Propositions 4.2 and 4.4].  $\square$

**Lemma 3.14** (after Furstenberg [19]). *Let  $M$  be a compact metrizable space, and let  $\Gamma \curvearrowright M$  act minimally and strongly proximally. Let  $\{U_x \mid x \in X\}$  be a measurable family of open subsets of  $M$  satisfying (3.11) for a ME-cocycle  $\alpha : \Gamma \times X \rightarrow \Gamma$  over an ergodic coupling. Then either  $U_x = \emptyset$  or  $U_x = M$  for a.e.  $x \in X$ .*

*Proof of Lemma 3.14.* We first reduce the question to the trivial cocycle. To distinguish the two copies of  $\Gamma$  acting on  $\Omega$  denote them by  $\Gamma_1$  and  $\Gamma_2$ . Let  $i : \Gamma_2 \times X \cong \Omega$  be a measure space isomorphism as in (1.1); in particular

$$(g_1, g_2) : i(\gamma, x) \mapsto i(g_2 \gamma \alpha(g_1, x)^{-1}, g_1.x) \quad (g_i \in \Gamma_i).$$

Consider the measurable family  $\{O_\omega\}$  of open subsets of  $M$  indexed by  $\omega \in \Omega$ , defined by  $O_{i(g,x)} = gU_x$ . Then for  $\omega = i(\gamma, x)$  and  $g_i \in \Gamma_i$  we have

$$O_{(g_1, g_2)\omega} = g_2 \gamma \alpha(g_1, x)^{-1} U_{g_1.x} = g_2 \gamma U_x = g_2 O_\omega.$$

Note that  $\omega \rightarrow O_\omega$  is invariant under the action of  $\Gamma_1$ . Therefore it descends to a measurable family of open sets  $\{V_y\}$  indexed by  $y \in Y \cong \Omega/\Gamma_1$ , and satisfying a.e. on  $Y$

$$V_{\gamma.y} = \gamma V_y \quad (\gamma \in \Gamma).$$

The claim about  $\{U_x \mid x \in X\}$  is clearly equivalent to the similar claim about  $\{V_y \mid y \in Y\}$ . By ergodicity, it suffices to reach a contradiction from the assumption that  $V_y \neq \emptyset, M$  for a.e.  $y \in Y$ .

Denote by  $\mu$  the  $\Gamma$ -invariant and ergodic probability measure on  $Y$ . Since  $M$  has a countable base for its topology, while  $\mu(\{y \mid V_y \neq \emptyset\}) = 1$ , it follows that there exists a non-empty open set  $W \subset M$  for which the set

$$A = \{y \in Y \mid W \subset V_y\}$$

has  $\mu(A) > 0$ . Since  $M \setminus V_y \neq \emptyset$  for  $\mu$ -a.e.  $y \in Y$ , there exists a measurable map  $s : Y \rightarrow M$  with  $s(y) \notin V_y$  a.e. Let  $\sigma \in \text{Prob}(M)$  denote the distribution of  $s(y)$ , i.e.,  $\sigma(E) = \mu\{y \in Y \mid s(y) \in E\}$ . Then for any  $\gamma \in \Gamma$

$$\begin{aligned} \sigma(\gamma^{-1}W) &= \mu\{y \in Y \mid s(y) \in \gamma^{-1}W\} \\ &\leq \mu(Y \setminus \gamma^{-1}A) + \mu\{y \in \gamma^{-1}A \mid s(y) \in \gamma^{-1}V_{\gamma.y} = V_y\} \\ &= 1 - \mu(\gamma^{-1}A) = 1 - \mu(A). \end{aligned}$$

This contradicts the assumption that the action  $\Gamma \curvearrowright M$  is minimal and strongly proximal.  $\square$

**3.5. Preserving maximal simplices of the boundary.** Recall that a geodesic simplex in  $\bar{\mathbf{H}}^n = \mathbf{H}^n \cup \partial\mathbf{H}^n$  is called *regular* if any permutation of its vertices can be realized by an element in  $\text{Isom}(\mathbf{H}^n)$ . The set of ordered  $(n+1)$ -tuples on the boundary  $B$  that form the vertex set of an ideal regular simplex is denoted by  $\Sigma^{\text{reg}}$ . The set  $\Sigma^{\text{reg}}$  is a disjoint union  $\Sigma^{\text{reg}} = \Sigma_+^{\text{reg}} \cup \Sigma_-^{\text{reg}}$  of two subsets that correspond to the positively and negatively oriented ideal regular  $n$ -simplices, respectively.

**Lemma 3.15** (Key facts from Thurston's proof of Mostow rigidity).

- (i) *The diagonal  $G$ -action on  $\Sigma^{\text{reg}}$  is simply transitive. The diagonal  $G^0$ -action on  $\Sigma_+^{\text{reg}}$  and  $\Sigma_-^{\text{reg}}$  are simply transitive, respectively.*
- (ii) *An ideal simplex has non-oriented volume  $v_{\max}$  if and only if it is regular.*
- (iii) *Let  $n \geq 3$ . Let  $\sigma, \sigma'$  be two regular ideal simplices having a common face of codimension one. Let  $\rho$  be the reflection along the hyperspace spanned by this face. Then  $\sigma = \rho(\sigma')$ .*

*Proof.* (i) See the proof of [52, Theorem 11.6.4 on p. 568].

(ii) The statement is trivial for  $n = 2$ , as all non-degenerate ideal triangles in  $\bar{\mathbf{H}}^2$  are regular, and  $G$  acts simply transitively on them. The case  $n = 3$  is due to Milnor, and Haagerup and Munkholm [27] proved the general case  $n \geq 3$ .

(iii) This is a key feature distinguishing the  $n \geq 3$  case from the  $n = 2$  case where Mostow rigidity fails. See [52, Lemma 13 on p. 567].  $\square$

We shall need the following lemma, which in dimension  $n = 3$  is due to Thurston [58, p. 133/134]. Recall that  $B = \partial\mathbf{H}^n$  is considered equipped with the Lebesgue measure class. We consider the natural measure  $m_{\Sigma_+^{\text{reg}}}$  on  $\Sigma_+^{\text{reg}}$  corresponding to the Haar measure on  $G^0$  under the simply transitive action of  $G^0$  on  $\Sigma_+^{\text{reg}}$ .

**Lemma 3.16.** *Let  $n \geq 3$  and  $\phi : B \rightarrow B$  be a Borel map such that  $\phi^{n+1} = \phi \times \dots \times \phi$  maps a.e. point in  $\Sigma_+^{\text{reg}}$  into  $\Sigma_+^{\text{reg}}$ . Then there exists a unique  $g_0 \in G^0 = \text{Isom}_+(\mathbf{H}^n)$  with  $\phi(b) = g_0 b$  for a.e.  $b \in B$ .*

*Proof.* Fix a regular ideal simplex  $\sigma = (b_0, \dots, b_n) \in \Sigma_+^{\text{reg}}$ , and identify  $G^0$  with  $\Sigma_+^{\text{reg}}$  via  $g \mapsto g\sigma$ . Then there is a Borel map  $f : G^0 \rightarrow G^0$  such that for a.e.  $g \in G^0$

$$(3.12) \quad (\phi(gb_0), \dots, \phi(gb_n)) = (f(g)b_0, \dots, f(g)b_n).$$

Interchanging  $b_0, b_1$  identifies  $\Sigma_+^{\text{reg}}$  with  $\Sigma_-^{\text{reg}}$ , and allows to extend  $f$  to a measurable map  $G \rightarrow G$  satisfying (3.12) for a.e.  $g \in G$ . Let  $\rho_0, \dots, \rho_n \in G$  denote the reflections in the codimension one faces of  $\sigma$ . Then Lemma 3.15 (iii) implies that

$$f(g\rho) = f(g)\rho \quad \text{for a.e. } g \in G$$

for  $\rho$  in  $\{\rho_0, \dots, \rho_n\}$ . It follows that the same applies to each  $\rho$  in the countable group  $R < G$  generated by  $\rho_0, \dots, \rho_n$ . We claim that there exists  $g_0 \in G$  so that  $f(g) = g_0 g$  for a.e.  $g \in G$ , which implies that  $\phi(b) = g_0 b$  also holds a.e. on  $B$ .

The case  $n = 3$  is due to Thurston ([58, p. 133/134]). So hereafter we focus on  $n > 3$ , and will show that in this case the group  $R$  is dense in  $G$  (for  $n = 2, 3$  it forms a lattice in  $G$ ). Consequently the  $R$ -action on  $G$  is ergodic with respect to the Haar measure. Since  $g \mapsto f(g)g^{-1}$  is a measurable  $R$ -invariant map on  $G$ , it follows that it is a.e. a constant  $g_0 \in G^0$ , i.e.,  $f(g) = g_0 g$  a.e. proving the lemma.

It remains to show that for  $n > 3$ ,  $R$  is dense in  $G$ . Not being able to find a convenient reference for this fact, we include the proof here.

For  $i \in \{0, \dots, n\}$  denote by  $P_i < G$  the stabilizer of  $b_i \in \partial \mathbf{H}^n$ , and let  $U_i < P_i$  denote its unipotent radical. We shall show that  $U_i$  is contained in the closure  $\overline{R \cap P_i} < P_i$  (in fact,  $\overline{R \cap P_i} = P_i$  but we shall not need this). Since unipotent radicals of any two opposite parabolics, say  $U_0$  and  $U_1$ , generate the whole connected simple Lie group  $G^0$ , this would show  $G^0 < \overline{R} < G$ . Since  $R$  is not contained in  $G^0$ , it follows that  $\overline{R} = G$  as claimed.

Let  $f_i : \partial \mathbf{H}^n \rightarrow \mathbf{E}^{n-1} \cup \{\infty\}$  denote the stereographic projection taking  $b_i$  to the point at infinity. Then  $f_i P_i f_i^{-1}$  is the group of similarities  $\text{Isom}(\mathbf{E}^{n-1}) \rtimes \mathbb{R}_+^\times$  of the Euclidean space  $\mathbf{E}^{n-1}$ . We claim that the subgroup of translations  $\mathbb{R}^{n-1} \cong U_i < P_i$  is contained in the closure of  $R_i = R \cap P_i$ . To simplify notations we assume  $i = 0$ . The set of all  $n$ -tuples  $(z_1, \dots, z_n)$  in  $\mathbf{E}^{n-1}$  for which  $(b_0, f_0^{-1}(z_1), \dots, f_0^{-1}(z_n))$  is a regular ideal simplex in  $\overline{\mathbf{H}}^n$  is precisely the set of all regular Euclidean simplices in  $\mathbf{E}^{n-1}$  [52, Lemma 3 on p. 519]. So conjugation by  $f_0$  maps the group  $R_0 = R \cap P_0$  to the subgroup of  $\text{Isom}(\mathbf{E}^{n-1})$  generated by the reflections in the faces of the Euclidean simplex  $\Delta = (z_1, \dots, z_n)$ , where  $z_i = f_0(b_i)$ . For  $1 \leq j < k \leq n$  denote by  $r_{jk}$  the composition of the reflections in the  $j$ th and  $k$ th faces of  $\Delta$ ; it is a rotation leaving fixed the co-dimension two affine hyperplane  $L_{jk}$  containing  $\{z_i \mid i \neq j, k\}$ . The angle of this rotation is  $2\theta_n$ , where  $\theta_n$  is the dihedral angle of the simplex  $\Delta$ . One can easily check that  $\cos(\theta_n) = -1/(n-1)$ , using the fact that the unit normals  $v_i$  to the faces of  $\Delta$  satisfy  $v_1 + \dots + v_n = 0$  and  $\langle v_i, v_j \rangle = \cos(\theta_n)$  for all  $1 \leq i < j \leq n$ . Thus  $w = \exp(\theta_n \sqrt{-1})$  satisfies  $w + 1/w = -2/(n-1)$ . Equivalently,  $w$  is a root of

$$p_n(z) = (n-1)z^2 + 2z + (n-1).$$

This condition on  $w$  implies that  $\theta_n$  is not a rational multiple of  $\pi$ . Indeed, otherwise,  $w$  is a root of unit, and therefore is a root of some cyclotomic polynomial

$$c_m(z) = \prod_{k \in \{1..m-1 \mid \gcd(k,m)=1\}} (z - e^{\frac{2\pi ki}{m}})$$

whose degree is Euler's totient function  $\deg(c_m) = \phi(m)$ . The cyclotomic polynomials are irreducible over  $\mathbb{Q}$ . So  $p_n(z)$  and  $c_m(z)$  share a root only if they are proportional, which in particular implies  $\phi(m) = 2$ . The latter happens only for  $m = 3$ ,  $m = 4$  and  $m = 6$ ; corresponding to  $c_3(z) = z^2 + z + 1$ ,  $c_4(z) = z^2 + 1$ , and  $c_6(z) = z^2 - z + 1$ . The only proportionality between these polynomials is  $p_3(z) = 2c_2(z)$ ; and it is ruled out by the assumption  $n > 3$ .

Thus the image of  $R_0$  in  $\text{Isom}(\mathbf{E}^{n-1})$  is not discrete. Let

$$\pi : \overline{R \cap P_0} \rightarrow \text{Isom}(\mathbf{E}^{n-1}) \rightarrow \text{O}(\mathbb{R}^{n-1})$$

denote the homomorphism defined by taking the linear part. Then  $\pi(r_{jk})$  is an irrational rotation in  $\text{O}(\mathbb{R}^{n-1})$  leaving invariant the linear subspace parallel to  $L_{jk}$ . The closure of the subgroup generated by this rotation is a subgroup  $C_{jk} < \text{O}(\mathbb{R}^{n-1})$ , isomorphic to  $\text{SO}(2)$ . The group  $K < \text{O}(\mathbb{R}^{n-1})$  generated by all such  $C_{jk}$  acts irreducibly on  $\mathbb{R}^{n-1}$ , because there is no subspace orthogonal to all  $L_{jk}$ . Since  $\overline{R \cap P_0}$  is not compact (otherwise there would be a point in  $\mathbf{E}^{n-1}$  fixed by all reflections in faces of  $\Delta$ ), the epimorphism  $\pi : \overline{R \cap P_0} \rightarrow K$  has a non-trivial kernel  $V < \mathbb{R}^{n-1}$ , which is invariant under  $K$ . As the latter group acts irreducibly,  $V = \mathbb{R}^{n-1}$  or, equivalently,  $U_0 < \overline{R \cap P_0}$ . This completes the proof of the lemma.  $\square$

## 4. PROOFS OF THE MAIN RESULTS

In this section we use the results of Section 2 and Theorems B and C to prove the remaining results stated in the introduction.

**4.1. Measure equivalence rigidity: Theorem D.** Let  $G = \text{Isom}(\mathbf{H}^n)$ ,  $n \geq 3$ . Let  $\Gamma < G$  be a lattice, and  $\Lambda$  a finitely generated group which admits an integrable  $(\Gamma, \Lambda)$ -coupling  $(\Omega, m)$ . By Lemma A.2 the  $(\Gamma, \Gamma)$ -coupling  $\Omega \times_{\Lambda} \Omega^*$  is integrable.

By Theorem B and Proposition 2.8 the lattice  $\Gamma$  is 1-taut relative to the inclusion  $\Gamma < G$ . Hence the coupling  $\Omega \times_{\Lambda} \Omega^*$  is taut. By Example 2.3 the group  $G$  is strongly ICC relative to  $\Gamma < G$ . Applying Theorem 2.5 we obtain a continuous homomorphism  $\rho : \Lambda \rightarrow G$  with finite kernel  $F$ , image  $\bar{\Lambda} = \rho(\Lambda)$  being discrete in  $G$ , and a measurable  $\text{id}_{\Gamma} \times \rho$ -equivariant map  $\Psi : \Omega \rightarrow G$ .

To complete the proof of statement (1) of Theorem D and case  $n \geq 3$  of Theorem A it remains to show that  $\bar{\Lambda}$  is not merely discrete, but is actually a lattice in  $G$ . This can be deduced from the application of Ratner's theorem below which is needed for the precise description of the push-forward measure  $\Psi_* m$  on  $G$  as stated in part (2) of Theorem D. Let us also give the following direct argument which relies only on the strong ICC property of  $G$ .

Consider the composition  $(G, \Lambda)$ -coupling  $\tilde{\Omega} = G \times_{\Gamma} \Omega$ , and the  $(G, G)$ -coupling  $\tilde{\Omega} \times_{\Lambda} \tilde{\Omega}^*$ . Since  $\Gamma$  is an integrable lattice in  $G$  (Theorem 1.9) by Lemma A.2 both  $\tilde{\Omega}$  and  $\tilde{\Omega} \times_{\Lambda} \tilde{\Omega}^*$  are integrable couplings. Theorem B provides a unique tautening map

$$\tilde{\Phi} : \tilde{\Omega} \times_{\Lambda} \tilde{\Omega}^* \rightarrow G.$$

Applying Theorem 2.1 (a special case of Theorem 2.5 with  $\mathcal{G} = G$ ), we obtain a homomorphism  $\tilde{\rho} : \Lambda \rightarrow G$  with finite kernel and image being a lattice in  $G$ . There is also a  $\text{Id}_G \times \tilde{\rho}$ -equivariant measurable map

$$\tilde{\Psi} : \tilde{\Omega} = G \times_{\Gamma} \Omega \rightarrow G.$$

We claim that  $\rho, \tilde{\rho} : \Lambda \rightarrow G$  are conjugate representations. To see this observe that since  $G$  is strongly ICC, there is only one tautening map  $\tilde{\Omega} \times_{\Lambda} \tilde{\Omega}^* \rightarrow G$ . This implies the a.e. identity

$$\tilde{\Psi}([g_1, \omega_1]) \tilde{\Psi}([g_2, \omega_2])^{-1} = g_1 \Psi(\omega_1) \Psi(\omega_2)^{-1} g_2^{-1}.$$

Equivalently, we have a.e. identity

$$\Psi(\omega_1)^{-1} g_1^{-1} \tilde{\Psi}([g_1, \omega_1]) = \Psi(\omega_2)^{-1} g_2^{-1} \tilde{\Psi}([g_2, \omega_2]).$$

Hence the value of both sides are a.e. equal to a constant  $g_0 \in G$ . It follows that for a.e.  $g \in G$  and  $\omega \in \Omega$

$$g^{-1} \tilde{\Psi}([g, \omega]) = \Psi(\omega) g_0.$$

Finally, the fact that  $\Psi, \tilde{\Psi}$  are  $\rho$ -,  $\tilde{\rho}$ -equivariant respectively, implies:

$$\tilde{\rho}(\lambda) = g_0 \rho(\lambda) g_0^{-1} \quad (\lambda \in \Lambda).$$

In particular,  $\bar{\Lambda} = g_0^{-1} \tilde{\rho}(\Lambda) g_0$  is a lattice in  $G$ .

We proceed with the proof of statement (2): given the  $\text{Id}_{\Gamma} \times \rho$ -equivariant measurable map  $\Psi : \Omega \rightarrow G$  we shall describe the pushforward  $\Psi_* m$  on  $G$ . (We shall use the discreteness of  $\bar{\Lambda} = \rho(\Lambda)$ , but the fact that it is a lattice will not be needed; in fact, it will follow from the application of Ratner's theorem.) Recall that the

measure  $\Psi_*m$  is invariant under the action  $x \mapsto \gamma x \rho(\lambda)^{-1}$ , and descends to a finite  $\Gamma$ -invariant measure  $\mu$  on  $G/\bar{\Lambda}$  and to a finite  $\bar{\Lambda}$ -invariant measure  $\nu$  on  $\Gamma \backslash G$ . Assuming  $m$  was  $\Gamma \times \Lambda$ -ergodic,  $\mu$  and  $\nu$  are ergodic under the  $\Gamma$ - and  $\bar{\Lambda}$ -action, respectively. One can now apply Ratner's theorem [53] to describe  $\mu$ , and thereby  $\Psi_*m$ , as in [15, Lemma 4.6]. For the reader's convenience we sketch the arguments.

Let  $\Lambda^0 = \bar{\Lambda} \cap G^0$ ; so either  $\Lambda^0 = \bar{\Lambda}$  or  $[\bar{\Lambda} : \Lambda^0] = 2$ . In the first case we set  $\mu' = \mu$ , in the latter case let  $\mu'$  denote the 2-to-1 lift of  $\mu$  to  $G/\Lambda^0$ . Let  $\Gamma^0 = \Gamma \cap G^0$ , and let  $\mu^0$  be an ergodic component of  $\mu'$  supported on  $G^0/\Lambda^0$ . We consider the homogeneous space  $Z = G^0/\Gamma^0 \times G^0/\Lambda^0$  which is endowed with the following probability measure

$$\tilde{\mu}^0 = \int_{G^0/\Gamma^0} \delta_{g\Gamma^0} \times g_*\mu^0 dm_{G^0/\Gamma^0}.$$

Observe that  $\tilde{\mu}^0$  well defined because  $\mu^0$  is  $\Gamma^0$ -invariant. Moreover,  $\tilde{\mu}^0$  is invariant and ergodic for the action of the diagonal  $\Delta(G^0) \subset G^0 \times G^0$  on  $Z$ . Since  $G^0$  is a connected group generated by unipotent elements, Ratner's theorem shows that  $\tilde{\mu}^0$  is *homogeneous*. This means that there is a connected Lie subgroup  $L < G^0 \times G^0$  containing  $\Delta(G^0)$  and a point  $z \in Z$  such that the stabilizer  $L_z$  of  $z$  is a lattice in  $L$  and  $\tilde{\mu}^0$  is the push-forward of the normalized Haar measure  $m_{L/L_z}$  to the  $L$ -orbit  $Lz \subset Z$ . Since  $G^0$  is a simple group, there are only two possibilities for  $L$ : either (i)  $L = G^0 \times G^0$  or (ii)  $L = \Delta(G^0)$ .

In case (i),  $\tilde{\mu}^0$  is the Haar measure on  $G^0/\Gamma^0 \times G^0/\Lambda^0$ , and  $\mu^0$  is the Haar measure on  $G^0/\Lambda^0$ . (In particular,  $\Lambda^0$  is a lattice in  $G^0$ , and  $\Lambda$  is a lattice in  $G$ ). The original measure  $\mu$  may be either the  $G$ -invariant measure  $m_{G/\bar{\Lambda}}$ , or a  $G^0$ -invariant measure on  $G/\bar{\Lambda}$ . In the latter case, by possibly multiplying  $\Phi$  and conjugating  $\rho$  with some  $x \in G \backslash G^0$ , we may assume that  $\mu$  is the  $G^0$ -invariant probability measure on  $G^0/\bar{\Lambda}$ .

In case (ii), the fact that  $L_z$  is lattice in  $L = \Delta(G^0)$ , implies that  $\mu^0$  and the original measure  $\mu$  are atomic. Since  $\Gamma$  acts ergodically on  $(G/\bar{\Lambda}, \mu)$ , this atomic measure is necessarily supported and equidistributed on a finite  $\Gamma$ -orbit of some  $g_0\bar{\Lambda} \in G/\bar{\Lambda}$ . It follows that  $\Gamma \cap g_0^{-1}\bar{\Lambda}g_0$  has finite index in  $\Gamma$ . (This also implies that  $\bar{\Lambda}$  is a lattice in  $G$ ). Upon multiplying  $\Psi$  and conjugating  $\rho$  by  $g_0 \in G$ , we may assume that  $\Phi_*m$  is equidistributed on the double coset  $\Gamma e\bar{\Lambda}$  and that  $\Gamma, \bar{\Lambda}$  are commensurable lattices. This completes the proof of Theorem D.

**4.2. Convergence actions on the circle: case  $n = 2$  of Theorem A.** Let  $\Gamma$  be a uniform lattice in  $G = \text{Isom}(\mathbf{H}^2) \cong \text{PGL}_2(\mathbb{R})$ . The group  $G$  is a subgroup of  $\text{Homeo}(S^1)$  by the natural action of  $\text{PGL}_2(\mathbb{R})$  on  $S^1 \cong \mathbb{R}P^1$ . Consider a compactly generated unimodular group  $H$  that is  $L^1$ -measure equivalent to  $\Gamma$ . We will prove a more general statement than in Theorem A, which is formulated for discrete  $H = \Lambda$ . Since  $\Gamma$  is uniform, hence integrable in  $G$ , we can induce any integrable  $(\Gamma, H)$ -coupling to an integrable  $(G, H)$ -coupling (Lemma A.2). Let  $(\Omega, m)$  be an integrable  $(G, H)$ -coupling  $(\Omega, m)$ .

From Theorem 2.5 we obtain a continuous homomorphism  $\rho : H \rightarrow \text{Homeo}(S^1)$  with compact kernel and closed image  $\bar{H} < \text{Homeo}(S^1)$  and, by pushing forward  $m$ , a measure  $\bar{m}$  on  $\text{Homeo}(S^1)$  that is invariant under all bilateral translations on  $f \mapsto gf\rho(h)^{-1}$  with  $g \in G$  and  $h \in \text{Homeo}(S^1)$  and descends to a finite  $\bar{H}$ -invariant measure  $\mu$  on  $G \backslash \text{Homeo}(S^1)$  and a finite  $G$ -invariant measure  $\nu$  on  $\text{Homeo}(S^1)/\bar{H}$ .

The next step is to show that  $\bar{H}$  can be conjugated into  $G$ . To this end, we shall use the existence of the finite  $\bar{H}$ -invariant measure  $\mu$  on  $G \backslash \text{Homeo}(S^1)$ , which

may be normalized to a probability measure. We need the following theorem which we prove relying on the deep work by Gabai [20] and Casson-Jungreis [8] on the determination of convergence groups as Fuchsian groups.

**Theorem 4.1.** *Let  $\mu$  be a Borel probability measure on  $G \backslash \text{Homeo}(S^1)$ . Then the stabilizer  $H_\mu = \{f \in \text{Homeo}(S^1) \mid f_*\mu = \mu\}$  for the action by the right translations is conjugate to a closed subgroup of  $G$ .*

*Proof.* We fix a metric  $d$  on the circle, say  $d(x, y) = \angle(x, y)$ . Let  $\text{Trp} \subset S^1 \times S^1 \times S^1$  be the space of distinct triples on the circle. The group  $\text{Homeo}(S^1)$  acts diagonally on  $\text{Trp}$ . We denote elements in  $\text{Trp}$  by bold letters  $\mathbf{x} \in \text{Trp}$ ; the coordinates of  $\mathbf{x} \in \text{Trp}$  or  $\mathbf{y} \in \text{Trp}$  will be denoted by  $x_i$  or  $y_i$  where  $i \in \{1, 2, 3\}$ , respectively. For  $f \in \text{Homeo}(S^1)$  we write  $f(\mathbf{x})$  for  $(f(x_1), f(x_2), f(x_3))$ . We equip  $\text{Trp}$  with the metric, also denoted by  $d$ , given by

$$d(\mathbf{x}, \mathbf{y}) = \max_{i \in \{1, 2, 3\}} d(x_i, y_i).$$

The following lemma will eventually allow us to apply the work of Gabai-Casson-Jungreis.

**Lemma 4.2.** *For every compact subset  $K \subset \text{Trp}$  and every  $\epsilon > 0$  there is  $\delta > 0$  so that for all  $h, h' \in H_\mu$  and  $\mathbf{y} \in K \cap h^{-1}K$  and  $\mathbf{y}' \in K \cap h'^{-1}K$  one has the implication:*

$$d(\mathbf{y}, \mathbf{y}') < \delta \text{ and } d(h(\mathbf{y}), h'(\mathbf{y}')) < \delta \implies \sup_{x \in S^1} d(h(x), h'(x)) < \epsilon$$

*Proof.* For an arbitrary triple  $\mathbf{z} \in \text{Trp}$  and  $x \in S^1 \setminus \{z_3\}$  consider the real valued cross-ratio

$$[x, z_1; z_2, z_3] = \frac{(x - z_1)(z_2 - z_3)}{(x - z_3)(z_2 - z_1)}.$$

In this formula we view the circle as the one-point compactification of the real line. Denote by  $[z_1, z_2]_{z_3}$  the circle arc from  $z_1$  to  $z_2$  not including  $z_3$ . As a function in the first variable,  $[-, z_1; z_2, z_3]$  is a monotone homeomorphism between the closed arc  $[z_1, z_2]_{z_3}$  and the interval  $[0, 1]$ . For  $f \in \text{Homeo}(S^1)$  and  $\mathbf{z} \in \text{Trp}$  we define the function

$$F_{\mathbf{z}, f} : [z_1, z_2]_{z_3} \rightarrow [0, 1], \quad F_{\mathbf{z}, f}(x) = [f(x), f(z_1); f(z_2), f(z_3)].$$

Since the cross-ratio is invariant under  $G$  [52, Theorem 4.3.1 on p. 116], we have  $F_{\mathbf{z}, gf}(x) = F_{\mathbf{z}, f}(x)$  for any  $g \in G$ . Hence we may and will use the notation  $F_{\mathbf{z}, Gf}(x)$ . We now average  $F_{\mathbf{z}, Gf}(x)$  with regard to the measure  $\mu$  and obtain the function  $\bar{F}_{\mathbf{z}} : [z_1, z_2]_{z_3} \rightarrow [0, 1]$  with

$$\bar{F}_{\mathbf{z}}(x) = \int_{G \backslash \text{Homeo}(S^1)} F_{\mathbf{z}, Gf}(x) d\mu(Gf).$$

The  $H_\mu$ -invariance of  $\mu$  implies that

$$(4.1) \quad \bar{F}_{h(\mathbf{z})}(h(x)) = \bar{F}_{\mathbf{z}}(x)$$

for every  $h \in H_\mu$  and every  $x \in [z_1, z_2]_{z_3}$ . Let us introduce the following notation: Whenever  $K \subset \text{Trp}$  is a subset, we denote by  $\tilde{K}$  the subset

$$\tilde{K} = \{(x, \mathbf{z}) \mid \mathbf{z} \in K, x \in [z_1, z_2]_{z_3}\} \subset S^1 \times S^1 \times S^1 \times S^1.$$

Next let us establish the following continuity properties:

(1) For every compact  $K \subset \text{Trp}$  and every  $\epsilon > 0$  there is  $\eta > 0$  such that:

$$\forall (s, \mathbf{z}), (t, \mathbf{z}) \in \tilde{K}: |\bar{F}_{\mathbf{z}}(t) - \bar{F}_{\mathbf{z}}(s)| < \eta \Rightarrow d(t, s) < \frac{\epsilon}{5}$$

(2) For every compact  $K \subset \text{Trp}$  and every  $\eta > 0$  there is  $\delta > 0$  such that:

$$\forall (t, \mathbf{y}), (t, \mathbf{z}) \in \tilde{K}: d(\mathbf{y}, \mathbf{z}) < \delta \Rightarrow |\bar{F}_{\mathbf{y}}(t) - \bar{F}_{\mathbf{z}}(t)| < \frac{\eta}{2}$$

*Proof of (1):* Let  $K \subset \text{Trp}$  be compact and  $\epsilon > 0$ . Let  $f \in \text{Homeo}(S^1)$ . The family of homeomorphisms  $\bar{F}_{\mathbf{z}, Gf}: [z_1, z_2]_{z_3} \rightarrow [0, 1]$  depends continuously on  $\mathbf{z} \in \text{Trp}$ . The inverses of these functions are equicontinuous when  $\mathbf{z}$  ranges in a compact subset. Hence there exists  $\theta(Gf) > 0$  such that for every  $\mathbf{z} \in K$  and all  $t, s \in [z_1, z_2]_{z_3}$  we have the implication

$$|F_{\mathbf{z}, Gf}(t) - F_{\mathbf{z}, Gf}(s)| < \theta(Gf) \Rightarrow d(t, s) < \frac{\epsilon}{5}.$$

The set  $G \setminus \text{Homeo}(S^1)$  is the union of an increasing sequence of measurable sets

$$A_n = \left\{ Gf \in G \setminus H \mid \theta(Gf) > \frac{1}{n} \right\}.$$

Fix  $n$  large enough so that  $\mu(A_n) > 1/2$ . We claim that  $\eta = (2n)^{-1}$  satisfies (1). Suppose that  $\mathbf{z} \in K$  and  $t, s \in [z_1, z_2]_{z_3}$  satisfy  $d(t, s) > \epsilon/5$ . Up to exchanging  $t$  and  $s$ , we may assume that  $[s, z_1; z_2, z_3] \geq [t, z_1; z_2, z_3]$ . Then  $F_{\mathbf{z}, Gf}(s) \geq F_{\mathbf{z}, Gf}(t)$  for all  $f \in \text{Homeo}(S^1)$ , and

$$\bar{F}_{\mathbf{z}}(s) - \bar{F}_{\mathbf{z}}(t) \geq \int_{A_n} (F_{\mathbf{z}, Gf}(s) - F_{\mathbf{z}, Gf}(t)) d\mu > \mu(A_n) \cdot \frac{1}{n} > \eta.$$

*Proof of (2):* Let  $K \subset \text{Trp}$  be compact, and let  $\eta > 0$ . Let  $f \in \text{Homeo}(S^1)$ . Since  $\tilde{K}$  is compact,  $\bar{F}_{\mathbf{z}}(x)$  as a function on  $\tilde{K}$  is equicontinuous. Hence there is  $\delta(Gf) > 0$  such that for all  $(x, \mathbf{y}) \in \tilde{K}$  and  $(x, \mathbf{z}) \in \tilde{K}$  with  $d(\mathbf{y}, \mathbf{z}) < \delta(Gf)$  we have

$$|F_{\mathbf{y}, Gf}(x) - F_{\mathbf{z}, Gf}(x)| < \frac{\eta}{2}.$$

The set  $G \setminus \text{Homeo}(S^1)$  is the union of an increasing sequence of measurable sets

$$B_n = \left\{ Gf \in G \setminus H \mid \delta(Gf) > \frac{1}{n} \right\}.$$

We choose  $n \in \mathbb{N}$  with  $\mu(B_n) > 1 - \eta/2$  and set  $\delta = n^{-1}$ . Then for  $(x, \mathbf{y}) \in \tilde{K}$  and  $(x, \mathbf{z}) \in \tilde{K}$  with  $d(\mathbf{y}, \mathbf{z}) < \delta$  we have

$$|\bar{F}_{\mathbf{y}}(x) - \bar{F}_{\mathbf{z}}(x)| \leq \int_{B_n} |F_{\mathbf{y}, Gf}(x) - F_{\mathbf{z}, Gf}(x)| d\mu(Gf) + \frac{\eta}{2} < \eta,$$

proving (2).

We can now complete the proof of the lemma. Let  $K \subset \text{Trp}$  be a compact subset. Let  $\epsilon > 0$ . We can choose  $r > 0$  such that

$$K \subset \left\{ \mathbf{x} \in \text{Trp} \mid d(x_1, x_2), d(x_2, x_3), d(x_3, x_1) \geq r \right\}.$$

For the given  $\epsilon$  and  $K$  let  $\eta > 0$  be as in (1). For the given  $\epsilon$  and  $K$  and this  $\eta$  let  $\delta > 0$  be as in (2). We may also assume that

$$\delta < \frac{\epsilon}{5} < \frac{r}{3}.$$

Consider  $h, h' \in H_\mu$  and  $\mathbf{y}, \mathbf{y}' \in K$  where  $\mathbf{z} = h(\mathbf{y})$ ,  $\mathbf{z}' = h'(\mathbf{y}')$  are also in  $K$ , and assume that  $d(\mathbf{y}, \mathbf{y}') < \delta$  and  $d(\mathbf{z}, \mathbf{z}') < \delta$ . There are several possibilities for

the cyclic order of the points  $\{y_1, y'_1, y_2, y'_2, y_3, y'_3\}$ , but since the pairs  $\{y_i, y'_i\}$  of corresponding points in the triples  $\mathbf{y}, \mathbf{y}'$  are closer ( $d(y_i, y'_i) < \delta < r/3$ ) than the separation between the points in the triples ( $d(y_i, y_j), d(y'_i, y'_j) \geq r$ ), these points define a partition of the circle into three long arcs  $L_{ij}$  separated by three short arcs  $S_k$  (possibly degenerating into points) in the following cyclic order

$$S^1 = L_{12} \cup S_2 \cup L_{23} \cup S_3 \cup L_{31} \cup S_1.$$

The end points of the arc  $S_i$  are  $\{y_i, y'_i\}$ ; and if  $(i, j, k) = (1, 2, 3)$  up to a cyclic permutation, then

$$L_{ij} = [y_i, y_j]_{y_k} \cap [y'_i, y'_j]_{y'_k}.$$

Note that for any  $x \in L_{ij}$  we have

$$h(x), h'(x) \in [z_i, z_j]_{z_k} \cap [z'_i, z'_j]_{z'_k}.$$

Using (2) and (4.1) we obtain

$$\begin{aligned} |\bar{F}_{\mathbf{z}}(h(x)) - \bar{F}_{\mathbf{z}}(h'(x))| &\leq |\bar{F}_{\mathbf{z}}(h(x)) - \bar{F}_{\mathbf{z}'}(h'(x))| + |\bar{F}_{\mathbf{z}'}(h'(x)) - \bar{F}_{\mathbf{z}}(h'(x))| \\ &\leq |\bar{F}_{\mathbf{z}}(h(x)) - \bar{F}_{\mathbf{z}'}(h'(x))| + \frac{\eta}{2} \\ &= |\bar{F}_{\mathbf{y}}(x) - \bar{F}_{\mathbf{y}'}(x)| + \frac{\eta}{2} < \eta. \end{aligned}$$

By (1) it follows that  $d(h(x), h'(x)) < \epsilon/5$  for every  $x \in L_{12} \cup L_{23} \cup L_{31}$ . It remains to consider points  $x \in S_i$ ,  $i = 1, 2, 3$ , which can be controlled via the behavior of the endpoints  $y_i, y'_i$  of the short arc  $S_i$ .

First observe that the image  $h(S_i)$  of  $S_i$  is the short arc defined by  $h(y_i), h(y'_i)$ . Indeed, on one hand the two points are close:

$$d(h(y_i), h(y'_i)) \leq d(h(y_i), h'(y'_i)) + d(h'(y'_i), h(y'_i)) < \delta + \frac{\epsilon}{5} < \frac{2}{5}\epsilon.$$

On the other hand, the complement  $S^1 \setminus S_i$  of  $S_i$  contains a point  $y_j$  with  $j \in \{1, 2, 3\} \setminus \{i\}$ ; therefore  $h(y_j) \notin h(S_i)$ . Since  $h(\mathbf{y}) \in K$  we have

$$d(h(y_i), h(y_j)) \geq r > 2\epsilon/5.$$

Hence  $h(S_i)$  is the short arc defined by  $2\epsilon/3$ -close points  $h(y_i), h(y'_i)$ , implying

$$d(h(x), h(y_i)) < \frac{2}{5}\epsilon \quad (x \in S_i).$$

Similarly,  $h'(S_i)$  is the short arc defined by  $2\epsilon/5$ -close points  $h'(y_i), h'(y'_i)$ , and

$$d(h'(x), h'(y_i)) < \frac{2}{5}\epsilon \quad (x \in S_i).$$

Since  $y_i \in L_{ij}$ ,  $d(h(y_i), h'(y_i)) < \epsilon/5$ . Therefore for any  $x \in S_i$

$$d(h(x), h'(x)) \leq d(h(x), h(y_i)) + d(h(y_i), h'(y_i)) + d(h'(x), h'(y_i)) < \epsilon.$$

This completes the proof of the lemma.  $\square$

*Continuation of the proof of Theorem 4.1.* We claim that  $H_\mu < \text{Homeo}(S^1)$  is a convergence group, i.e., for any compact subset  $K \subset \text{Trp}$  the set

$$H(\mu, K) = \{h \in H_\mu \mid h^{-1}K \cap K \neq \emptyset\}$$

is compact. In particular, the Polish group  $H_\mu$  is locally compact. Let us fix a compact subset  $K \subset \text{Trp}$ . Since  $H(\mu, K)$  is a closed subset in the Polish group  $\text{Homeo}(S^1)$ , it suffices to show that any sequence  $\{h_n\}_{n=1}^\infty$  in  $H(\mu, K)$  contains a

Cauchy subsequence. Choose triples  $\mathbf{y}_n \in h_n^{-1}K \cap K$ . Upon passing to a subsequence, we may assume that the points  $\mathbf{y}_n$  converge to some  $\mathbf{y} \in K$  and the points  $\mathbf{z}_n = h_n(\mathbf{y}_n)$  converge to some  $\mathbf{z} \in K$ . Let  $\epsilon > 0$ . For the given  $\epsilon$  and  $K$  let  $\delta > 0$  be as in Lemma 4.2. Choose  $N \in \mathbf{N}$  be large enough to ensure that  $d(\mathbf{y}_n, \mathbf{y}_m) < \delta$  and  $d(\mathbf{z}_n, \mathbf{z}_m) < \delta$  for all  $n, m > N$ . It follows from Lemma 4.2 that  $h_n$  and  $h_m$  are  $\epsilon$ -close whenever  $n, m > N$ . This proves that  $H_\mu$  is a convergence group on the circle.

Finally, it follows that  $H_\mu$  is conjugate to a closed subgroup of  $G$ . For discrete groups this is a well known results of Gabai [20] and Casson – Jungreis [8]. The case of non-discrete convergence group  $H_\mu < \text{Homeo}(S^1)$  can be argued more directly. The closed convergence group  $H_\mu$  is a locally compact subgroup of  $\text{Homeo}(S^1)$ ; the classification of all such groups is well known, and the only ones with convergence property are conjugate to  $\text{PGL}_2(\mathbb{R})$  [16, pp. 51–54; 24, pp. 345–348].  $\square$

We return to the proof of Theorem A in case of  $n = 2$ . Starting from an integrable  $(G, H)$ -coupling  $(\Omega, m)$  between  $G = \text{PGL}_2(\mathbb{R})$  and an unknown compactly generated unimodular group  $H$  a continuous representation  $\rho : H \rightarrow \text{Homeo}(S^1)$  with compact kernel and closed image was constructed. Theorem 4.1 implies that, up to conjugation, we may assume that

$$\bar{H} = \rho(H) < G = \text{PGL}_2(\mathbb{R}).$$

Since  $\bar{H}$  is measure equivalent to  $G = \text{PGL}_2(\mathbb{R})$ , it is non-amenable.

Case (1):  $\bar{H} < G = \text{PGL}_2(\mathbb{R})$  is non-discrete. (This does not occur in the original formulation of Theorem A, but is included in the broader context of lsc  $H$  adapted in this proof). There are only two non-discrete non-amenable closed subgroups of  $G$ : the whole group  $G$  and its index two subgroup  $G^0 = \text{PSL}_2(\mathbb{R})$ . Both of these groups may appear as  $\bar{H}$ ; in fact, direct products of the form  $H \cong G \times K$  or  $H \cong G^0 \times K$  with compact  $K$  and certain almost direct products  $G' \times K'/C$  as in [16, Theorem A] give rise to an integrable measure equivalence between  $H$  and  $G$  (cf. [16, Theorem C]).

Case (2):  $\bar{H}$  is discrete. We claim that such  $\bar{H}$  is a cocompact lattice in  $G$ . Indeed, every finitely generated discrete non-amenable subgroup of  $G$  is either cocompact or is virtually a free group  $\mathbb{F}_2$ . The latter possibility is ruled out by the following.

**Lemma 4.3.** *The free group  $\mathbb{F}_2$  is not  $L^1$ -measure equivalent to  $G$ .*

Note that these groups are measure equivalent since  $\mathbb{F}_2$  forms a lattice in  $G$ .

*Proof.* Assuming  $\mathbb{F}_2$  is  $L^1$ -measure equivalent to  $G$ , one can construct an integrable measure equivalence between  $G$  and the automorphism group  $H = \text{Aut}(\text{Tree}_4)$  of the 4-regular tree, which contains  $\mathbb{F}_2$  as a cocompact lattice. By Theorems C and 2.5 this would yield a continuous homomorphism  $H \rightarrow \text{Homeo}(S^1)$  with closed image. This leads to a contradiction, because  $H$  is totally disconnected and virtually simple [57, Théorème 4.5], while  $\text{Homeo}(S^1)$  has no non-discrete totally disconnected subgroups [24, Theorem 4.7 on p. 345].  $\square$

## APPENDIX A. MEASURE EQUIVALENCE

The appendix contains some general facts related to measure equivalence (Definition 1.1), the strong ICC property (Definition 2.2), and the notions of taut couplings and groups (Definition 1.3).

**A.1. The category of couplings.** Measure equivalence is an equivalence relation on unimodular lsc groups. Let us describe explicitly the constructions which show reflexivity, symmetry and transitivity of measure equivalence.

**A.1.1. Tautological coupling.** The tautological coupling is the  $(G \times G)$ -coupling  $(G, m_G)$  given by  $(g_1, g_2) : g \mapsto g_1 g g_2^{-1}$ . It demonstrates reflexivity of measure equivalence.

**A.1.2. Duality.** Symmetry is implied by the following: Given a  $(G, H)$ -coupling  $(\Omega, m)$  the dual  $(\Omega^*, m^*)$  is the  $(H, G)$ -coupling  $\Omega^*$  with the same underlying measure space  $(\Omega, m)$  and the  $H \times G$ -action  $(h, g) : \omega^* \mapsto (g, h)\omega^*$ .

**A.1.3. Composition of couplings.** Compositions defined below shows that measure equivalence is a transitive relation. Let  $G_1, H, G_2$  be unimodular lsc groups, and  $(\Omega_i, m_i)$  be a  $(G_i, H)$ -coupling for  $i \in \{1, 2\}$ . We describe the  $(G_1, G_2)$ -coupling  $\Omega_1 \times_H \Omega_2^*$  modeled on the space of  $H$ -orbits on  $(\Omega_1 \times \Omega_2, m_1 \times m_2)$  with respect to the diagonal  $H$ -action. Consider measure isomorphisms for  $(\Omega_i, m_i)$  as in (1.1): For  $i \in \{1, 2\}$  there are finite measure spaces  $(X_i, \mu_i)$  and  $(Y_i, \nu_i)$ , measure-preserving actions  $G_i \curvearrowright (X_i, \mu_i)$  and  $H \curvearrowright (Y_i, \nu_i)$ , measurable cocycles  $\alpha_i : G_i \times X_i \rightarrow H$  and  $\beta_i : H \times Y_i \rightarrow G_i$ , and measure space isomorphisms  $G_i \times Y_i \cong \Omega_i \cong H \times X_i$  with respect to which the  $G_i \times H$ -actions are given by

$$\begin{aligned} (g_i, h) &: (h', x) \mapsto (hh'\alpha_i(g_i, x)^{-1}, g_i \cdot x), \\ (g_i, h) &: (g', y) \mapsto (g_i g' \beta(h, y)^{-1}, h \cdot y). \end{aligned}$$

The space  $\Omega_1 \times_H \Omega_2^*$  with its natural  $G_1 \times G_2$ -action is equivariantly isomorphic to  $(X_1 \times X_2 \times H, \mu_1 \times \mu_2 \times m_H)$  endowed with the  $G_1 \times G_2$ -action

$$(g_1, g_2) : (x_1, x_2, h) \mapsto (g_1 \cdot x_1, g_2 \cdot x_2, \alpha_1(g_1, x_1) h \alpha_2(g_2, x_2)^{-1}).$$

To see that it is a  $(G_1, G_2)$ -coupling, we identify this space with  $Z \times G_1$  equipped with the action

$$(g_1, g_2) : (g', z) \mapsto (g_1 g' c(g_2, z)^{-1}, g_2 \cdot z) \quad (g' \in G_1, z \in Z)$$

where  $Z = X_2 \times Y_1$ , while the action  $G_2 \curvearrowright Z$  and the cocycle  $c : G_2 \times Z \rightarrow G_1$  are given by

$$(A.1) \quad \begin{aligned} g_2 &: (x, y) \mapsto (g_2 \cdot x, \alpha_2(g_2, x) \cdot y), \\ c(g_2, (x, y)) &= \beta_1(\alpha_2(g_2, x), y). \end{aligned}$$

Similarly,  $\Omega_1 \times_H \Omega_2^* \cong W \times G_2$ , for  $W \cong X_1 \times Y_2$ .

**A.1.4. Morphisms.** Let  $(\Omega_i, m_i)$ ,  $i \in \{1, 2\}$ , be two  $(G, H)$ -couplings. Let  $F : \Omega_1 \rightarrow \Omega_2$  be a measurable map such that for  $m_1$ -a.e.  $\omega \in \Omega_1$  and every  $g \in G$  and every  $h \in H$

$$F((g, h)\omega) = (g, h)F(\omega).$$

Such maps are called *quotient maps* or *morphisms*.

**A.1.5. Compact kernels.** Let  $(\Omega, m)$  be a  $(G, H)$ -coupling, and let

$$\{1\} \rightarrow K \rightarrow G \rightarrow \bar{G} \rightarrow \{1\}$$

be a short exact sequence where  $K$  is compact. Then the natural quotient space  $(\bar{\Omega}, \bar{m}) = (\Omega, m)/K$  is a  $(\bar{G}, H)$ -coupling, and the natural map  $F : \Omega \rightarrow \bar{\Omega}$ ,  $F : \omega \mapsto K\omega$ , is equivariant in the sense of  $F((g, h)\omega) = (\bar{g}, h)F(\omega)$ . This may be considered as an *isomorphism of couplings up to compact kernel*.

**A.1.6. Passage to lattices.** Let  $(\Omega, m)$  be a  $(G, H)$ -coupling, and let  $\Gamma < G$  be a lattice. By restricting the  $G \times H$ -action on  $(\Omega, m)$  to  $\Gamma \times H$  we obtain a  $(\Gamma, H)$ -coupling. Formally, this follows by considering  $(G, m_G)$  as a  $\Gamma \times G$ -coupling and considering the composition  $G \times_G \Omega$  as  $\Omega$  with the  $\Gamma \times H$ -action.

**A.2.  $L^p$ -integrability conditions.** Let  $G$  and  $H$  be compactly generated unimodular lcsc groups equipped with proper norms  $|\cdot|_G$  and  $|\cdot|_H$ . Let  $c : G \times X \rightarrow H$  be a measurable cocycle, and fix some  $p \in [1, \infty)$ . For  $g \in G$  we define

$$\|g\|_{c,p} = \left( \int_X |c(g,x)|_H^p d\mu(x) \right)^{1/p}.$$

For  $p = \infty$  we use the essential supremum. Assume that  $\|g\|_{c,p} < \infty$  for a.e.  $g \in G$ . We claim that there are constants  $a, A > 0$  so that for every  $g \in G$

$$(A.2) \quad \|g\|_{c,p} \leq A \cdot |g|_G + a.$$

Hence  $c$  is  $L^p$ -integrable in the sense of Definition 1.5. The key observation here is that  $\|\cdot\|_{c,p}$  is subadditive. Indeed, by the cocycle identity, subadditivity of the norm  $|\cdot|_H$ , and the Minkowski inequality, for any  $g_1, g_2 \in G$  we get

$$\begin{aligned} \|g_2 g_1\|_{c,p} &\leq \left( \int_X (|c(g_2, g_1 \cdot x)|_H + |c(g_1, x)|_H)^p d\mu(x) \right)^{1/p} \\ &\leq \left( \int_X |c(g_2, -)|_H^p d\mu \right)^{1/p} + \left( \int_X |c(g_1, -)|_H^p d\mu \right)^{1/p} \\ &= \|g_2\|_{c,p} + \|g_1\|_{c,p}. \end{aligned}$$

For  $t > 0$  denote  $E_t = \{g \in G : \|g\|_{c,p} < t\}$ . We have  $E_t \cdot E_s \subseteq E_{s+t}$  for any  $t, s > 0$ . Fix  $t$  large enough so that  $m_G(E_t) > 0$ . By [11, Corollary 12.4 on p. 235],  $E_{2t} \supseteq E_t \cdot E_t$  has a non-empty interior. Hence any compact subset of  $G$  can be covered by finitely many translates of  $E_{2t}$ . The subadditivity implies that  $\|g\|_{c,p}$  is bounded on compact sets. This gives (A.2).

**Lemma A.1.** *Let  $G, H, L$  be compactly generated groups,  $G \curvearrowright (X, \mu)$ ,  $H \curvearrowright (Y, \nu)$  be finite measure-preserving actions, and  $\alpha : G \times X \rightarrow H$  and  $\beta : H \times Y \rightarrow L$  be  $L^p$ -integrable cocycles for some  $1 \leq p \leq \infty$ . Consider  $Z = X \times Y$  and  $G \curvearrowright Z$  by  $g : (x, y) \mapsto (g \cdot x, \alpha(g, x) \cdot y)$ . Then the cocycle  $\gamma : G \times Z \rightarrow L$  given by*

$$\gamma(g, (x, y)) = \beta(\alpha(g, x), y).$$

*is  $L^p$ -integrable.*

*Proof.* For  $p = \infty$  the claim is obvious. Assume  $p < \infty$ . Let  $A, a, B, b$  be constants such that  $\|h\|_{\beta,p} \leq B \cdot |h|_H + b$  and  $\|g\|_{\alpha,p} \leq A \cdot |g|_G + a$ . Then

$$\begin{aligned} \|g\|_{\gamma,p}^p &= \int_{X \times Y} |\beta(\alpha(g, x), y)|_L^p d\mu(x) d\nu(y) \\ &\leq \int_X (B \cdot |\alpha(g, x)|_H + b)^p d\mu(x) \\ &\leq \max(B, b)^p \cdot \|g\|_{\alpha,p}^p \leq (C \cdot |g|_G + c)^p \end{aligned}$$

for appropriate constants  $c > 0$  and  $C > 0$ . □

**Lemma A.2.** *Let  $G_1, H, G_2$  be compactly generated unimodular lcsc groups. For  $i \in \{1, 2\}$  let  $(\Omega_i, m_i)$  be an  $L^p$ -integrable  $(G_i, H)$ -coupling. Then  $\Omega_1 \times_H \Omega_2^*$  is an  $L^p$ -integrable  $(G_1, G_2)$ -coupling.*

*Proof.* This follows from Lemma A.1 using the explicit description (A.1) of the cocycles for  $\Omega_1 \times_H \Omega_2^*$ .  $\square$

We conclude that for each  $1 \leq p \leq \infty$ ,  $L^p$ -measure equivalence is an equivalence relation between compactly generated unimodular lcsc groups.

### A.3. Tautening maps.

**Lemma A.3.** *Let  $G$  be a lcsc group,  $\Gamma$  a countable group and  $j_1, j_2 : \Gamma \rightarrow G$  be homomorphisms with  $\Gamma_i = j_i(\Gamma)$  being lattices in  $G$ . Assume that  $G$  is taut (resp.  $p$ -taut and  $\Gamma_i$  are  $L^p$ -integrable). Then there exists  $g \in G$  so that*

$$j_2(\gamma) = g j_1(\gamma) g^{-1} \quad (\gamma \in \Gamma).$$

*If  $\pi : G \rightarrow \mathcal{G}$  is a continuous homomorphism into a Polish group and  $G$  is taut relative to  $\pi : G \rightarrow \mathcal{G}$  (resp.  $G$  is  $p$ -taut relative to  $\pi : G \rightarrow \mathcal{G}$  and  $\Gamma_i$  are  $L^p$ -integrable) then there exists  $y \in \mathcal{G}$  with*

$$\pi(j_2(\gamma)) = y \pi(j_1(\gamma)) y^{-1} \quad (\gamma \in \Gamma).$$

*Proof.* We prove the more general second statement. The group

$$\Delta = \{(j_1(\gamma), j_2(\gamma)) \in G \times G \mid \gamma \in \Gamma\}$$

is a closed discrete subgroup in  $G \times G$ . The homogeneous  $G \times G$ -space  $\Omega = G \times G / \Delta$  equipped with the  $G \times G$ -invariant measure is easily seen to be a  $(G, G)$ -coupling. It will be  $L^p$ -integrable if  $\Gamma_1$  and  $\Gamma_2$  are  $L^p$ -integrable lattices. Let  $\Phi : \Omega \rightarrow \mathcal{G}$  be the tautening map. There are  $a, b \in G$  and  $x \in \mathcal{G}$  such that for all  $g_1, g_2 \in G$

$$\Phi((g_1 a, g_2 b) \Delta_f) = \pi(g_1) x \pi(g_2)^{-1}.$$

Since  $(a, b)$  and  $(j_1(\gamma)^a a, j_2(\gamma)^b b)$  are in the same  $\Delta$ -coset, where  $g^h = h g h^{-1}$ , we get for all  $g_1, g_2 \in G$  and every  $\gamma \in \Gamma$

$$\pi(g_1) x \pi(g_2)^{-1} = \pi(g_1) \pi(j_1(\gamma)^a) x \pi(j_2(\gamma)^b)^{-1} \pi(g_2)^{-1}.$$

This implies that  $j_1$  and  $j_2$  are conjugate homomorphisms.  $\square$

The following lemma relates tautening maps  $\Phi : \Omega \rightarrow \mathcal{G}$  and cocycle rigidity for ME-cocycles.

**Lemma A.4.** *Let  $G$  be a unimodular lcsc group,  $\mathcal{G}$  be a Polish group,  $\pi : G \rightarrow \mathcal{G}$  a continuous homomorphism. Let  $(\Omega, m)$  be a  $(G, G)$ -coupling and  $\alpha : G \times X \rightarrow G$ ,  $\beta : G \times Y \rightarrow G$  be the corresponding ME-cocycles. Then  $\Omega$  is taut relative to  $\pi$  iff the  $\mathcal{G}$ -valued cocycle  $\pi \circ \alpha$  is conjugate to  $\pi$ , that is,*

$$\pi \circ \alpha(g, x) = f(g.x)^{-1} \pi(g) f(x)$$

*for a unique measurable map  $f : X \rightarrow \mathcal{G}$ . This is also equivalent to  $\pi \circ \beta$  being uniquely conjugate to  $\pi : G \rightarrow \mathcal{G}$ .*

*Proof.* Let  $\alpha : G \times X \rightarrow G$  be the ME-cocycle associated to a measure space isomorphism  $i : (G, m_G) \times (X, \mu) \rightarrow (\Omega, m)$  as in (1.1). In particular,

$$(g_1, g_2) : i(g, x) \mapsto i(g_2 g \alpha(g_1, x)^{-1}, g_1 \cdot x).$$

We shall now establish a 1-to-1 correspondence between Borel maps  $f : X \rightarrow \mathcal{G}$  with

$$\pi \circ \alpha(g, x) = f(g \cdot x)^{-1} \pi(g) f(x)$$

and tautening maps  $\Phi : \Omega \rightarrow \mathcal{G}$ . Given  $f$  as above one verifies that

$$\Phi : \Omega \rightarrow \mathcal{G}, \quad \Phi(i(g, x)) = f(x) \pi(g)^{-1}$$

is  $G \times G$ -equivariant.

For the converse direction, suppose  $\Phi : \Omega \rightarrow G$  is a tautening map. Thus,

$$g_1 \Phi(g_0, x) g_2^{-1} = \Phi((g_1, g_2)(g_0, x)) = \Phi(g_2 g_0 \alpha(g_1, x)^{-1}, g_1 \cdot x).$$

For  $\mu$ -a.e.  $x \in X$  and a.e.  $g \in G$  the value of  $\Phi(g, x)g$  is constant  $f(x)$ . The above identity implies the required identity  $\alpha(g, x) = f(g \cdot x)^{-1} g f(x)$ .  $\square$

#### A.4. Strong ICC property.

**Lemma A.5.** *Let  $G$  be a unimodular lcsc group,  $\mathcal{G}$  a Polish group,  $\pi : G \rightarrow \mathcal{G}$  a continuous homomorphism. Suppose that  $\mathcal{G}$  is not strongly ICC relative to  $\pi(G)$ . Then there is a  $(G, G)$ -coupling  $(\Omega, m)$  with two distinct tautening maps to  $\mathcal{G}$ .*

*Proof.* Let  $\mu$  be a Borel probability measure on  $\mathcal{G}$  invariant under conjugations by  $\pi(G)$ . Consider  $\Omega = G \times \mathcal{G}$  with the measure  $m = m_G \times \mu$  where  $m_G$  denotes the Haar measure, and measure-preserving  $G \times G$ -action

$$(g_1, g_2) : (g, x) \mapsto (g_1 g g_2^{-1}, \pi(g_2) x \pi(g_2)^{-1}).$$

This is clearly a  $(G, G)$ -coupling and the following measurable maps  $\Phi_i : \Omega \rightarrow \mathcal{G}$ ,  $i \in \{1, 2\}$ , are  $G \times G$ -equivariant:  $\Phi_1(g, x) = \pi(g)$  and  $\Phi_2(g, x) = \pi(g) \cdot x$ . Note that  $\Phi_1 = \Phi_2$  on a conull set iff  $\mu = \delta_e$ .  $\square$

**Lemma A.6.** *Let  $G$  be a unimodular lcsc group and  $\mathcal{G}$  a Polish group. Assume that  $\mathcal{G}$  is strongly ICC relative to  $\pi(G)$ . Let  $(\Omega, m)$  be a  $(G, G)$ -coupling. Then:*

- (1) *There is at most one tautening map  $\Phi : \Omega \rightarrow \mathcal{G}$ .*
- (2) *Let  $F : (\Omega, m) \rightarrow (\Omega_0, m_0)$  be a morphism of  $(G, G)$ -couplings and suppose that there exists a tautening map  $\Phi : \Omega \rightarrow \mathcal{G}$ . Then it descends to  $\Omega_0$ , i.e.,  $\Phi = \Phi_0 \circ F$  for a unique tautening map  $\Phi_0 : \Omega_0 \rightarrow \mathcal{G}$ .*
- (3) *If  $\Gamma_1, \Gamma_2 < G$  are lattices, then  $\Phi : \Omega \rightarrow \mathcal{G}$  is unique as a  $\Gamma_1 \times \Gamma_2$ -equivariant map.*
- (4) *If  $\Gamma_1, \Gamma_2 < G$  are lattices, and  $(\Omega, m)$  admits a  $\Gamma_1 \times \Gamma_2$ -equivariant map  $\Phi : \Omega \rightarrow \mathcal{G}$ , then  $\Phi$  is  $G \times G$ -equivariant.*
- (5) *If  $\eta : \Omega \rightarrow \text{Prob}(\mathcal{G})$ ,  $\omega \mapsto \eta_\omega$ , is a measurable  $G \times G$ -equivariant map to the space of Borel probability measures on  $\mathcal{G}$  endowed with the weak topology, then it takes values in Dirac measures: We have  $\eta_\omega = \delta_{\Phi(\omega)}$ , where  $\Phi : \Omega \rightarrow \mathcal{G}$  is the unique tautening map.*

*Proof.* We start from the last claim and deduce the other ones from it.

(5). Given an equivariant map  $\eta : \Omega \rightarrow \text{Prob}(\mathcal{G})$  consider the convolution

$$\nu_\omega = \check{\eta}_\omega * \eta_\omega,$$

namely the image of  $\eta_\omega \times \eta_\omega$  under the map  $(a, b) \mapsto a^{-1} \cdot b$ . Then

$$\nu_{(g,h)\omega} = \nu_\omega^{\pi(g)} \quad (g, h \in G),$$

where the latter denotes the push-forward of  $\nu_\omega$  under the conjugation

$$a \mapsto a^{\pi(g)} = \pi(g)^{-1} a \pi(g).$$

In particular, the map  $\omega \mapsto \nu_\omega$  is invariant under the action of the second  $G$ -factor. Therefore  $\nu_\omega$  descends to a measurable map  $\tilde{\nu} : \Omega/G \rightarrow \text{Prob}(\mathcal{G})$ , satisfying

$$\tilde{\nu}_{g \cdot x} = \tilde{\nu}_x^{\pi(g)} \quad (x \in X = \Omega/G, g \in G).$$

Here we identify  $\Omega/G$  with a finite measure space  $(X, \mu)$  as in (1.1). Consider the center of mass

$$\bar{\nu} = \mu(X)^{-1} \int_X \tilde{\nu}_x d\mu(x).$$

It is a probability measure on  $\mathcal{G}$ , which is invariant under conjugations. By the strong ICC property relative to  $\pi(G)$  we get  $\bar{\nu} = \delta_e$ . Since  $\delta_e$  is an extremal point of  $\text{Prob}(\mathcal{G})$ , it follows that  $m$ -a.e.  $\nu(\omega) = \delta_e$ . This implies that  $\eta_\omega = \delta_{\Phi(\omega)}$  for some measurable  $\Phi : \Omega \rightarrow \mathcal{G}$ . The latter is automatically  $G \times G$ -equivariant.

(1). If  $\Phi_1, \Phi_2 : \Omega \rightarrow \mathcal{G}$  are tautening maps, then  $\eta_\omega = \frac{1}{2}(\delta_{\Phi_1(\omega)} + \delta_{\Phi_2(\omega)})$  is an equivariant map  $\Omega \rightarrow \text{Prob}(\mathcal{G})$ . By (5) it takes values in Dirac measures, which is equivalent to the  $m$ -a.e. equality  $\Phi_1 = \Phi_2$ .

(2). Disintegration of  $m$  with respect to  $m_0$  gives a  $G \times G$ -equivariant measurable map  $\Omega_0 \rightarrow \mathcal{M}(\Omega)$ ,  $\omega \mapsto m_{\omega_0}$ , to the space of finite measures on  $\Omega$ . Then the map  $\eta : \Omega_0 \rightarrow \text{Prob}(\mathcal{G})$ , given by

$$\eta_{\omega_0} = \|m_{\omega_0}\|^{-1} \cdot \Phi_*(m_{\omega_0}).$$

is  $G \times G$ -equivariant. Hence by (5),  $\eta_{\omega_0} = \delta_{\Phi_0(\omega_0)}$  for the unique tautening map  $\Phi_0 : \Omega_0 \rightarrow \mathcal{G}$ . The relation  $\Phi = \Phi_0 \circ F$  follows from the fact that Dirac measures are extremal.

(3) follows from (4) and (1).

(4). The claim is equivalent to: For  $m$ -a.e.  $\omega \in \Omega$  the map  $F_\omega : G \times G \rightarrow \mathcal{G}$  with

$$F_\omega(g_1, g_2) = \pi(g_1)^{-1} \Phi((g_1, g_2)\omega) \pi(g_2)$$

is  $m_G \times m_G$ -a.e. constant  $\Phi_0(\omega)$ . Note that the family  $\{F_\omega\}$  has the following equivariance property: For  $g_1, g_2, h_1, h_2 \in G$  we have

$$\begin{aligned} F_{(h_1, h_2)\omega}(g_1, g_2) &= \pi(g_1)^{-1} \Phi((g_1 h_1, g_2 h_2)\omega) \pi(g_2) \\ &= \pi(h)_1^{-1} F_\omega(g_1 h_1, g_2 h_2) \pi(h_2). \end{aligned}$$

Since  $\Phi$  is  $\Gamma_1 \times \Gamma_2$ -equivariant, for  $m$ -a.e.  $\omega \in \Omega$  the map  $F_\omega$  descends to  $G/\Gamma_1 \times G/\Gamma_2$ . Let  $\eta_\omega \in \text{Prob}(\mathcal{G})$  denote the distribution of  $F_\omega(\cdot, \cdot)$  over the probability space  $G/\Gamma_1 \times G/\Gamma_2$ , that is, for a Borel subset  $E \subset \mathcal{G}$

$$\eta_\omega(E) = m_{G/\Gamma_1} \times m_{G/\Gamma_2} \{(g_1, g_2) \mid F_\omega(g_1, g_2) \in E\}.$$

Since this measure is invariant under translations by  $G \times G$ , it follows that  $\eta_\omega$  is a  $G \times G$ -equivariant maps  $\Omega \rightarrow \text{Prob}(\mathcal{G})$ . By (5) one has  $\eta_\omega = \delta_{f(\omega)}$  for some measurable  $G \times G$ -equivariant map  $f : \Omega \rightarrow \mathcal{G}$ . Hence  $F_\omega(g_1, g_2) = f(\omega)$  for a.e.  $g_1, g_2 \in G$ ; it follows that

$$(A.3) \quad \Phi((g_1, g_2)\omega) = \pi(g_1)\Phi(\omega)\pi(g_2)^{-1}$$

holds for  $m_G \times m_G \times m$ -a.e.  $(g_1, g_2, \omega)$ .  $\square$

**Corollary A.7.** *Let  $\pi : G \rightarrow \mathcal{G}$  be as above and assume that  $\mathcal{G}$  is strongly ICC relative to  $\pi(G)$ . Then the collection of all  $(G, G)$ -couplings which are taut relative to  $\pi : G \rightarrow \mathcal{G}$  is closed under the operations of taking the dual, compositions, quotients and extensions.*

*Proof.* The uniqueness of tautening maps follow from the relative strong ICC property (Lemma A.6.(1)). Hence we focus on the existence of such maps.

Let  $\Phi : \Omega \rightarrow \mathcal{G}$  be a tautening map. Then  $\Psi(\omega^*) = \Phi(\omega)^{-1}$  is a tautening map  $\Omega^* \rightarrow \mathcal{G}$ .

Let  $\Phi_i : \Omega_i \rightarrow \mathcal{G}$ ,  $i = 1, 2$ , be tautening maps. Then  $\Psi([\omega_1, \omega_2]) = \Phi(\omega_1) \cdot \Phi(\omega_2)$  is a tautening map  $\Omega_1 \times_G \Omega_2 \rightarrow \mathcal{G}$ .

If  $F : (\Omega_1, m_1) \rightarrow (\Omega_2, m_2)$  is a quotient map and  $\Phi_1 : \Omega_1 \rightarrow \mathcal{G}$  is a tautening map, then, by Lemma A.6.(2),  $\Phi_1$  factors as  $\Phi_1 = \Phi_2 \circ F$  for a tautening map  $\Phi_2 : \Omega_2 \rightarrow \mathcal{G}$ . On the other hand, given a tautening map  $\Phi_2 : \Omega_2 \rightarrow \mathcal{G}$ , the map  $\Phi_1 = \Phi_2 \circ F$  is tautening for  $\Omega_1$ .  $\square$

## APPENDIX B. BOUNDED COHOMOLOGY

Our background references for bounded cohomology, especially for the functorial approach to it, are [6, 41]. We summarize what we need from Burger-Monod's theory of bounded cohomology. Since we restrict to discrete groups, some results we quote from this theory are already go back to Ivanov [31].

**B.1. Banach modules.** All Banach spaces are over the field  $\mathbb{R}$  of real numbers. By the dual of a Banach space we understand the normed topological dual. The dual of a Banach space  $E$  is denoted by  $E^*$ . Let  $\Gamma$  be a discrete and countable group. A *Banach  $\Gamma$ -module* is a Banach space  $E$  endowed with a group homomorphism  $\pi$  from  $\Gamma$  into the group of isometric linear automorphisms of  $E$ . We use the module notation  $\gamma \cdot e = \pi(\gamma)(e)$  or just  $\gamma e = \pi(\gamma)(e)$  for  $\gamma \in \Gamma$  and  $e \in E$  whenever the action is clear from the context. The submodule of  $\Gamma$ -invariant elements is denoted by  $E^\Gamma$ . Note that  $E^\Gamma \subset E$  is closed.

If  $E$  and  $F$  are Banach  $\Gamma$ -modules, a  $\Gamma$ -*morphism*  $E \rightarrow F$  is a  $\Gamma$ -equivariant continuous linear map. The space  $\mathcal{B}(E, F)$  of continuous, linear maps  $E \rightarrow F$  is endowed with a natural Banach  $\Gamma$ -module structure via

$$(B.1) \quad (\gamma \cdot f)(e) = \gamma f(\gamma^{-1}e).$$

The *contragredient* Banach  $\Gamma$ -module structure  $E^\sharp$  associated to  $E$  is by definition  $\mathcal{B}(E, \mathbb{R}) = E^*$  with the  $\Gamma$ -action (B.1). A *coefficient  $\Gamma$ -module* is a Banach  $\Gamma$ -module  $E$  contragredient to some separable continuous Banach  $\Gamma$ -module denoted by  $E^b$ . The choice of  $E^b$  is part of the data. The specific choice of  $E^b$  defines a weak-\* topology on  $E$ . The only examples that appear in this paper are  $E = L^\infty(X, \mu)$  with  $E^b = L^1(X, \mu)$  and  $E = E^b = \mathbb{R}$ .

For a coefficient  $\Gamma$ -module  $E$  let  $C_b^k(\Gamma, E)$  be the Banach  $\Gamma$ -module  $L^\infty(\Gamma^{k+1}, E)$  consisting of bounded maps from  $\Gamma^{k+1}$  to  $E$  endowed with the supremum norm and the  $\Gamma$ -action:

$$(B.2) \quad (\gamma \cdot f)(\gamma_0, \dots, \gamma_k) = \gamma \cdot f(\gamma^{-1}\gamma_0, \dots, \gamma^{-1}\gamma_k).$$

For a coefficient  $\Gamma$ -module  $E$  and a standard Borel  $\Gamma$ -space  $S$  with quasi-invariant measure let  $L_{w*}^\infty(S, E)$  be the space of weak-\* measurable essentially bounded maps from  $S$  to  $E$ , where maps are identified if they only differ on a null set. The space  $L_{w*}^\infty(S, E)$  is endowed with the essential supremum norm and the  $\Gamma$ -action (B.2).

For a measurable space  $X$  the Banach space  $\mathcal{B}^\infty(X, E)$  is the space of weak- $*$ -measurable bounded maps from  $X$  to  $E$  endowed with supremum norm [4, Section 2] and the  $\Gamma$ -action (B.2).

**B.2. Injective resolutions.** Let  $\Gamma$  be a discrete group and  $E$  be a Banach  $\Gamma$ -module. The sequence of Banach  $\Gamma$ -modules  $C_b^k(\Gamma, E)$ ,  $k \geq 0$ , becomes a chain complex of Banach  $\Gamma$ -modules via the standard homogeneous coboundary operator

$$(B.3) \quad d(f)(\gamma_0, \dots, \gamma_k) = \sum_{i=0}^k (-1)^i f(\gamma_0, \dots, \hat{\gamma}_i, \dots, \gamma_k).$$

The *bounded cohomology*  $H_b^\bullet(\Gamma, E)$  of  $\Gamma$  with coefficients  $E$  is the cohomology of the complex of  $\Gamma$ -invariants  $C_b^\bullet(\Gamma, E)^\Gamma$ . The bounded cohomology  $H_b^\bullet(\Gamma, E)$  inherits a semi-norm from  $C_b^\bullet(\Gamma, E)$ : The *(semi-)norm* of an element  $x \in H_b^\bullet(\Gamma, E)$  is the infimum of the norms of all cocycles in the cohomology class  $x$ .

Next we briefly recall the functorial approach to bounded cohomology as introduced by Ivanov [31] for discrete groups and further developed by Burger-Monod [6, 41]. We refer for the definition of *relative injectivity* of a Banach  $\Gamma$ -module to [41, Definition 4.1.2 on p. 32]. A *strong resolution*  $E^\bullet$  of  $E$  is a resolution, i.e. an acyclic complex,

$$0 \rightarrow E \rightarrow E^0 \rightarrow E^1 \rightarrow E^2 \rightarrow \dots$$

of Banach  $\Gamma$ -modules that has chain contraction which is contracting with respect to the Banach norms. The key to the functorial definition of bounded cohomology are the following two theorems:

**Theorem B.1** ([6, Proposition 1.5.2]). *Let  $E$  and  $F$  be Banach  $\Gamma$ -modules. Let  $E^\bullet$  be a strong resolution of  $E$ . Let  $F^\bullet$  be a resolution  $F$  by relatively injective Banach  $\Gamma$ -modules. Then any  $\Gamma$ -morphism  $E \rightarrow F$  extends to a  $\Gamma$ -morphism of resolutions  $E^\bullet \rightarrow F^\bullet$  which is unique up to  $\Gamma$ -homotopy. Hence  $E \rightarrow F$  induces functorially continuous linear maps  $H^\bullet(E^{\bullet\Gamma}) \rightarrow H^\bullet(F^{\bullet\Gamma})$ .*

**Theorem B.2** ([41, Corollary 7.4.7 on p. 80]). *Let  $E$  be a Banach  $\Gamma$ -module. The complex  $E \rightarrow C_b^\bullet(\Gamma, E)$  with  $E \rightarrow C_b^0(\Gamma, E)$  being the inclusion of constant functions is a strong, relatively injective resolution.*

For a coefficient  $\Gamma$ -module, a measurable space  $X$  with measurable  $\Gamma$ -action, and a standard Borel  $\Gamma$ -space  $S$  with quasi-invariant measure we obtain chain complexes  $\mathcal{B}^\infty(X^{\bullet+1}, E)$  and  $L_{w*}(S^{\bullet+1}, E)$  of Banach  $\Gamma$ -modules via the standard homogeneous coboundary operators (similar as in (B.3)).

The following result is important for expressing induced maps in bounded cohomology in terms of boundary maps [4].

**Proposition B.3** ([4, Proposition 2.1]). *Let  $E$  be a coefficient  $\Gamma$ -module. Let  $X$  be a measurable space with measurable  $\Gamma$ -action. The complex  $E \rightarrow \mathcal{B}^\infty(X^{\bullet+1}, E)$  with  $E \rightarrow \mathcal{B}^\infty(X, E)$  being the inclusion of constant functions is a strong resolution of  $E$ .*

The next theorem is one of the main results of the functorial approach to bounded cohomology by Burger-Monod:

**Theorem B.4** ([6, Corollary 2.3.2; 41, Theorem 7.5.3 on p. 83]). *Let  $S$  be a regular  $\Gamma$ -space and be  $E$  a coefficient  $\Gamma$ -module. Then  $E \rightarrow L_{w*}(S^{\bullet+1}, E)$  with  $E \rightarrow L_{w*}(S^{\bullet+1}, E)$  being the inclusion of constant functions is a strong resolution. If,*

in addition,  $S$  is amenable in the sense of Zimmer [41, Definition 5.3.1], then each  $L_{w*}(S^{k+1}, E)$  is relatively injective, and according to Theorem B.1 the cohomology groups  $H^\bullet(L_{w*}(S^{\bullet+1}, E)^\Gamma)$  are canonically isomorphic to  $H_b^\bullet(\Gamma, E)$ .

**Definition B.5.** Let  $S$  be a standard Borel  $\Gamma$ -space with a quasi-invariant probability measure  $\mu$ . Let  $E$  be a coefficient  $\Gamma$ -module. The *Poisson transform*  $PT^\bullet : L_{w*}(S^{\bullet+1}, E) \rightarrow C_b^\bullet(\Gamma, E)$  is the  $\Gamma$ -morphism of chain complexes defined by

$$PT^k(f)(\gamma_0, \dots, \gamma_k) = \int_{S^{k+1}} f(\gamma_0 s_0, \dots, \gamma_k s_k) d\mu(s_0) \dots d\mu(s_k).$$

If  $S$  is amenable, then the Poisson transform induces a canonical isomorphism in cohomology (Theorem B.4). By the same theorem this isomorphism does not depend on the choice of  $\mu$  within a given measure class.

#### REFERENCES

- [1] S. Adams, G. A. Elliott, and T. Giordano, *Amenable actions of groups*, Trans. Amer. Math. Soc. **344** (1994), no. 2, 803–822.
- [2] U. Bader, A. Furman, and R. Sauer, *Efficient subdivision in hyperbolic groups and applications* (2010), 24 pages pp., available at [arxiv:math/1003.1562](https://arxiv.org/abs/math/1003.1562).
- [3] M. R. Bridson, M. Tweeddale, and H. Wilton, *Limit groups, positive-genus towers and measure-equivalence*, Ergodic Theory Dynam. Systems **27** (2007), no. 3, 703–712.
- [4] M. Burger and A. Iozzi, *Boundary maps in bounded cohomology. Appendix to: “Continuous bounded cohomology and applications to rigidity theory” [Geom. Funct. Anal. **12** (2002), no. 2, 219–280] by Burger and N. Monod*, Geom. Funct. Anal. **12** (2002), no. 2, 281–292.
- [5] ———, *A useful formula in bounded cohomology*, to appear in *Seminaires et Congres*, nr. 18, available at <http://www.math.ethz.ch/iozzi/grenoble.ps>.
- [6] M. Burger and N. Monod, *Continuous bounded cohomology and applications to rigidity theory*, Geom. Funct. Anal. **12** (2002), no. 2, 219–280.
- [7] M. Burger and S. Mozes, *CAT(-1)-spaces, divergence groups and their commensurators*, J. Amer. Math. Soc. **9** (1996), no. 1, 57–93.
- [8] A. Casson and D. Jungreis, *Convergence groups and Seifert fibered 3-manifolds*, Invent. Math. **118** (1994), no. 3, 441–456.
- [9] K. Corlette, *Archimedean superrigidity and hyperbolic geometry*, Ann. of Math. (2) **135** (1992), no. 1, 165–182.
- [10] K. Corlette and R. J. Zimmer, *Superrigidity for cocycles and hyperbolic geometry*, Internat. J. Math. **5** (1994), no. 3, 273–290.
- [11] J. M. G. Fell and R. S. Doran, *Representations of \*-algebras, locally compact groups, and Banach \*-algebraic bundles. Vol. 1*, Pure and Applied Mathematics, vol. 125, Academic Press Inc., 1988. Basic representation theory of groups and algebras.
- [12] D. Fisher and T. Hitchman, *Cocycle superrigidity and harmonic maps with infinite-dimensional targets*, Int. Math. Res. Not. (2006).
- [13] D. Fisher, D. W. Morris, and K. Whyte, *Nonergodic actions, cocycles and superrigidity*, New York J. Math. **10** (2004), 249–269 (electronic).
- [14] A. Furman, *Orbit equivalence rigidity*, Ann. of Math. (2) **150** (1999), no. 3, 1083–1108.
- [15] ———, *Gromov’s measure equivalence and rigidity of higher rank lattices*, Ann. of Math. (2) **150** (1999), no. 3, 1059–1081.
- [16] ———, *Mostow-Margulis rigidity with locally compact targets*, Geom. Funct. Anal. **11** (2001), no. 1, 30–59.
- [17] ———, *A survey of Measured Group Theory*, Geometry, Rigidity and Group Actions, Univ. of Chicago Press, Chicago. to appear.
- [18] H. Furstenberg, *Boundary theory and stochastic processes on homogeneous spaces*, Harmonic analysis on homogeneous spaces (Proc. Sympos. Pure Math., Vol. XXVI, Williams Coll., Williamstown, Mass., 1972), Amer. Math. Soc., Providence, R.I., 1973, pp. 193–229.
- [19] ———, *Rigidity and cocycles for ergodic actions of semisimple Lie groups (after G. A. Margulis and R. Zimmer)*, Bourbaki Seminar, Vol. 1979/80, Lecture Notes in Math., vol. 842, Springer, Berlin, 1981, pp. 273–292.

- [20] D. Gabai, *Convergence groups are Fuchsian groups*, Ann. of Math. (2) **136** (1992), no. 3, 447–510.
- [21] D. Gaboriau, *Coût des relations d'équivalence et des groupes*, Invent. Math. **139** (2000), no. 1, 41–98.
- [22] ———, *Invariants  $l^2$  de relations d'équivalence et de groupes*, Publ. Math. Inst. Hautes Études Sci. **95** (2002), 93–150 (French).
- [23] ———, *Examples of groups that are measure equivalent to the free group*, Ergodic Theory Dynam. Systems **25** (2005), no. 6, 1809–1827.
- [24] É. Ghys, *Groups acting on the circle*, Enseign. Math. (2) **47** (2001), no. 3-4, 329–407.
- [25] M. Gromov, *Asymptotic invariants of infinite groups*, Geometric group theory, Vol. 2 (Sussex, 1991), London Math. Soc. Lecture Note Ser., vol. 182, Cambridge Univ. Press, Cambridge, 1993, pp. 1–295.
- [26] G. Hjorth, *A converse to Dye's theorem*, Trans. Amer. Math. Soc. **357** (2005), no. 8, 3083–3103 (electronic).
- [27] U. Haagerup and H. J. Munkholm, *Simplices of maximal volume in hyperbolic  $n$ -space*, Acta Math. **147** (1981), no. 1-2, 1–11.
- [28] A. Ioana, J. Peterson, and S. Popa, *Amalgamated free products of weakly rigid factors and calculation of their symmetry groups*, Acta Math. **200** (2008), no. 1, 85–153.
- [29] A. Ioana, *Cocycle superrigidity for profinite actions of property (T) groups*, available at [arxiv:0805.2998](https://arxiv.org/abs/0805.2998).
- [30] A. Iozzi, *Bounded cohomology, boundary maps, and rigidity of representations into  $\text{Homeo}_+(\mathbf{S}^1)$  and  $\text{SU}(1, n)$* , Rigidity in dynamics and geometry (Cambridge, 2000), Springer, 2002, pp. 237–260.
- [31] N. V. Ivanov, *Foundations of the theory of bounded cohomology*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **143** (1985), 69–109, 177–178 (Russian, with English summary).
- [32] A. S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, 1995.
- [33] Y. Kida, *The mapping class group from the viewpoint of measure equivalence theory*, Mem. Amer. Math. Soc. **196** (2008), no. 916, viii+190.
- [34] ———, *Measure Equivalence rigidity of the Mapping Class Group*, available at [arxiv:0607600](https://arxiv.org/abs/0607600).
- [35] Y. Kida, *Orbit equivalence rigidity for ergodic actions of the mapping class group*, Geom. Dedicata **131** (2008), 99–109.
- [36] Y. Kida, *Rigidity of amalgamated free products in measure equivalence theory*, available at [arXiv:0902.2888](https://arxiv.org/abs/0902.2888).
- [37] C. Löh, *Measure homology and singular homology are isometrically isomorphic*, Math. Z. **253** (2006), no. 1, 197–218.
- [38] G. A. Margulis, *Non-uniform lattices in semisimple algebraic groups*, Lie groups and their representations (Proc. Summer School on Group Representations of the Bolyai János Math. Soc., Budapest, 1971), Halsted, New York, 1975, pp. 371–553.
- [39] ———, *Discrete groups of motions of manifolds of nonpositive curvature*, Proceedings of the International Congress of Mathematicians (Vancouver, B.C., 1974), Vol. 2, Canad. Math. Congress, Montreal, Que., 1975, pp. 21–34 (Russian).
- [40] ———, *Discrete subgroups of semisimple Lie groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 17, Springer-Verlag, Berlin, 1991.
- [41] N. Monod, *Continuous bounded cohomology of locally compact groups*, Lecture Notes in Mathematics, vol. 1758, Springer-Verlag, Berlin, 2001.
- [42] N. Monod and Y. Shalom, *Orbit equivalence rigidity and bounded cohomology*, Ann. of Math. (2) **164** (2006), no. 3, 825–878.
- [43] N. Monod and Y. Shalom, *Cocycle superrigidity and bounded cohomology for negatively curved spaces*, J. Differential Geom. **67** (2004), no. 3, 395–455.
- [44] G. D. Mostow, *Strong rigidity of locally symmetric spaces*, Princeton University Press, Princeton, N.J., 1973. Annals of Mathematics Studies, No. 78.
- [45] S. Popa, *On a class of type  $\text{II}_1$  factors with Betti numbers invariants*, Ann. of Math. (2) **163** (2006), no. 3, 809–899.

- [46] ———, *Strong rigidity of  $\text{II}_1$  factors arising from malleable actions of  $w$ -rigid groups. I*, Invent. Math. **165** (2006), no. 2, 369–408.
- [47] ———, *Strong rigidity of  $\text{II}_1$  factors arising from malleable actions of  $w$ -rigid groups. II*, Invent. Math. **165** (2006), no. 2, 409–451.
- [48] ———, *Cocycle and orbit equivalence superrigidity for malleable actions of  $w$ -rigid groups*, Invent. Math. **170** (2007), no. 2, 243–295.
- [49] ———, *Deformation and rigidity for group actions and von Neumann algebras*, International Congress of Mathematicians. Vol. I, Eur. Math. Soc., Zürich, 2007, pp. 445–477.
- [50] ———, *On the superrigidity of malleable actions with spectral gap*, J. Amer. Math. Soc. **21** (2008), no. 4, 981–1000.
- [51] G. Prasad, *Strong rigidity of  $\mathbf{Q}$ -rank 1 lattices*, Invent. Math. **21** (1973), 255–286.
- [52] J. G. Ratcliffe, *Foundations of hyperbolic manifolds*, Graduate Texts in Mathematics, vol. 149, Springer-Verlag, New York, 1994.
- [53] M. Ratner, *Interactions between ergodic theory, Lie groups, and number theory*, 2 (Zürich, 1994), Birkhäuser, Basel, 1995, pp. 157–182.
- [54] Y. Shalom, *Rigidity, unitary representations of semisimple groups, and fundamental groups of manifolds with rank one transformation group*, Ann. of Math. (2) **152** (2000), no. 1, 113–182.
- [55] ———, *Measurable group theory*, European Congress of Mathematics, Eur. Math. Soc., Zürich, 2005, pp. 391–423.
- [56] M. Stroppel, *Locally compact groups*, EMS Textbooks in Mathematics, European Mathematical Society (EMS), Zürich, 2006.
- [57] J. Tits, *Sur le groupe des automorphismes d'un arbre*, Essays on topology and related topics (Mémoires dédiés à Georges de Rham), Springer, 1970.
- [58] W. P. Thurston, *The Geometry and Topology of Three-Manifolds* (1978), available at <http://www.msri.org/publications/books/gt3m>.
- [59] A. Zastrow, *On the (non)-coincidence of Milnor-Thurston homology theory with singular homology theory*, Pacific J. Math. **186** (1998), no. 2, 369–396.
- [60] R. J. Zimmer, *Strong rigidity for ergodic actions of semisimple Lie groups*, Ann. of Math. (2) **112** (1980), no. 3, 511–529.
- [61] ———, *Ergodic theory and semisimple groups*, Monographs in Mathematics, vol. 81, Birkhäuser Verlag, Basel, 1984.

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