Coarse-geometric perspective on negatively curved manifolds and groups

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Abstract. Let (X, g) be a compact negatively curved Riemannian manifold with the fundamental group Γ. Restricting the lifted metric on the universal cover $(\tilde{X}, \tilde{g})$ of (X, g) to a Γ-orbit Γx one obtains a left invariant metric $d_{g,x}$ on Γ, which is well defined up to a bounded amount, depending on the choice of the orbit Γx. Motivated by this geometric example, we study classes $[d]$ of general left-invariant metrics d on general Gromov hyperbolic groups Γ, where $[d_1] = [d_2]$ if $d_1 - d_2$ is bounded. It turns out that many of the geometric objects associated with (X, g), such as: marked length spectrum, cross-ratio, Bowen-Margulis measure - can be defined in the general coarse-geometric setting. The main result of the paper is a characterization of the compact negatively curved locally symmetric spaces within this coarse-geometric setting.

1 Introduction

Compact connected Riemannian manifolds (X, g) with strictly negative sectional curvature are important and much studied objects in Riemannian geometry and Dynamics. Given such a manifold (X, g) consider the isometric action of the fundamental group $\Gamma = \pi_1(X)$ on the universal cover $(\tilde{X}, \tilde{g})$ and let $d_{g,x}$ denote the following metric on Γ

$$d_{g,x}(\gamma_1, \gamma_2) := \mathrm{dist}_{\tilde{g}}(\gamma_1 x,\, \gamma_2 x) \tag{1}$$

where $x \in \tilde{X}$ is some fixed point. This is just a restriction of the path metric $\mathrm{dist}_{\tilde{g}}$ on $\tilde{X}$ to the Γ-orbit Γx. As the Γ-action on $(\tilde{X}, \mathrm{dist}_{\tilde{g}})$ is isometric, proper and cocompact, the metric $d_{g,x}$ on Γ is left-invariant and $(\Gamma, d_{g,x})$ is *roughly-isometric* to the universal cover $(\tilde{X}, \mathrm{dist}_{\tilde{g}})$ (hereafter *rough*-isometry means a quasi-isometry with multiplicative constant one). Moreover, changing the base point x to y amounts to a *bounded* change in $d_{g,x}$, in fact

$$|d_{g,x}(\gamma_1, \gamma_2) - d_{g,y}(\gamma_1, \gamma_2)| \le 2\,\mathrm{diam}_{\tilde{g}}(X)$$

Note also that as the Riemannian structure g on X varies, the metrics $d_{g,x}$ remain *quasi-isometric* (generally with non-trivial multiplicative constants) to each other and to any word metric on the Gromov hyperbolic group Γ.

In this paper we shall consider the following general *coarse-geometric* setup:

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- Γ is an infinite discrete non-elementary torsion-free Gromov hyperbolic group.
- D_Γ is the collection of all left invariant metrics d on Γ, which are quasi-isometric to a word metric on Γ.
- $\mathcal{D}_\Gamma$ the factor space of D_Γ consisting of equivalence classes $\delta = [d]$ of metrics $d \in D_\Gamma$, where $d, d' \in D_\Gamma$ belong to the same class $\delta = [d] = [d']$ if $d - d'$ is bounded.
- $\mathcal{PD}_\Gamma$ is the projective version of $\mathcal{D}_\Gamma$, consisting of classes $\bar{\delta}$ where $d, d' \in D_\Gamma$ belong to the same class $\bar{\delta}$ if there exists a constant λ so that $d' - \lambda \cdot d$ is bounded.

The basic object of this paper are Gromov hyperbolic groups Γ equipped with a class $\delta \in \mathcal{D}_\Gamma$ or $\bar{\delta} \in \mathcal{PD}_\Gamma$ as above. Our discussion is motivated and guided by the *geometric setup* where $\Gamma = \pi_1(X)$ is a fundamental group of a compact manifold X which admits a negatively curved Riemannian structure g, and $\delta_g \in \mathcal{D}_\Gamma$ denotes the class $[d_{g,x}]$ as in (1). The Marked Length Rigidity Conjecture and the results cited below indicate that much, conjecturally all, of the Riemannian geometry of (X, g) is determined by $\delta_g \in \mathcal{D}_\Gamma$, because the marked length spectrum ℓ_g of (X, g) is an invariant of δ_g. However the general coarse-geometric framework of $\delta \in \mathcal{D}_\Gamma$ on Γ allows to consider other metrics and more general groups, including the following examples: metrics on hyperbolic groups $\Gamma = \pi_1(X)$ corresponding to general (not necessarily negatively curved) Riemannian or Finsler metrics on compact aspherical manifolds X with $\Gamma = \pi_1(X)$; Gromov hyperbolic groups Γ and metrics induced by convex cocompact isometric Γ-actions on $CAT(-1)$ spaces; general Gromov hyperbolic groups with word metrics, etc.

Our goal in this paper is to present the coarse-geometric point of view on some known constructions and facts about compact negatively curved manifolds and try to generalize them to this broader coarse-geometric setup. The paper contains some basic constructions and results, with further analysis to be presented elsewhere.

2 The Geometric Setup

Let X be a compact connected manifold without boundary equipped with a Riemannian structure g on X with strictly negative sectional curvature. Let $(\tilde{X}, \tilde{g})$ denote the universal cover with the lifted Riemannian structure and $\Gamma = \pi_1(X)$ be the fundamental group of X acting by isometries of $(\tilde{X}, \tilde{g})$. Let us recall some basic constructions and facts of this setup.

Marked Length Spectrum. For $\gamma \in \Gamma \setminus \{e\}$ denote by c_γ the corresponding conjugacy class in Γ. The set $\mathcal{C}_\Gamma = \{c_\gamma \mid \gamma \in \Gamma \setminus e\}$ parameterizes *free homotopy* classes of closed loops in X, with each such class containing a *unique* loop of minimal g-length (= a closed geodesic) on X. Denoting by

$\ell_g(c_\gamma)$ its length one obtains the function $\ell_g : \mathcal{C}_\Gamma \to \mathbb{R}_+$ which is called the *marked length spectrum* of (X, g).

MARKED LENGTH SPECTRUM RIGIDITY CONJECTURE ([BK] (3.1)) asserts that *negatively curved Riemannian structure g on X is determined by the Marked Length Spectrum ℓ_g*. More precisely, if g, g' are two negatively curved Riemannian structures on a compact X so that $\ell_g = \ell_{g'}$ then there exists a diffeomorphism $\phi : X \to X$ isotopic to the identity which carries g to g'.

The conjecture was stimulated by the deformation rigidity results of Guillemin and Kazhdan [GK]. J.-P. Otal [Ot1] and independently C. Croke [Cr] proved the conjecture (and more general results) for the case $\dim X = 2$, i.e. surfaces. In higher dimensions U. Hamenstädt [Ha] proved (using Besson-Courtios-Gallot results [BCG]) the conjecture for the case where one of the metrics, say g on X, is *locally symmetric*.

Busemann functions $\beta : \tilde{X} \times \tilde{X} \times \partial\tilde{X} \to \mathbb{R}$ and $B_x : \partial\tilde{X} \times \partial\tilde{X} \to \mathbb{R}_+ \cup \{\infty\}$ are defined as follows

$$\beta(x, y, \xi) := \lim_{z \to \xi} \left(\mathrm{dist}_{\tilde{g}}(x, z) - \mathrm{dist}_{\tilde{g}}(y, z)\right) \tag{2a}$$

$$B_x(\xi, \eta) := \beta(x, w, \xi) + \beta(x, w, \eta) \tag{2b}$$

where w in (2b) is an arbitrary point on the geodesic line $[\xi, \eta]$ connecting ξ and η. As soon as one verifies that these functions are well defined, it is easy to deduce that for any isometry γ of $(\tilde{X}, \tilde{g})$

$$B_{\gamma x}(\gamma \cdot \xi, \gamma \cdot \eta) = B_x(\xi, \eta) \tag{3a}$$

$$B_x(\xi, \eta) - B_y(\xi, \eta) = \beta(x, y, \xi) + \beta(x, y, \eta) \tag{3b}$$

Cross-Ratio. The classical notion of the cross-ratio on $S^1 = \mathbb{R} \cup \{\infty\} \cong \partial\mathbb{H}^2$ can be generalized (see J.-P. Otal [Ot2] and Hamnestädt [Ha]) as follows: given a negatively curved (X, g) with the universal cover $(\tilde{X}, \tilde{g})$ and the associated Busemann functions B_x define the (so called symplectic) *cross-ratio* on the boundary $\partial\tilde{X}$ by

$$[\xi, \eta, \xi', \eta']_g := \frac{1}{2}\left(B_x(\xi, \eta) + B_x(\xi', \eta') - B_x(\xi', \eta) - B_x(\xi, \eta')\right) \tag{4}$$

It follows from (3b) that the cross-ratio does not depend on the choice of $x \in \tilde{X}$, and (3a) implies that the cross-ratio is preserved under the natural action by any isometry γ of $(\tilde{X}, \tilde{g})$. One of the results (Thm 2.2) in [Ot2] states that for two negatively curved Riemannian structures g, g' on a compact manifold X the conditions $\ell_g = \ell_{g'}$ and $[,,,]_g = [,,,]_{g'}$ are equivalent.

Bowen-Margulis measure. The geodesic flow $\Phi = \{\Phi^t\}$ on the unit tangent bundle SX to X (associated to the metric g) is a topologically mixing Anosov flow. Its topological entropy $h_{top}(SX, \Phi^1)$ equals to the *volume growth*

$$h_g := \lim_{R\to\infty} \frac{1}{R} \log \mathrm{Vol}_{\tilde{g}} \mathsf{B}(x, R) \tag{5}$$

(the limit exists and does not depend on $x \in \tilde{X}$). For such flows Bowen [Bow] showed that the topological entropy is achieved by the measure-theoretical entropy of a unique Φ-invariant probability measure, which can be constructed as the weak limit of periodic orbits (i.e. closed geodesics) weighted according to their lengths $\ell_g(c_\gamma)$. This measure μ_{BM}, known as Bowen-Margulis measure, was constructed in a different form by Margulis [Mar] in his proof of the asymptotic

$$\#\{c_\gamma \in \mathcal{C}'_\Gamma \mid \ell_g(c_\gamma) \le t\} \sim \frac{e^{h_g\, t}}{h_g\, t}$$

of the number of *primitive* closed geodesics corresponding to the *indivisible* conjugacy classes $\mathcal{C}'_\Gamma \subset \mathcal{C}_\Gamma$. The key property of Margulis' construction of μ_{BM}, needed in his proof, is a uniform (by a factor of $e^{\pm h_g \cdot t}$) expansion/contraction of the conditional measures of μ_{BM} along unstable/stable foliations of the geodesic flow $\{\Phi^t\}$ on SX.

Paterson-Sullivan measures. Given a point $x \in \tilde{X}$ consider the Poincaré series

$$\Pi(x, s) := \sum_{\gamma\in\Gamma} e^{-s\cdot \mathrm{dist}_{\tilde{g}}(x,\gamma x)} \tag{6}$$

which are readily seen to converge for $s > h_g$ and to diverge for $s < h_g$, and, in our setting, the series *diverges* at the *critical exponent* $s = h_g$. For $x \in \tilde{X}$ the Patterson-Sullivan measure ν_x is constructed as a weak limit, as $s \searrow h_g$, of the probability measures

$$\nu_{x,s} := \frac{1}{\Pi(x, s)} \sum_{\gamma\in\Gamma} e^{-s\cdot \mathrm{dist}_{\tilde{g}}(x,\gamma x)} \mathrm{Dirac}(\gamma x) \tag{7}$$

on the compactification $\bar{X} \subset \tilde{X} \cup \partial\tilde{X}$. Due to the divergence of (6) at $s = h_g$, the weak limits are supported on $\partial\tilde{X}$. In fact, the weak limit ν_x is unique, has no atoms and has $\mathrm{supp}(\nu_x) = \partial\tilde{X}$. All ν_x, $x \in \tilde{X}$, are in the same measure class $[\nu_*]$ and satisfy

$$\frac{d\nu_x}{d\nu_y}(\xi) = e^{-h_g\cdot\beta(x,y,\xi)} \tag{8a}$$

$$\gamma_*\nu_x = \nu_{\gamma x} \tag{8b}$$

for all isometries γ of $(\tilde{X}, \tilde{g})$. The Radon measure m on $\partial^2\tilde{X}$ defined by

$$dm(\xi, \eta) := e^{-h_g\, B_x(\xi,\eta)}\, d\nu_x(\xi)\, d\nu_x(\eta) \tag{9}$$

does not depend on the choice of $x \in \tilde{X}$ (this follows from (8a) and (3b)), and is invariant under the diagonal Γ-action (follows from (8b) and (3a)). Moreover, the Γ-action on $(\partial^2\tilde{X}, m)$ is *ergodic*. Clearly any positive multiple of such a measure shares the same properties.

Geodesic currents, introduced by Bonahon [Bon], are Radon measures on the space $\partial^2\tilde{X} := \{(\xi,\eta) \mid \xi \neq \eta \in \partial\tilde{X}\}$ of pairs of distinct points on the boundary $\partial\tilde{X}$ of $\tilde{X}$, which are invariant for the diagonal Γ-action $\gamma : (\xi,\eta) \mapsto (\gamma\cdot\xi, \gamma\cdot\eta)$ and the *flip* $F : (\xi,\eta) \mapsto (\eta,\xi)$. The space $\partial^2\tilde{X}$ is naturally identified with the space of *geodesic lines* $[\xi,\eta]$ connecting $\xi \neq \eta \in \partial\tilde{X}$. Geodesic currents come about in the following consideration: the geodesic flow $\tilde{\Phi} = \{\tilde{\Phi}^t\}$ on the unit tangent bundle $S\tilde{X}$ of the universal cover $\tilde{X}$ commutes with the Γ-action. The natural $\{\tilde{\Phi}^t\}$-action on the quotient $S\tilde{X}/\Gamma \cong SX$ gives the geodesic flow (SX, Φ^t); while the quotient $S\tilde{X}/\tilde{\Phi}$, being naturally identified with the space of geodesic lines and hence with $\partial^2\tilde{X}$, has a natural Γ-action on $S\tilde{X}/\tilde{\Phi}$ corresponding to the (diagonal) Γ-action on $\partial^2\tilde{X}$.

This amounts to a natural affine correspondence between the cones of (i) Φ-invariant finite measures on SX, (ii) $\tilde{\Phi}\times\Gamma$-invariant Radon measures on $S\tilde{X}$, (iii) Γ-invariant Radon measures on $\partial^2\tilde{X}$. Therefore finite (positive) measures on SX which are invariant under the geodesic flow Φ (and time reversal) stand with one-to-one correspondence with (positive) geodesic currents on $\partial^2\tilde{X}$. In [Ka] Kaimanovich identified the geodesic currents corresponding to the natural geodesic-flow invariant measures on SX (the Lebesgue-Liouville measure, the harmonic measure and the Bowen-Margulis measure), in particular showing that the geodesic current corresponding to the Bowen-Margulis measure μ_{BM} on SX is precisely a positive multiple $C\cdot m$ of the measure m on $\partial^2\tilde{X}$ defined in (9), where the constant C is chosen according to the normalization $\mu_{BM}(SX) = 1$.

Locally symmetric manifolds. If (X,g) is a locally symmetric negatively curved manifold, then $(\tilde{X},\tilde{g})$ is a symmetric space $\mathbb{H}$ $(= \mathbb{H}^n_{\mathbb{R}}, \mathbb{H}^k_{\mathbb{C}}, \mathbb{H}^l_{\mathbb{Q}}$ or $\mathbb{H}^2_{\mathbb{O}})$ of a simple real rank-one Lie group $G = \mathrm{Isom}_+(\tilde{X},\tilde{g})$ $(\cong \mathrm{PO}(n,1), \mathrm{PU}(k,1), \mathrm{PSp}(l,1)$, or $\mathrm{F}^4_{-20})$ and $\Gamma = \pi_1(X)$ is a cocompact torsion free lattice in G. In the constant curvature case $\mathbb{H} = \mathbb{H}^n_{\mathbb{R}}$, $\Gamma \subset G = \mathrm{PO}(n,1)$, and identifying the sphere at infinity $\partial\mathbb{H}^n = S^{n-1}$ with $\mathbb{R}^{n-1}\cup\{\infty\}$ via the stereographic projection, the cross-ratio $[,,,]_g$ above is proportional to the logarithm of the "usual" cross-ratio

$$[x,y,x',y'] := \log\frac{\|x-y\|\cdot\|x'-y'\|}{\|x'-y\|\cdot\|x-y'\|} \qquad (x,y,x',y' \in \mathbb{R}^{n-1})$$

while the measure m on $\partial^2\mathbb{H}^n$ can be written as $dm(x,y) = dx\,dy/\|x-y\|^2$. Moreover the ambient Lie group G can be identified (see Sullivan [Su]) as follows

$$G = \{h \in \mathrm{Homeo}(\partial\mathbb{H}^n) \mid [,,,]_g \circ h = [,,,]_g\} \tag{10a}$$

$$= \{h \in \mathrm{Homeo}(\partial\mathbb{H}^n) \mid (h\times h)_* m = m\} \tag{10b}$$

The latter characterizations remain valid in other locally symmetric spaces.

2.1 Preliminaries on Gromov hyperbolic groups

The ideal boundary $\partial\Gamma$. Gromov hyperbolic group Γ acts by homeomorphisms on its ideal boundary $\partial\Gamma$. Assuming that Γ is non-elementary and torsion free, the Γ-action is faithful and every $\gamma\in\Gamma$ has two fixed points $\gamma_-\neq\gamma_+$ on $\partial\Gamma$ so that: $\{\gamma^n, n\to\infty\}$ contract $\partial\Gamma\setminus\{\gamma_-\}$ to γ_+, and $\{\gamma^n, n\to-\infty\}$ contract $\partial\Gamma\setminus\{\gamma_+\}$ to γ_-, uniformly on compact sets. These general properties can be formulated and proved without any reference to a specific metric $d\in D_\Gamma$ or a class $\delta\in\mathcal{D}_\Gamma$. For a positive integer p denote by $\partial^p\Gamma\subset(\partial\Gamma)^p$ the set of *distinct* p-tuples of points on the boundary $\partial\Gamma$, which is always a locally compact space, but not compact one unless $p=1$. The Γ-action on $\partial^1\Gamma=\partial\Gamma$ is *minimal*; the (diagonal) Γ-action on $\partial^2\Gamma$ is *topologically transitive*; and the Γ-action on $\partial^3\Gamma$ is *proper* and *cocompact.* In fact, there exist Γ-equivariant proper maps $\pi:\partial^3\Gamma\to\Gamma$ (equivariance implies that such a map is surjective).

Gromov product and Gromov metric. Fix a $d\in D_\Gamma$, so that (Γ,d) is a Gromov-hyperbolic space with a transitive isometric action of Γ. Recall the notion of the Gromov product $(x\mid y)_g$ on (Γ,d) relative to some $g\in\Gamma$:

$$(x\mid y)_g:=\frac{1}{2}\,\left(d(x,g)+d(y,g)-d(x,y)\right) \tag{11}$$

which extends to the boundary $\partial\Gamma$ by

$$(\xi\mid\eta)_g:=\inf\{\liminf_{i\to\infty}(x_i\mid y_i)_g\mid x_i\to\xi,\ y_i\to\eta\}$$

In [Gr] Gromov constructs a family ρ_g^ϵ of metrics on $\partial\Gamma$, where ϵ belongs to some non-trivial range $(0,\epsilon_0)$, so that for each $\epsilon\in(0,\epsilon_0)$ the metric ρ_g^ϵ induces the "standard" compact topology on $\partial\Gamma$. The metrics ρ_g^ϵ and $\rho_{\gamma g}^\epsilon$ are related by $\rho_{\gamma g}^\epsilon(\gamma\cdot\xi,\gamma\cdot\eta)=\rho_g^\epsilon(\xi,\eta)$, so it suffices to understand ρ_e^ϵ, and it turns out that

$$C^{-1}\,e^{-\epsilon\cdot(\xi\mid\eta)_e}\leq\rho_e^\epsilon(\xi,\eta)\leq C\,e^{-\epsilon\cdot(\xi\mid\eta)_e} \tag{12}$$

for some constant $C=C(\epsilon)$.

Patterson-Sullivan measures. A version of Patterson-Sullivan theory for Gromov hyperbolic groups was carried out by Coornaert [Co]. Let us summarize some of his results which are needed below.

The Poincaré series $\Pi_d(s):=\sum_{\gamma\in\Gamma}e^{-s\cdot d(\gamma,e)}$ converge for all $s>h_d$ and diverge for all $s<h_d$, where the *critical exponent* h_d is given by the *growth*

$$h_d:=\lim_{R\to\infty}\frac{1}{R}\log\#\{\gamma\in\Gamma\mid d(\gamma,e)\leq R\} \tag{13}$$

In fact, Coornaert shows that there exist constants $0 < C_1 \leq C_2 < \infty$ so that

$$C_1 \, e^{h_\delta \, R} \leq \#\{\gamma \in \Gamma \mid d(\gamma, e) \leq R\} \leq C_2 \, e^{h_\delta \, R} \tag{14}$$

which in particular implies that $\Pi_d(h_d) = \infty$. Denote by $PS(\Gamma, d)$ the set of all weak limits, as $s \searrow h_\delta$, of the probability measures

$$\nu_s := \frac{1}{\Pi_d(s)} \sum_{\gamma \in \Gamma} e^{-s \cdot d(\gamma, e)} \, \mathrm{Dirac}(\gamma)$$

on $\bar{\Gamma} = \Gamma \cup \partial\Gamma$. Due to the divergence of the Poincaré series at the critical exponent h_d, the limit measures $\nu \in PS(\Gamma, d)$ are supported on $\partial\Gamma$. Although in this general setup there is no claim of uniqueness for the limit measure $\nu \in PS(\Gamma, d)$, all such limit measures are in the same measure class $[\nu_*]$ and have bounded Radon-Nikodym derivatives with respect to each other. Furthermore, every $\nu \in PS(\Gamma, d)$ has no atoms and has full support on $\partial\Gamma$. For $\nu \in PS(\Gamma, d)$, the measure m_ν on $\partial^2\Gamma$ defined by

$$dm_\nu(\xi, \eta) = e^{2h_\delta \cdot (\xi|\eta)_e} \, d\nu(\xi) \, d\nu(\eta) \tag{15}$$

is locally finite and quasi-invariant for the diagonal Γ-action. In fact, the Radon-Nikodym derivatives $\frac{d(\gamma \times \gamma)_*^{-1} m_\nu}{dm_\nu}$ are uniformly bounded. The diagonal Γ-action on $(\partial^2\Gamma, m_\nu)$ is *ergodic*.

3 Basic Notions of the Coarse-geometric setup

Let Γ be a Gromov hyperbolic group and $\delta \in \mathcal{D}_\Gamma$ be a class of left-invariant metrics as defined in the introduction. Fix some metric $d \in D_\Gamma$ from $\delta \in \mathcal{D}_\Gamma$. Noting that the growth h_d does not really depend on d but only on δ, hereafter we shall denote it by h_δ. Associated with δ there is also the *stable length* function $\ell_\delta : \mathcal{C} \to \mathbb{R}_+$ defined by

$$\ell_\delta(c_\gamma) := \lim_{n \to \infty} \frac{d(\gamma^n, e)}{n} \tag{16}$$

Observe that $\ell_\delta : \mathcal{C}_\Gamma \to \mathbb{R}_+$ is well defined, i.e. the limits exist and are independent of the choices of $d \in \delta$ and $\gamma \in c_\gamma$. Moreover, ℓ_δ is homogeneous in $c \in \mathcal{C}_\Gamma$, namely $\ell_\delta(c_{\gamma^n}) = |n| \cdot \ell_\delta(c_\gamma)$ so that ℓ_δ is determined by its values on the subset $\mathcal{C}'_\Gamma \subset \mathcal{C}_\Gamma$ of the indivisible classes.

Observe that ℓ_δ can be considered as a generalization of the marked length spectrum, because in the geometric case of compact negatively curved manifold (X, g) and $\Gamma = \pi_1(X)$ the marked length spectrum can also be computed as

$$\ell_g(c_\gamma) = \lim_{n \to \infty} \frac{\mathrm{dist}_{\tilde{g}}(\gamma^n x, x)}{n}$$

Proposition 1 *To every $\bar{\delta} \in \mathcal{PD}_\Gamma$ one can canonically associate a geodesic current, i.e. Γ- and flip-invariant locally finite measure, $m_{\bar{\delta}}$ on $\partial^2\Gamma$, of the form*

$$dm_{\bar{\delta}}(\xi,\eta) = e^{2h_\delta \cdot f_\nu(\xi,\eta)}\, d\nu(\xi)\, d\nu(\eta) \tag{17}$$

where $\nu \in \mathcal{P}(\partial\Gamma)$ is a Patterson-Sullivan measure corresponding to some $d \in \delta$ and $f_\nu : \partial^2\Gamma \to \mathbb{R}_+$ is a symmetric Borel function, uniquely defined up to sets of $\nu \times \nu$-measure zero. Moreover such measurable function f_ν is within bounded distance from the functions $(\xi \mid \eta)_e$ and $\epsilon^{-1} \cdot \log \rho_e^\epsilon(\xi,\eta)$.

Note that the ergodic geodesic current $m_{\bar{\delta}}$ in the Proposition depends only on $\bar{\delta}$ and not on the choices of $d \in \delta$ or $\nu \in PS(\Gamma, d)$. In the geometric context where $\Gamma = \pi_1(X)$ is a fundamental group of a compact negatively curved manifold (X, g) with $d = d_{g,x}$ and $\nu = \nu_x$ being *the* Patterson-Sullivan measure, one has $f_\nu = B_x(\xi,\eta) = 2\,(\xi \mid \eta)_e$ and the geodesic current $m_{\bar{\delta}_g}$ on $\partial^2\Gamma \cong \partial^2\tilde{X}$ corresponds precisely to the Bowen-Margulis measure μ_{BM} on SX (including the normalization, which is explained below). Hence for general $\bar{\delta} \in \mathcal{PD}_\Gamma$ we shall refer to $m_{\bar{\delta}}$ as a (generalized) *Bowen-Margulis geodesic current* associated to $\bar{\delta} \in \mathcal{PD}_\Gamma$.

Proof of Proposition 1. Fix a $\delta \in \mathcal{D}_\Gamma$ and choose a representative $d \in D_\Gamma$ of δ and a Patterson-Sullivan measure $\nu \in PS(\Gamma, d)$. Since ν has no atoms, the measure class $[\nu \times \nu]$ on $\partial^2\Gamma$ can be considered on $\partial\Gamma \times \partial\Gamma$. This measure class contains an *invariant* measure for the diagonal Γ-action iff there is a $[\nu \times \nu]$-measurable solution $f_\nu(\xi,\eta)$ to the cohomological equation:

$$2h_\delta \cdot [f_\nu(\gamma \cdot \xi, \gamma \cdot \eta) - f_\nu(\xi,\eta)] = \sigma_\nu(\gamma,\xi) + \sigma_\nu(\gamma,\eta) \tag{18}$$

where $\sigma_\nu(\gamma,\xi) = \log \frac{d\gamma^{-1}\nu}{d\nu}(\xi)$ is the logarithmic Radon-Nikodym cocycle. Recall that Coornaert shows that $f_0(\xi,\eta) = (\xi \mid \eta)_e$ "almost" satisfies (18) in the sense that the cocycle

$$\Delta(\gamma,(\xi,\eta)) := \sigma_\nu(\gamma,\xi) + \sigma_\nu(\gamma,\eta) - 2h_\delta \cdot [f_0(\gamma \cdot \xi, \gamma \cdot \eta) - f_0(\xi,\eta)]$$

is bounded in $L^\infty([\nu \times \nu])$ *uniformly* over Γ. It is a general fact that every *uniformly bounded* measurable cocycle $\Delta : \Gamma \times \Omega \to \mathbb{R}$ over any group action (Ω, Γ) with a quasi-invariant measure is an L^∞-*coboundary.* Indeed taking

$$\phi(\omega) := \sup_{\gamma' \in \Gamma} \Delta(\gamma', \omega)$$

one verifies that $\Delta(\gamma,\omega) = \phi(\omega) - \phi(\gamma \cdot \omega)$ using the cocycle equation

$$\Delta(\gamma'\gamma, \omega) = \Delta(\gamma', \gamma \cdot \omega) + \Delta(\gamma, \omega)$$

Hence, the cohomological equation (18) has a solution f_ν which is within bounded L^∞ distance from $f_0(\xi,\eta) = (\xi \mid \eta)_e$, which in turn is within bounded distance from $\epsilon^{-1} \cdot \log \rho_e^\epsilon(\xi, \eta)$ by (12). This proves the existence of Γ-invariant measures m in the class $[\nu \times \nu]$. Due to the ergodicity such a solution f_ν is unique up to an additive constant, and therefore all such measures m are positively proportional i.e. form a ray $\overline{m} := \{\lambda \cdot m\}_{\lambda > 0}$. Observe that the choices of the scaling for δ in $\bar{\delta}$ and of the representatives $d \in \delta$ and $\nu \in PS(\Gamma, d)$ have no effect on the measure classes $[\nu]$ and $[\nu \times \nu]$, so that the ray $\overline{m}$ of Γ-invariant measures in the class $[\nu \times \nu]$ is independent of these choices. In the proposition we have singled out a particular measure $m_{\bar{\delta}} \in \overline{m}$. This is done by an appropriate normalization which will be explained below (the normalization is not entirely obvious because $\overline{m}$ consists of infinite measures). □

The abstract Geodesic Flow of $\delta \in \mathcal{D}_\Gamma$. Let $m \in \bar{m}$ be one of the measures constructed above. It is of the form $dm = e^{2h_\delta \cdot f_\nu}\, \nu \times \nu$. Consider the space $\tilde{Y} = \partial^2 \Gamma \times \mathbb{R}$ equipped with the locally finite measure $d\tilde{m} = dm(\xi, \eta)\, dt$ and let Γ-act on $(\tilde{Y}, \tilde{m})$ by

$$\gamma : (\xi, \eta, t) \mapsto \left(\gamma \cdot \xi,\, \gamma \cdot \eta,\, t + \frac{1}{h_\delta}\, \sigma_\nu(\gamma, \xi) \right)$$

where $\sigma_\nu(\gamma, \xi) = \log(d\gamma^{-1}\nu / d\nu(\xi))$ is the logarithmic Radon-Nikodym cocycle of ν. This is a measure-preserving Γ-action, which commutes with the following $\mathbb{R}$-flow $\tilde{\Phi} = \{\tilde{\Phi}^t\}$ acting only in the $\mathbb{R}$-coordinate

$$\tilde{\Phi}^s \;:\; (\xi, \eta, t) \mapsto (\xi, \eta, t + s)$$

Observe that although the choice of ν and hence that of σ_ν needed for the definition of the Γ-action on $\tilde{Y}$ are not canonical, the measure class $[\nu]$ is uniquely determined (by $\bar{\delta}$). Therefore any other measure ν' in the class $[\nu]$ would define a *cohomologous* Radon-Nikodym cocycle

$$\sigma_{\nu'}(\gamma, \xi) = \sigma_\nu(\gamma, \xi) + \phi(\gamma \cdot \xi) - \phi(\xi)$$

where $\phi = \log(d\nu'/d\nu)$, and the Γ-action on $\tilde{Y}$ defined using $\sigma_{\nu'}$ would be measurably isomorphic to the one for σ_ν by the isomorphism $(\xi, \eta, t) \mapsto (\xi, \eta, t + \phi(\xi))$, which commutes with $\tilde{\Phi}$. Therefore, in the *measure-theoretical* sense the $\Gamma \times \tilde{\Phi}$-actions on $(\tilde{Y}, \tilde{m})$ are canonically defined.

The coarse-geometric interpretation of σ_ν being (roughly) a h_δ-multiple of a "Busemann function" β (in the general coarse-geometric context the latter can be defined only *roughly*) allows to show that the Γ-action on $\tilde{Y}$ admits a *bounded* fundamental domain $Y \subset \tilde{Y}$. Therefore $0 < \tilde{m}(Y) < \infty$ and $\tilde{m}$, and hence m on $\partial^2 \Gamma$, can be canonically normalized by $\tilde{m}(Y) = 1$. It is this normalization of $m \in \bar{m}$ which is denoted by $m_{\bar{\delta}}$ in Proposition 1.

With thus normalized $m_{\bar{\delta}}$ on $\partial^2 \Gamma$ (and $\bar{m}_{\bar{\delta}}$ on $\tilde{Y}$) one can define (measure-theoretically) the "geodesic flow" $\Phi = \{\Phi^t\}$ on Y as the quotient of the

$\tilde{\Phi}$-action on $\tilde{Y}$ by Γ, so that the restriction μ_δ of $\tilde{m}_{\bar{\delta}}$ to Y becomes a Φ-invariant *probability* measure on Y. In fact, (Y, m_δ, Φ^t) is an *ergodic* flow, because $m_{\bar{\delta}}$ was Γ-ergodic, and moreover one can show that the metric entropy $h(Y, \mu_\delta, \Phi^1)$ of this flow is precisely h_δ. (Note that μ_δ depends on h_δ and is therefore scale dependent, although $m_{\bar{\delta}}$ is not).

In the geometric context, where $\Gamma = \pi_1(X)$ and $\delta = [d_{g,x}]$ arise from a negatively curved Riemannian structure g on X, one has by (8a) $\sigma_\nu(\gamma, \xi) = h_\delta \cdot \beta(x, \gamma x, \xi)$ and (Y, μ_δ, Φ^t) is naturally (measurably) isomorphic to the usual geodesic flow (SX, μ_{BM}, Φ^t) with the Bowen-Margulis measure μ_{BM}.

Let us also point out, that the geodesic current $m_{\bar{\delta}}$ admits an alternative construction, analogous to the Bowen's definition of μ_{BM}, namely as the weak limit $m_{\bar{\delta}} = \lim_{t\to\infty} m_t$ of the atomic geodesic currents

$$m_t := \left(\sum_{\ell_\delta(c_\gamma)<t} \ell_\delta(c_\gamma) \right)^{-1} \cdot \sum_{\{\gamma\in\Gamma \mid \ell_\delta(c_\gamma)<t\}} \ell_\delta(c_\gamma) \cdot \mathrm{Dirac}(\gamma_-, \gamma_+) \qquad (19)$$

which precisely correspond to the closed geodesics in the geometric context of Bowen's construction. We shall not pursue this analogy here, however.

The Cross-Ratio. Let us generalize the notion of the (symplectic) cross-ratio to the coarse-geometric setup as follows. Given a hyperbolic group Γ and $\delta \in \mathcal{D}_\Gamma$ write the associated Bowen-Margulis geodesic current $m_{\bar{\delta}}$ in the form (17) and define the *cross-ratio* to be the measurable function $[,,,]_\delta : \partial^4\Gamma \to \mathbb{R}_+$

$$[\xi, \eta, \xi', \eta']_\delta := f_\nu(\xi, \eta) + f_\nu(\xi', \eta') - f_\nu(\xi, \eta') - f_\nu(\xi', \eta) \qquad (20)$$

First let us check that the definition depends only on δ and not on ν. If ν' is a measure on $\partial\Gamma$ so that $dm_{\bar{\delta}}(\xi, \eta) = \exp(2h_\delta \cdot f_{\nu'}(\xi, \eta))\, d\nu'(\xi)\, d\nu'(\eta)$ then ν' and ν are in the same measure class and $f_{\nu'}$ is a measurable solution to the cohomological equation

$$\begin{aligned} 2h_\delta\ [\ f_{\nu'}(\gamma\cdot\xi, \gamma\cdot\eta) - f_{\nu'}(\xi, \eta)] &= \sigma_{\nu'}(\gamma, \xi) + \sigma_{\nu'}(\gamma, \eta) \\ &= \sigma_\nu(\gamma, \xi) + \sigma_\nu(\gamma, \eta) + \phi(\gamma\cdot\xi) - \phi(\xi) + \phi(\gamma\cdot\eta) - \phi(\eta) \\ &= 2h_\delta \cdot [f_\nu(\gamma\cdot\xi, \gamma\cdot\eta) + \phi(\gamma\cdot\xi) + \phi(\gamma\cdot\eta) \\ &\qquad - f_\nu(\xi, \eta) - \phi(\xi) - \phi(\eta)] \end{aligned}$$

where $\phi = \log(d\nu'/d\nu)$. By ergodicity such a solution $f_{\nu'}$ is unique up to a constant, and one deduces that $\nu \times \nu$-a.e.

$$f_{\nu'}(\xi, \eta) = f_\nu(\xi, \eta) + \phi(\xi) + \phi(\eta) + C.$$

Now one easily verifies that the substitution of $f_{\nu'}$ instead of f_ν in (20) does not change the value of the expression. Therefore the cross-ratio $[,,,]_\delta : \partial^4\Gamma \to \mathbb{R}$ is well defined, up to ν^4-null sets. From Proposition 1 the cross-ratio $[\xi, \eta, \xi', \eta']_\delta$ stays within bounded distance from

$$(\xi \mid \eta)_e + (\xi' \mid \eta')_e - (\xi' \mid \eta)_e - (\xi \mid \eta')_e$$

One can also prove the standard identities for the cross-ration $[,,,]_\delta$, and verify that $[,,,]_\delta$ is Γ-invariant.

Let us point out that in the geometric context of a compact negatively curved manifold (X,g) or, more generally, in the case of δ arising from an isometric convex cocompact Γ-action on a CAT(-1) space, the cross-ratio is a continuous function on $\partial^4\Gamma$, since it can be defined using the Busemann function $B_x(\xi,\eta)$ which is *continuous*. However in the general situation above, our definition of the cross-ratio gives only a measurable function.

4 Some Results

Theorem 2 *For Γ and $\delta,\delta'\in\mathcal{D}_\Gamma$ the following are equivalent:*

(a) *$m_{\bar\delta}$ and $m_{\bar\delta'}$ are not mutually singular.*
(b) *$m_{\bar\delta}=m_{\bar\delta'}$.*
(c) *$h_\delta\cdot\ell_\delta(c_\gamma)=h_{\delta'}\cdot\ell_{\delta'}(c_\gamma)$ for $\gamma\in\Gamma$.*
(d) *$h_\delta\cdot\delta=h_{\delta'}\cdot\delta'$ in $\mathcal{D}_\Gamma$.*
(e) *$\bar\delta=\bar\delta'$ in $\mathcal{PD}_\Gamma$.*

Note that in the geometric setup, where X is a compact manifold, g and g' are two negatively curved Riemannian structures on X, $\Gamma=\pi_1(X)$ and $\delta=\delta_g$, $\delta'=\delta_{g'}$ in $\mathcal{D}_\Gamma$, the Theorem asserts that the condition $\ell_g=\ell_{g'}$ (which by [Ot2] Thm 2.2 is equivalent to $[,,,]_g=[,,,]_{g'}$) is also equivalent to: $\mathrm{dist}_{\tilde g}-\mathrm{dist}_{\tilde g'}$ being bounded on $\tilde X$. Hence the following is a reformulation of the Marked Length Spectrum Rigidity Conjecture:

For a compact manifold X with negatively curved Riemannian metrics g and g' the universal covers $(\tilde X,\tilde g)$ and $(\tilde X,\tilde g')$ are roughly isometric if and only if they are isometric.

Proof of Theorem 2. The implications (e) $\Leftrightarrow$ (d) $\Rightarrow$ (c) are straightforward, the equivalence (a) $\Leftrightarrow$ (b) follows from the ergodicity of the corresponding "geodesic flows", while (c) $\Rightarrow$ (b) follows from (19). Here let us show that (b) $\Rightarrow$ (d). After a rescaling we can assume that $h_\delta=h_{\delta'}=1$, which together with $m_{\bar\delta}=m_{\bar\delta'}$ yields $[\xi_1,\eta_1,\xi_2,\eta_2]_\delta=[\xi_1,\eta_1,\xi_2,\eta_2]_{\delta'}$. Now choose metrics $d\in\delta$ and $d'\in\delta'$ and let $(\cdot\mid\cdot)_e$ and $(\cdot\mid\cdot)'_e$ denote the associated Gromov products. Since $[,,,]_\delta$ and $[,,,]_{\delta'}$ can be uniformly approximated by the four term expressions in the corresponding Gromov products we conclude that for a.e. ξ,ξ_0,η,η_0 the difference

$$\begin{gathered}\left[(\xi\mid\eta)_e-(\xi\mid\eta)'_e\right]+\\+\left[(\xi_0\mid\eta_0)_e-(\xi_0\mid\eta_0)'_e\right]-\left[(\xi_0\mid\eta)_e-(\xi_0\mid\eta)'_e\right]-\left[(\xi\mid\eta_0)_e-(\xi\mid\eta_0)'_e\right]\end{gathered}$$

is uniformly bounded. Fix $\xi_0\neq\eta_0$ and focus on ξ, η away from some fixed small neighborhoods of ξ_0 and η_0. Then the three difference, except the one in the first line above, are bounded, and therefore $\left|(\xi\mid\eta)_e-(\xi\mid\eta)'_e\right|$ is bounded for such ξ, η. Applying this argument to another pair $\xi_0 \neq \eta_0$ which has "excluded" neighborhoods which do not overlap with those of the previous ξ_0, η_0, we conclude that there is C so that for all $\xi \neq \eta$

$$\left|(\xi \mid \eta)_e - (\xi \mid \eta)'_e\right| \leq C \tag{21}$$

Now recall that there exist Γ-equivariant (hence onto) proper maps $\pi : \partial^3\Gamma \to \Gamma$. We claim that on $\partial^3\Gamma$ the difference

$$d(e, \pi(\xi_1, \xi_2, \xi_3)) \; - \max_{i \neq j \in \{1,2,3\}} (\xi_i \mid \xi_j)_e \tag{22}$$

is uniformly bounded (look at the case of a tree as a convincing example). Hence the bound (21) on Gromov-products together with (22) imply that $d(e, \gamma) - d'(e, \gamma)$ is bounded. Since d and d' are left invariant we deduce that $d - d'$ is bounded and $\delta = \delta'$. □

Theorem 3 *Given Γ, $\bar{\delta} \in \mathcal{PD}_\Gamma$ and $m_{\bar{\delta}}$ on $\partial^2\Gamma$ as above, let $H_{\bar{\delta}}$ denote the topological group*

$$H_{\bar{\delta}} \; := \; \{ h \in \mathrm{Homeo}\,(\partial\Gamma) \,|\, (h \times h)_* m_{\bar{\delta}} = m_{\bar{\delta}} \}$$

with the open-compact topology induced from Homeo$\,(\partial\Gamma)$. *Then $H_{\bar{\delta}}$ is a locally compact group which contains Γ as a cocompact lattice.*

Remark 4 Fix a non-elementary torsion free Gromov hyperbolic group Γ, and consider the collection $\mathcal{H}_\Gamma$ of the locally compact groups $H_{\bar{\delta}}$ and the cocompact lattice embeddings $j_{\bar{\delta}} : \Gamma \hookrightarrow H_{\bar{\delta}}$, for all $\bar{\delta} \in \mathcal{PD}_\Gamma$. It can be shown that $\mathcal{H}_\Gamma$ is the *universal* object for Γ in the following sense: given any embedding $i : \Gamma \hookrightarrow L$ of Γ in a locally compact group L as a cocompact lattice, there exists $\bar{\delta} \in \mathcal{PD}_\Gamma$ and a continuous homomorphism $p : L \to H_{\bar{\delta}}$, so that $\mathrm{Ker}\, p$ is a compact normal subgroup in L disjoint from $i(\Gamma)$, and $p(L) \subseteq H_{\bar{\delta}}$ is a closed cocompact subgroup containing $j_{\bar{\delta}}(\Gamma)$, and furthermore $p \circ i = j_{\bar{\delta}}$. This can be shown by choosing a left-invariant proper quasi-metric d on L and restricting it to Γ to form $d \in D_\Gamma$ and $\delta = [d]$. Then one can apply the arguments from [Fu1] producing a Γ-equivariant continuous homomorphism $p : L \to \mathrm{Homeo}\,(\partial\Gamma)$ with locally compact image $p(L)$ and compact kernel, and verifying that $p(L)$ preserves $m_{\bar{\delta}}$.

Examples 5 (i) Let (X, g_0) be a *locally symmetric* negatively curved manifold, so that $(\tilde{X}, \tilde{g}_0)$ is a symmetric space $\mathbb{H}$, $G = \mathrm{Isom}_+(\mathbb{H})$ is a simple connected real Lie group and $\bar{G} = \mathrm{Isom}\,(\mathbb{H})$ contains G as a subgroup of index two. Then $\Gamma = \pi_1(X)$ is a cocompact lattice in G acting on $\mathbb{H}$ $(X = \mathbb{H}/\Gamma)$. Let $\delta_0 = [d_{x,g_0}] \in \mathcal{D}_\Gamma$ be the corresponding class of metrics on Γ. Then there is a natural isomorphism between $H_{\bar{\delta}_0}$ and $\bar{G}$ under which $\Gamma \subset H_{\bar{\delta}_0}$ corresponds to the embedding of Γ in $G \subset \bar{G}$.

(ii) Let $\Gamma = F_k$ be a free group on $1 < k < \infty$ free generators $\gamma_1, \ldots, \gamma_k$, and $w = (w_1, \ldots, w_k)$ be some positive numbers. Denote by d_w the weighted left invariant word metric on Γ, defined by $d_w(\gamma, \gamma\gamma_i^{\pm 1}) = w_i$, and denote $\delta_w = [d_w] \in \mathcal{D}_\Gamma$. Then the group $H_{\bar{\delta}_w}$ can be naturally identified with the group $\mathrm{Isom}\,(T_w)$ of the isometries of the metric tree T_w which is the Cayley graph of F_k where the edges $(\gamma, \gamma\gamma_i^{\pm 1})$ have length w_i. Observe that $\mathrm{Isom}\,(T_w)$ is a totally disconnected locally compact group containing F_k as a cocompact lattice (since F_k acts transitively on T_w the quotient $\mathrm{Isom}\,(T_w)/F_k$ can be identified with the stabilizer of a vertex $o \in T_w$ which is a compact group). If $w_1 = w_2 = \ldots = w_k$ then $H_{\bar{\delta}_w}$ is $\mathrm{Aut}\,(T)$ - automorphism group of the $2k$-regular tree. The classification of locally compact groups containing F_k as a cocompact lattice obtained by Mosher-Sageev-Whyte [MSW] basically shows that that the above examples of $H_{\bar{\delta}_w}$ describe all possible $H_{\bar{\delta}}$ for the free group $\Gamma = F_k$.

Proof of Theorem 3. Since $h \in H_{\bar{\delta}}$ preserve $m_{\bar{\delta}} = e^{2h_\delta \cdot f_\nu}\, \nu \times \nu$ the measure ν is $H_{\bar{\delta}}$-quasi-invariant and the logarithmic Radon-Nikodym cocycle $\sigma_\nu : H_{\bar{\delta}} \times (\partial\Gamma, \nu) \to \mathbb{R}$ defined by $\sigma_\nu(h, \xi) = \log \frac{dh^{-1}\nu}{d\nu}(\xi)$ satisfies the cohomological equation

$$2h_\delta \cdot [f_\nu(h \cdot \xi, h \cdot \eta) - f_\nu(\xi, \eta)] = \sigma_\nu(h, \xi) + \sigma_\nu(h, \eta) \tag{23}$$

Recall that the functions $f_\nu(\xi, \eta)$, $(\xi \mid \eta)_e$ and $\epsilon^{-1} \cdot \log \rho_e^\epsilon(\xi, \eta)$ differ by at most bounded amount, which gives the estimate

$$\frac{\rho_e^\epsilon(h \cdot \xi, h \cdot \eta)}{\rho_e^\epsilon(\xi, \eta)} \asymp \left(e^{\sigma_\nu(h,\xi) + \sigma_\nu(h,\eta)}\right)^{\epsilon/2h_\delta} \tag{24}$$

for $\nu \times \nu$-a.e. (ξ, η). Thus if $h \in H_{\bar{\delta}}$ has a finite norm $\|h\|_\sigma := \|\sigma_\nu(h, \cdot)\|_\infty$ it is (a ν-a.e. and hence everywhere by continuity) Lipschitz map of $(\partial\Gamma, \rho_e^\epsilon)$. Moreover every subset $V \subset H_{\bar{\delta}}$ with $\sup_{h \in V} \|h\|_\sigma < \infty$ has a uniformly bounded Lipschitz norm and is therefore precompact in $\mathrm{Homeo}\,(\partial\Gamma)$ by Arzela-Ascoli theorem. We claim that for $r > 0$ sufficiently small, the neighborhood

$$V_r := \{h \in H_{\bar{\delta}} \mid \forall \xi \in \partial\Gamma : \ \rho_e^\epsilon(h \cdot \xi, \xi) < r\}$$

of the identity in $H_{\bar{\delta}}$ satisfies $\sup_{h \in V_r} \|h\|_\sigma < \infty$. By the previous argument this would imply that V_r is precompact, and as $H_{\bar{\delta}}$ is a closed subgroup of the complete group $\mathrm{Homeo}\,(\partial\Gamma)$ it would follow that $H_{\bar{\delta}}$ is *locally compact.* For $t > 0$ let $E_t := \{(\xi, \eta) \in \partial^2\Gamma \mid \rho_e^\epsilon(\xi, \eta) > t\}$. Observe that $(\xi \mid \eta)_e$ and hence $f_\nu(\xi, \eta)$ is bounded, say by $c_1(t)$, on E_t. The triangle inequality implies that $h \in V_r$ map E_{3r} into E_r, which gives the bound

$$|f_\nu(h \cdot \xi, h \cdot \eta) - f_\nu(\xi, \eta)| \leq c_2(r) := c_1(3r) + c_1(r)$$

for $\nu \times \nu$-a.e. $(\xi, \eta) \in E_{3r}$. In view of (23) for $h \in V_r$ we have

$$|\sigma_\nu(h, \xi) + \sigma_\nu(h, \eta)| \leq c_3(r) := 2h_\delta c_2(r) \tag{25}$$

$\nu \times \nu$-a.e. on E_{3r}. For $\xi \in \partial\Gamma$ and $t > 0$ denote $A_t(\xi) = \{\eta \mid \rho_e^\epsilon(\xi,\eta) > t\}$. Since ν has no atoms and has full support on $\partial\Gamma$, the function $a(\xi) := \nu(A_{3r}(\xi))$ is continuous and positive, so that $\alpha := \min_\xi a(\xi) > 0$. Hence for ν-a.e. ξ

$$\frac{1}{\alpha} \geq \frac{1}{\nu(A_{3r}(\xi))} \geq \frac{\nu(hA_{3r}(\xi))}{\nu(A_{3r}(\xi))} \geq \frac{1}{\nu(A_{3r}(\xi))} \int_{A_{3r}(\xi)} e^{\sigma_\nu(h,\eta)}\, d\nu(\eta)$$
$$\geq \inf_{\eta \in A_{3r}(\xi)} e^{\sigma_\nu(h,\eta)} \geq e^{-c_3(r)-\sigma_\nu(h,\xi)}$$

This shows that for $h \in V_r$ and ν-a.e. $\xi \in \partial\Gamma$ one has $\sigma_\nu(h,\cdot) \geq \log\alpha - c_3(r)$ which, combined with (25), gives $\sup_{h\in V_r} \|h\|_\sigma < \infty$. This proves that $H_{\bar\delta}$ is a locally compact group.

Finally, since Γ is a discrete subgroup of $\mathrm{Homeo}\,(\partial\Gamma)$ which preserves $m_{\bar\delta}$, it forms a discrete subgroup of $H_{\bar\delta}$, and it remains to show that $H_{\bar\delta}/\Gamma$ is compact. Recalling that the Γ-action on $\partial^3\Gamma$ is *cocompact* (and proper as well) it suffices to verify that the $H_{\bar\delta}$-action on $\partial^3\Gamma$ is *proper*. This readily follows from the already established characterization: $h_n \to \infty$ in $H_{\bar\delta}$ if (and only if) $\|h_n\|_\sigma \to \infty$, and the relation (24). □

Examples 5 and Theorem 3 suggest interesting lattice embeddings of hyperbolic groups in locally compact groups of the form $H_{\bar\delta}$. However it seems that such examples are rare and, moreover can be used to single out rank one lattices Γ (i.e. fundamental groups of locally symmetric negatively curved manifolds) among other groups, and among all possible left invariant metrics $\bar\delta \in \mathcal{D}_\Gamma$ on such Γ to single out the metrics coming from the corresponding symmetric space. This is maid precise in the following Theorems 6 and 7.

Let X be a compact manifold which admits a *locally symmetric* negatively curved Riemannian metric g_0. Then the universal Riemannian cover $(\tilde X, \tilde g_0)$ is a symmetric rank-one space $\mathbb{H}$ and the natural $\Gamma = \pi_1(X)$ action on $\mathbb{H}$ gives rise to an embedding $\Gamma \subset G = \mathrm{Isom}_+(\mathbb{H})$. The embedding depends on the choice of a base point (a change of the base point amounts to a conjugation of Γ in G) but does not effect the class $\delta_{g_0} \in \mathcal{D}_\Gamma$. Furthermore the projective class $\bar\delta_{g_0} \in \mathcal{PD}_\Gamma$ is not sensitive to the scaling of the locally symmetric metric g_0 on X. If $\dim X > 2$ then by Mostow rigidity there is a unique, up to conjugation, embedding $\Gamma \subset G$ and therefore a unique $\bar\delta_{g_0} \in \mathcal{PD}_\Gamma$; while in the case of surfaces $G = \mathrm{PSL}_2(\mathbb{R})$ and there is a moduli space of "symmetric" $\bar\delta_{g_0} \in \mathcal{PD}_\Gamma$ on the surface group Γ.

Theorem 6 *Let $\Gamma = \pi_1(X)$ be a fundamental group of a compact manifold X which admits a locally symmetric negatively curved Riemannian metric. Then for $\bar\delta \in \mathcal{PD}_\Gamma$ the associated group $H_{\bar\delta}$ is discrete and contains Γ as a finite index subgroup, unless $\bar\delta = \bar\delta_{g_0}$ where g_0 is a locally symmetric Riemannian metric on X. In the latter case there is an isomorphism $H_{\bar\delta} \cong G = \mathrm{Isom}\,(\tilde X, \tilde g_0)$ which is compatible with the embeddings of Γ in $H_{\bar\delta}$ and in $\mathrm{Isom}\,(\tilde X, \tilde g_0)$. Furthermore, for any Riemannian metric g on X the* *associated* $\bar{\delta}_g \in \mathcal{PD}_\Gamma$ *has non-discrete* $H_{\bar{\delta}_g}$ *iff the metric* g *is locally symmetric.*

Hence *non-triviality* of $H_{\bar{\delta}}$ (in the sense of not being non-discrete) characterizes locally symmetric metrics on X in $\mathcal{PD}_\Gamma$.

Theorem 6 will be deduced from the following

Theorem 7 *Let* Γ *be a Gromov hyperbolic group,* $\bar{\delta} \in \mathcal{PD}_\Gamma$ *and* $H_{\bar{\delta}}$ *be as in Theorem 3. Assume condition (*) below. Then one of the following two alternatives holds:*

(A) *either* $H_{\bar{\delta}}$ *is a discrete group containing* Γ *as a finite index subgroup, or*
(B) Γ *is a torsion free cocompact lattice in a rank-one group* $\bar{G} = \mathrm{Isom}\,(\mathbb{H})$ *where* $\mathbb{H}$ *is a negatively curved symmetric space,* $H_{\bar{\delta}} \cong \bar{G}$ *and* $\bar{\delta} = \bar{\delta}_{g_0}$ *corresponds to a* $\bar{G}$*-invariant metric on the symmetric space* $\mathbb{H}$*.*

The assumption (*) in Theorem 7 stands for "$H_{\bar{\delta}}$ has No Small Subgroups (NSS) property", which means that there is a neighborhood U of the identity in $H_{\bar{\delta}}$ which contains no non-trivial subgroups. This assumption can be verified in the following cases:

(i) Γ is a cocompact lattice in a real Lie group G of rank one. This allows to deduce Theorem 6 from the one above.
(ii) $\Gamma = \pi_1(X)$ where X is a compact n-manifold, $\delta = \delta_g$ where g is a negatively curved Riemannian metric on X satisfying the following pinching condition
$$p(g) \leq 1 + \frac{2}{n-1} \tag{26}$$
where $p(g) = b/a$ with $-b^2 \leq K_g \leq -a^2 < 0$ are bounds on the sectional curvature of g.
(iii) Γ is quasi-isometric to a fundamental group Γ' of a compact n-manifold X which admits a negatively curved Riemannian structure g so that
$$\lambda(\bar{\delta}, \overline{\delta'_g}) \cdot p(g) \leq 1 + \frac{2}{n-1} \tag{27}$$
where $p(g) = b/a$ is a bound on the pinching $-b^2 \leq K_g \leq -a^2 < 0$ of the sectional curvatures of g, and $\lambda(\bar{\delta}, \overline{\delta'_g}) = \lambda_2/\lambda_1$ is a bound on the *multiplicative* constants λ_1 λ_2
$$\lambda_1 \cdot d(\gamma_1, \gamma_2) - C \leq d'(q(\gamma_1), q(\gamma_2)) \leq \lambda_2 \cdot d(\gamma_1, \gamma_2) + C \tag{28}$$
of representatives $d \in D_\Gamma$, $d' \in D_{\Gamma'}$ for $\bar{\delta}$ and $\overline{\delta'_g}$ and the given quasi-isometry $q : \Gamma \to \Gamma'$.

Let us recall the following generalization of Hilbert's 5-th problem

Conjecture 8 (Hilbert-Smith) *A locally compact group acting faithfully by homeomorphisms on a topological manifold has No Small Subgroups.*

Hilbert-Smith Conjecture implies that assumption (*) in Theorem 7 is satisfied whenever the boundary $\partial\Gamma$ of Γ is a topological manifold, in particular in the case of fundamental groups $\Gamma = \pi_1(X)$ of compact manifolds X which admit negatively curved Riemannian structure (in which case $\partial\Gamma = \partial\tilde{X}$ is homeomorphic to a sphere). Hilbert-Smith Conjecture has been proven for actions by C^1-diffeomorphisms (Bochner Montgomery [BM]), and more recently for actions by Lipschitz maps (Repovš and Ščepin [RS]), and Hölder homeomorphisms with exponent $\dim X/(\dim X + 2)$ (Maleshich [Mal]). This is used below.

Before proving Theorem 7 let us deduce condition (*) (i.e. $H_{\bar{\delta}}$ has NSS) in cases (i)-(iii). In case (i) Γ is a cocompact lattice both in a rank-one Lie group $\bar{G} = \mathrm{Isom}\,(\mathbb{H})$ and in $H_{\bar{\delta}}$. The general structure of locally compact groups H containing Γ as a cocompact lattice is described in [Fu1] Theorems B and C and it follows that in our particular situation where $H_{\bar{\delta}} \subseteq \mathrm{Homeo}\,(\partial\Gamma)$ such $H_{\bar{\delta}}$ is either discrete or is isomorphic to $\mathrm{Isom}\,(\mathbb{H})$ or $\mathrm{Isom}_+(\mathbb{H})$. In all these cases it has the NSS property.

Consider case (iii), which contains (ii) as a particular case. The quasi-isometry q defines a homeomorphism $\partial q : \partial\Gamma \to \partial\Gamma' \cong \partial\tilde{X}$ which is a topological sphere S^{n-1}. Choose d, d' left invariant metrics on Γ and Γ' as in (28), and choose $\epsilon > 0$, $\epsilon' > 0$ so that one can form Gromov metrics $\rho := \rho_e^{\epsilon}$ and $\rho' := \rho_{e'}^{\epsilon'}$ on $\partial\Gamma$ and $\partial\Gamma'$, respectively. Let us identify these boundaries with the sphere S^{n-1} and let ρ_0 denote the standard round metric on S^{n-1}. These metrics are related by the following inequalities

$$C^{-1}\rho_0^{b/\epsilon'} \le \rho' \le C\cdot\rho_0^{a/\epsilon'} \qquad C^{-1}\rho^{\lambda_2\epsilon/\epsilon'} \le \rho' \le C\cdot\rho^{\lambda_1\epsilon/\epsilon'}$$

where the first one follows from comparison theorems and the second from (12) and (28). Combining these inequalities and the fact that a neighborhood V of the identity in $H_{\bar{\delta}}$ acts by Lipschitz homeomorphisms with respect to the metric ρ, one estimates

$$\begin{aligned}\rho_0(h\cdot\xi, h\cdot\eta) &\le C_1\cdot\rho'(h\cdot\xi, h\cdot\eta)^{\epsilon'/b} \le C_2\cdot\rho(h\cdot\xi, h\cdot\eta)^{(\epsilon'/b)\cdot(\lambda_1\epsilon/\epsilon')}\\ &\le C_3\cdot\rho(\xi,\eta)^{\epsilon\lambda_1/b} \le C_4\cdot\rho'(\xi,\eta)^{(\epsilon\lambda_1/b)\cdot(\epsilon'/\lambda_2\epsilon)}\\ &\le C_5\cdot\rho_0(\xi,\eta)^{(a/b)\cdot(\lambda_1/\lambda_2)}\end{aligned}$$

The result [Mal] of Maleshich proving Hilbert-Smith conjecture for $\frac{k}{k+2}$–Hölder actions on a compact k-manifold, which is S^{n-1} in our case, combined with condition (27) guarantees that the locally compact group $H_{\bar{\delta}}$ has NSS.

Proof of Theorem 7. Recall that Γ acts *minimally* and *strongly proximally* on $\partial\Gamma$, which means that for any $\mu \in \mathcal{P}(\partial\Gamma)$ and any $\xi \in \partial\Gamma$ there is a sequence $\gamma_n \in \Gamma$ so that the point measure $\mathrm{Dirac}(\xi)$ is a weak limit of $(\gamma_n)_*\mu$. These properties of minimality and strong proximality are inherited by the *locally compact* group $H_{\bar{\delta}} \supset \Gamma$, which still acts faithfully on the space $X = \partial\Gamma$. Such actions are discussed in [Fu2], and it follows from the results there that if $H_{\bar{\delta}}$

is assumed to have No Small Subgroups, then either (A) $H_{\bar{\delta}}$ is a countable discrete group, or (B) $H_{\bar{\delta}}$ is, up to index two, a simple connected rank-one Lie group G with trivial center. In the first case (A), the discrete group $H_{\bar{\delta}}$ has to contain Γ as a finite index subgroup, because Γ is a lattice in $H_{\bar{\delta}}$.

In case (B) Γ is a cocompact lattice in $G \cong \mathrm{Isom}_+(\mathbb{H})$ where $\mathbb{H} \cong G/K$ is the corresponding symmetric space, and Γ can be thought of a fundamental group $\pi_1(X)$ of the locally symmetric space $X = \mathbb{H}/\Gamma$. Let $\delta_0 \in \mathcal{D}_\Gamma$ be the (class of) metrics corresponding to the symmetric metric on $\mathbb{H}$ and the given embedding $\Gamma \subset H_{\bar{\delta}} \simeq \mathrm{Isom}_+(\mathbb{H})$, and let $m_{\bar{\delta}_0}$ denote the corresponding Bowen-Margulis current which can be written as

$$dm_{\bar{\delta}_0} = e^{h_0 \cdot B_o(\xi,\eta)}\, d\nu_o(\xi)\, d\nu_o(\eta)$$

where ν_o is the (geometric) Patterson-Sullivan measure of $o \in \mathbb{H}$. This measure $m_{\bar{\delta}_0}$ is preserved by all of $\mathrm{Isom}(\mathbb{H}) \supset G$. Moreover, ν_o is preserved by the maximal compact subgroup $K_o = \{g \in G \mid go = o\}$. The original measure $m_{\bar{\delta}} = \exp(2h_\delta \cdot f_\nu)\ \nu \times \nu$ is preserved by $G \subseteq H_{\bar{\delta}}$, and ν is G-quasi-invariant. Since G (already K_o) acts transitively on $\partial\Gamma \cong \partial\mathbb{H}$ the measures ν and ν_o are in the same measure class, and therefore $m_{\bar{\delta}}$ and $m_{\bar{\delta}_0}$ are absolutely continuous with respect to each other. Hence Theorem 2 gives $\bar{\delta} = \bar{\delta}_0$.

□

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