

PRODUCT GROUPS ACTING ON MANIFOLDS

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ABSTRACT. We analyse volume-preserving actions of product groups on Riemannian manifolds. To this end, we establish a new superrigidity theorem for ergodic cocycles of product groups ranging in linear groups. There are no *a priori* assumptions on the acting groups, except a spectral gap assumption on their action.

Our main application to manifolds concerns irreducible actions of Kazhdan product groups. We prove the following dichotomy: Either the action is infinitesimally linear, which means that the derivative cocycle arises from unbounded linear representations of all factors. Otherwise, the action is measurably isometric, in which case there are at most two factors in the product group.

As a first application, this provides lower bounds on the dimension of the manifold in terms of the number of factors in the acting group. Another application is a strong restriction for actions of non-linear groups.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

1.A. Actions on Manifolds. Consider a group Γ acting on a compact Riemannian manifold M by volume-preserving diffeomorphisms. What restrictions, if any, does the structure of Γ impose upon the dimension of M and the dynamics of the action?

When Γ is a lattice in a semi-simple group of higher rank, this investigation is the object of Zimmer's programme [58], which aims at a non-linear (or infinite-dimensional) extension of Margulis' work [35], [37] on superrigidity. (See *e.g.* [54], [55], [56], [57], [30], [18], [16].) Zimmer conjectures that there is no (infinite) action when the manifold has lower dimension than any linear representation of the semi-simple group. In this setting, Zimmer's cocycle superrigidity theorem establishes the existence of an invariant measurable metric. Had it been a smooth metric, the original action would have been smoothly conjugated to translations on a homogeneous space of a compact group, thus answering the question. In [59], Zimmer provides a *measurable* conjugation using Kazhdan property (T).

In this paper, we shall focus on the case where $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ is a product of (at least two) groups or perhaps a lattice in suitable product groups. This apparently weak hypothesis has recently been found to be a fertile ground for very diverse rigidity phenomena (see *e.g.* [1], [2], [3], [6], [7], [23], [27], [38], [39], [40], [43], [50]).

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One should of course discard the case where the action is a combination of independent actions of the factors, for instance when M is a product $M_1 \times \cdots \times M_n$ with component-wise action. We shall impose the following condition, hereafter called *ergodic irreducibility*: Each factor Γ_i act ergodically on M .

Very specific examples of ergodically irreducible actions are provided by algebraic actions on homogeneous manifolds through linear representations of the acting groups (see Section 1.C below for a description).

The main motivation of this paper is to show that *from the measurable cocycle viewpoint* these linear examples are in fact essentially the only possibilities when the groups have Kazhdan's property (T). More precisely:

Theorem A. *Let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ be a product of $n \geq 2$ groups with property (T) with a smooth volume-preserving action on a compact Riemannian manifold M (of non-zero dimension) such that each Γ_i acts ergodically. Then, either:*

- (Lin₁) *There are for each i finite index subgroups $\Gamma_i^* < \Gamma_i$ with Zariski-dense representations $\Gamma_i^* \rightarrow H_i$ in connected non-compact semi-simple Lie groups H_i ;*
- (Lin₂) *The product $H = H_1 \times \cdots \times H_n$ is a subquotient of $\mathrm{SL}_{\dim(M)}(\mathbf{R})$; in particular, $\dim(M) \geq 3n$;*
- (Lin₃) *The derivative cocycle of $\Gamma \curvearrowright M$ is virtually semi-conjugated modulo an amenable group to the product representation $\prod_i \Gamma_i^* \rightarrow H$;*

Or otherwise, the following hold:

- (Ism₁) *There are only two factors: $\Gamma = \Gamma_1 \times \Gamma_2$;*
- (Ism₂) *There are homomorphisms $\tau_i : \Gamma_i \rightarrow K$ with dense images in some compact group K .*
- (Ism₃) *The Γ -action on M is measurably isomorphic to the bilateral action on K :*

$$(g_1, g_2)k = \tau_1(g_1)k\tau_2(g_2)^{-1}.$$

Remarks 1.1. (1) In case (Lin), the most visible consequence of our statement is perhaps that each factor Γ_i of the group Γ must admit an unbounded real linear representation, and the restriction on the dimension. As for the dynamical conclusion (Lin₃), it will be made more precise below; it involves a sequence of reductions such as passing to an algebraic hull and dividing by the amenable radical. This procedure still retains some basic characteristics of the derivative cocycle which enables one to describe, for example, the Kolmogorov–Sinai entropy $h(g, M)$ of individual diffeomorphisms g of M :

$$(1.i) \quad h(g, M) = \sum_k m_k \lambda_k \quad (g \in \prod_i \Gamma_i^*),$$

where the λ_k are the positive Lyapunov exponents of $\varrho(g)$ and m_k the corresponding total multiplicities as explained in Section 3.B.

(2) The proof of Theorem A combines our superrigidity theorem (Theorem D below) with Zimmer's [59]. In our setting, the (Ism) alternative takes a particularly precise form (only two factors, no isotropy subgroup of the compact group). The problem of the smoothness of the conjugation remains.

Theorem A is a rigidity statement describing suitable actions of product groups as arising from linear representations and in particular providing dimension bounds. As an immediate by-product, it rules out certain actions, see Section 1.C. For non-Kazhdan groups, a number of the conclusions still hold as long as at least one factor has property (T):

Variation (On Theorem A). *Let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ be a product of $n \geq 2$ groups with a smooth volume-preserving action on a compact Riemannian manifold M (of non-zero dimension) such that each Γ_i acts ergodically. Assume that Γ_i has property (T) for $1 \leq i \leq k$, where $k \geq 1$. Then, either:*

- (Lin'\$_1\$) *For $i \leq k$, there are finite index subgroups $\Gamma_i^* < \Gamma_i$ with Zariski-dense representations $\Gamma_i^* \rightarrow H_i$ in connected non-compact semi-simple Lie groups H_i ;*
- (Lin'\$_2\$) *The product $H = H_1 \times \cdots \times H_k$ is a subquotient of $\mathrm{SL}_{\dim(M)}(\mathbf{R})$; in particular, $\dim(M) \geq 3k$;*
- (Lin'\$_3\$) *The derivative cocycle of $\Gamma_1 \times \cdots \times \Gamma_k \curvearrowright M$ is virtually semi-conjugated modulo an amenable group to the product representation $\prod_{i \leq k} \Gamma_i^* \rightarrow H$;*

Or otherwise, (Ism\$_{1,2,3}\$) hold as in Theorem A.

The possibility of non-Kazhdan factors actually restricts further the alternative (Ism), since for instance a compact group cannot contain both an infinite Kazhdan group and a dense commutative (or more generally soluble) subgroup. For example:

Let $\Gamma = \Gamma_1 \times \Gamma_2$ with Γ_1 non-linear Kazhdan and Γ_2 soluble. Then Γ admits no ergodically irreducible smooth volume-preserving action on any compact Riemannian manifold of non-zero dimension.

We now turn to actions of groups Γ which, rather than being products, are lattices in a product $G = G_1 \times \cdots \times G_n$ of $n \geq 2$ locally compact groups. Such lattices are assumed *irreducible* in the sense that $G_i \cdot \Gamma$ is dense in G for all i .

If the lattice is not cocompact, we shall need to assume it to be *integrable*; this condition will be explained in more detail below and means that the canonical cocycle class $G \times G/\Gamma \rightarrow \Gamma$ has a representative c such that the word-length of $c(g, -)$ is in $L^1(G/\Gamma)$ for every $g \in G$.

Theorem B (Lattices in Products). *Let $G = G_1 \times \cdots \times G_n$ be a product of $n \geq 2$ locally compact second countable groups with property (T). Let $\Gamma < G$ be a cocompact or integrable irreducible lattice with a mixing smooth volume-preserving action on a compact Riemannian manifold M .*

Then G admits an unbounded continuous real linear representation.

Moreover, this G -representation is virtually semi-conjugated modulo an amenable group to the cocycle induced to G from the derivative cocycle of $\Gamma \curvearrowright M$.

The *induced* cocycle in the statement refers to the following construction. Let $G^* < G$ be a closed subgroup such that G/G^* carries a G -invariant probability measure (e.g. G^* is a lattice in G , or G^* is closed of finite index). The *canonical*

class is realized by the cocycles

$$c : G \times G/G^* \longrightarrow G^*, \quad c(g, hG^*) = \sigma(ghG^*)^{-1}g\sigma(hG^*)$$

where $\sigma : G/G^* \rightarrow G$ is a measurable cross-section; different choices of σ give rise to cohomologous cocycles. To any G^* -space X corresponds the *induced* G -space $G/G^* \ltimes X$ with product measure and action $g(hG^*, x) = (ghG^*, c(g, hG^*)x)$. To any cocycle $\alpha : G^* \times X \rightarrow H$, e.g. a homomorphism $G^* \rightarrow H$, corresponds the *induced cocycle*

$$G \times (G/G^* \ltimes X) \longrightarrow H, \quad (g, (hG^*, x)) \longmapsto \alpha(c(g, hG^*), x).$$

1.B. Cocycle Superrigidity. The main new tool in the proofs of the above results is a new cocycle superrigidity result for Lie-group-valued cocycles of ergodically irreducible actions of product groups on probability spaces. This result does not use property (T) of the acting group, but the weaker property of *spectral gap* for the action. Our cocycle superrigidity theorem also requires some integrability or boundedness condition on the values of the cocycle, which can always be assumed for derivative cocycles arising from actions on compact manifolds.

We first state a simpler version of this result, although for the above applications we shall need the more general statement of Theorem D below.

Theorem C (Cocycle Superrigidity — Semi-Simple Hull). *Let $G = G_1 \times \cdots \times G_n$ be a product of $n \geq 2$ locally compact second countable groups with a measure-preserving action on a standard probability space X . Let $\alpha : G \times X \rightarrow H$ be a measurable cocycle ranging in a connected centre-free semi-simple Lie group H without compact factors. Assume:*

- (Zd) *α is Zariski-dense in H , i.e. it is not cohomologous to a cocycle ranging in a proper algebraic subgroup of H ;*
- (SG) *Each of the actions $G_i \curvearrowright X$ is ergodic and has a spectral gap;*
- (L^1) *$\log \|\alpha(g, -)\| \in L^1(X)$ for every $g \in G$.*

Then α is cohomologous to a continuous homomorphism $\varrho : G \rightarrow H$.

Remark 1.2. The assumption that H has no compact factors is necessary, see Section 9.B. On the other hand, our proof shows that one can drop the spectral gap assumption for one of the factors G_i . This is also the case for the general form in Theorem D below.

Our proof of Theorem C uses notably random walks and Oseledets' theorem (an influence of Margulis' work [34],[35]). In Section 9.A, we also point to an alternative geometric proof using the CAT(0) techniques of [38]. We also mention that S. Popa established very general cocycle superrigidity results for products with spectral gap [43]. A very striking feature of his results is that they have no restriction at all on the target groups. On the other hand, they are about a specific class of actions and therefore cannot be used for our present purposes.

The general result (Theorem D below) does not assume Zariski-density of the cocycle. Therefore, its statement requires that we recall a few other basic constructions.

Radical. For any topological group L one can define the *amenable radical* $\text{Ramen}(L)$ to be the maximal closed normal amenable subgroup in L (in analogy to Zassenhaus' classical definition [53]). If L is a connected real algebraic group, then the quotient $L/\text{Ramen}(L)$ can be obtained by dividing first by the soluble radical of L , then by the centre of the resulting reductive group, and then by the product of all the compact factors of the resulting connected semi-simple centre-free Lie group.

Semi-conjugacy. Let $\alpha : G \times X \rightarrow V$ be a cocycle. If some conjugate of α ranges in a subgroup $L < V$ and $p : L \rightarrow H$ is (the canonical morphism to) a quotient group, consider the corresponding cocycle $\vartheta : G \times X \rightarrow H$. We shall say that α is *semi-conjugated* to ϑ modulo $\text{Ker}(p)$.

Hull. For a cocycle $\alpha : G \times X \rightarrow V$ ranging in an algebraic group V (over \mathbf{R} in our case) there is a *minimal* algebraic subgroup $L < V$ into which some conjugate $\beta = \alpha^f : G \times X \rightarrow L < V$ ranges; if $G \curvearrowright X$ is ergodic, this group V is unique up to conjugacy and is called the *algebraic hull* [55, 9.2]. The neutral component (connected component of the identity) L^0 of L is an algebraic subgroup of finite index in L . There is an ergodic finite extension $\pi : Y \rightarrow X$ of the original action such that the lift $\tilde{\beta} : G \times Y \rightarrow L$ of β is cohomologous to a cocycle into L^0 (this construction from [55, 9.2.6] is recalled in the proof of Theorem D below).

We are ready to state the general result; for the application to Theorem A, one can assume that G is discrete.

Theorem D (Cocycle Superrigidity — Unrestricted Hull). *Let $G = G_1 \times \cdots \times G_n$ be a product of $n \geq 2$ locally compact second countable groups with a measure-preserving action on a standard probability space X . Let $\alpha : G \times X \rightarrow \text{GL}_d(\mathbf{R})$ be a measurable cocycle. Assume:*

- (SG) *Each of the actions $G_i \curvearrowright X$ is ergodic and has a spectral gap;*
- (L^∞) *$\|\alpha(g, -)\| \in L^\infty(X)$ for every $g \in G$.*

Let L be the algebraic hull of α , $L^0 \triangleleft L$ the neutral component, and $p : L^0 \rightarrow H = L^0/\text{Ramen}(L^0)$ the quotient by the amenable radical. Then we have:

- (1) *A splitting $H = H_1 \times \cdots \times H_n$ into (possibly trivial) connected centre-free semi-simple real Lie groups without compact factors;*
- (2) *Finite index open subgroups $G_i^* < G_i$ and continuous homomorphisms $\varrho_i^* : G_i^* \rightarrow H_i$ with Zariski-dense image.*

Upon lifting α to a finite ergodic G -extension $Y \rightarrow X$, it is semi-conjugated to the cocycle induced from the product representation $\prod_i \varrho_i^ : G^* \rightarrow H$ of $G^* = \prod_i G_i^*$.*

(The conclusion of this theorem is a precise formulation of what we called *virtual semi-conjugacy* in Theorem A above.)

1.C. Further comments. There are three basic algebraic examples of volume-preserving, ergodically irreducible actions on homogenous compact manifolds *via* linear representations: semi-simple, nilpotent and compact.

Semi-simple. Let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$, where each Γ_i admits a representation $\Gamma_i \rightarrow H_i$ with unbounded image in a real semi-simple Lie group H_i , and let $H = H_1 \times \cdots \times H_n$. Embed H into some real semi-simple Lie group L (e.g. $L = \mathrm{SL}_N(\mathbf{R})$ with $N \geq \sum \dim \mathrm{Lie}(H_i)$) and choose an irreducible cocompact lattice Λ in L . Then the Γ -action on $M = L/\Lambda$ by left translations *via* $\Gamma \rightarrow H < L$ is ergodically irreducible, and even mixing (Howe-Moore theorem).

Nilpotent. Let N be a connected real nilpotent group with a lattice $\Lambda < N$; such lattices are always cocompact and arithmetic by a theorem of Mal'cev, see e.g. [33] and [44]. If a group Γ admits a homomorphism $\rho : \Gamma \rightarrow \mathrm{Aut}(N)$ where $\rho(\Gamma)$ normalizes Λ , one obtains a volume-preserving Γ -action on the compact nil-manifold $M = N/\Lambda$. The standard $\mathrm{SL}_d(\mathbf{Z})$ -action on the torus \mathbf{T}^d is a prime example of this setting, and indeed it is the critical case to investigate ergodic irreducibility since any compact nil-manifold admits such a torus as an equivariant quotient. Let thus $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ with infinite homomorphisms $\rho_i : \Gamma_i \rightarrow \mathrm{SL}_{d_i}(\mathbf{Z})$. Upon passing to finite index subgroups and reducing the dimensions d_i one may assume that $\varrho_i(\Gamma_i^*)$ is irreducible over \mathbf{Q} for all finite index subgroups $\Gamma_i^* < \Gamma_i$. Let $\varrho = \varrho_1 \otimes \cdots \otimes \varrho_n$ be the tensor representation taking values in $\mathrm{SL}_d(\mathbf{Z})$ where $d = \prod d_i$. Then the corresponding Γ -action on $M = \mathbf{T}^d$ is ergodically irreducible. Indeed, a Fourier transform argument shows that the ergodicity of Γ_i is equivalent to $\varrho(\Gamma_i^*)$ having no invariant vectors in $\mathbf{Q}^d \setminus \{0\}$ for finite index subgroups $\Gamma_i^* < \Gamma_i$, which follows by construction. Note that the dimension $d = \dim(M)$ will typically exceed that of semi-simple examples and in addition the required linearity assumption is more stringent: Γ_i must have unbounded representations defined over \mathbf{Z} , rather than over \mathbf{R} .

Compact (isometric). Let $\Gamma = \Gamma_1 \times \Gamma_2$ and suppose there are homomorphisms $\tau_i : \Gamma_i \rightarrow K$ with dense images in some compact group K . Then the Γ -action on $M = K$ given by

$$(g_1, g_2)k = \tau_1(g_1)k\tau_2(g_2)^{-1} \quad (g_i \in \Gamma_i, k \in K)$$

is ergodically irreducible. If K is moreover a connected Lie group, we have an ergodically irreducible volume-preserving action on a manifold, and this action is even isometric.

An example where Theorem A rules out altogether certain actions is as follows.

Corollary 1.3. *Let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ be a product of $n \geq 2$ infinite groups with property (T), where one of the factors does not admit unbounded real linear representations.*

Then Γ has no smooth mixing volume-preserving actions on compact manifolds.

Mixing is a convenient strengthening of ergodic irreducibility, but the latter more natural assumption suffices in various cases. For instance, the statement of Theorem A shows that ergodic irreducibility suffices in Corollary 1.3 if in addition $n \geq 3$, or if Γ_1 and Γ_2 cannot not embed densely in the same compact group. Further:

Corollary 1.4. *Let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ be a product of $n \geq 2$ groups with property (T), where one of the factors admits only finitely many finite quotients.*

Then Γ has no ergodically irreducible smooth volume-preserving actions on compact manifolds of non-zero dimension.

We recall in Section 3.B that there are many infinite groups with Kazhdan's property (T) that do not admit unbounded linear representations over \mathbf{R} and some are known not to have finite quotients.

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2. GENERAL NOTATIONS

Throughout the paper, a *lcsc group* will mean a locally compact second countable topological group. We denote by m_G a choice of left Haar measure and recall that its measure class depends only on G . If G is a lcsc group, a *probability G -space* refers to a standard (Lebesgue) probability space endowed with a measurable G -action that preserves the measure. All (ergodic-theoretical) cocycles are assumed measurable. If $\alpha : G \times X \rightarrow H$ is a cocycle and $f : X \rightarrow H$ a measurable map, the corresponding cocycle α^f *cohomologous* (or *conjugated*) to α is defined by

$$\alpha^f(g, x) = f(gx)^{-1}\alpha(g, x)f(x).$$

For any subgroup $L < G$, we denote simply by $\alpha|_L$ the restriction of α to $L \times X$. If Y is some H -space, one defines the *skew product G -space* $X \ltimes Y$ by endowing the product $X \times Y$ with the action $g(x, y) = (gx, \alpha(g, x)y)$. Notice that *induced actions*, as defined in the introduction, are a special case of this construction.

A *norm* on a group H will mean a map $\|\cdot\| : H \rightarrow [1, \infty)$ such that $\|gh\| \leq \|g\| \cdot \|h\|$ and $\|g^{-1}\| = \|g\|$ for all $g, h \in H$; by default, we think of $\mathrm{GL}_d(\mathbf{R})$ as endowed with $\|g\| = \|g\|_{\mathrm{op}} \cdot \|g^{-1}\|_{\mathrm{op}}$, wherein $\|\cdot\|_{\mathrm{op}}$ is the operator norm. (The terminology conflicts of course with normed vector spaces.) The following fact is well-known both in the measurable setting for lcsc groups and for Borel norms on Baire topological groups:

Lemma 2.1. *Any measurable norm on a lcsc group is bounded on compact sets.*

□

We write $\mathrm{GL}_d^1 < \mathrm{GL}_d$ for the subgroup of determinant ± 1 .

Let G be a lcsc group and (X, μ) an ergodic probability G -space. We say that the G -action has a *spectral gap* if the G -representation on

$$L_0^2(X) = L^2(X, \mu) \ominus \mathbf{C} = \left\{ f \in L^2(X, \mu) : \int f \, d\mu = 0 \right\}$$

does not almost have invariant vectors. This representation is the (Koopman) representation given by $gf = f \circ g^{-1}$. Explicitly, the spectral gap means that there is a compact subset $K \subset G$ and an $\epsilon > 0$ such that

$$(2.i) \quad \forall F \in L_0^2(X), \exists g \in K : \quad \|gF - F\| \geq \epsilon \cdot \|F\|.$$

The action $G \curvearrowright X$ is called *mixing* if the matrix coefficients of the representation $L_0^2(X)$ are C_0 (vanish at infinity).

We use \succ for the weak containment of unitary representations; the trivial representation of a lcsc group G is denoted by $\mathbf{1}_G$. Thus for instance the above spectral gap property is $L_0^2(X) \not\succ \mathbf{1}_G$. This terminology is due to the following classical characterization à la Kesten for arbitrary unitary representations π , see *e.g.* [4, G.4.2].

Lemma 2.2. *Let σ be a probability measure on G that is absolutely continuous with respect to Haar measures and whose support generates G . Then $\pi \not\succ \mathbf{1}_G$ if and only if the spectral radius of $\pi(\sigma)$ satisfies $\|\pi(\sigma)\|_{\text{sp}} < 1$.* \square

All ergodic-theoretical statements regarding actions on compact Riemannian manifolds are understood to refer to the normalised measure defined by the volume form. We endow by default compact groups with their normalised Haar measure. More generally, when $H < G$ is a closed subgroup of a lcsc group such that K/L admits a non-zero G -invariant measure, we endow it with such a measure which we denote by $m_{G/H}$. We normalise $m_{G/H}$ whenever it is finite.

3. FROM COCYCLE SUPERRIGIDITY TO THEOREM A

In this section, we deduce Theorem A and its corollaries from the general cocycle superrigidity Theorem D. To this end, we use also the main result of Zimmer's [59] which states that a smooth volume-preserving action of a property (T) group on a compact manifold preserving a measurable Riemannian structure has discrete spectrum. This result uses Kazhdan's property (T) and smoothness in an essential way (more than just the spectral gap for the action on the manifold).

3.A. Proof of Theorem A. A volume-preserving smooth action $\Gamma \curvearrowright M$ defines a Γ -action on the tangent bundle TM . The tangent bundle can be measurably trivialised, thus defining the *derivative cocycle* (unique up to cohomology) $\alpha : \Gamma \times M \rightarrow \text{GL}_d^1(\mathbf{R})$ where $d = \dim(M)$. Moreover, one can assume that the norm $\|\alpha(g, -)\|$ is essentially bounded over M for each $g \in \Gamma$.

To see this, choose a measurable family $\{V \xrightarrow{p_x} U_x : x \in M\}$ of volume-preserving charts, where V and U_x are neighbourhoods of $0 \in \mathbf{R}^d$ and $x \in M$ respectively, such that $p_x(0) = x$ (it suffices to require $|\text{Jac}(p_x)(0)| = 1$), with $\|Dp_x(0)\|$ being uniformly bounded over $x \in M$. This is possible by compactness of M . Given such a family, one defines the measurable cocycle

$$(3.i) \quad \alpha : \Gamma \times M \longrightarrow \text{GL}_d^1(\mathbf{R}) \quad \text{by} \quad \alpha(g, x) = D(p_{gx}^{-1} \circ g \circ p_x)(0).$$

Observe that changing $\{p_x\}_{x \in M}$ would yield cohomologous cocycles. Note that for each $g \in \Gamma$ the cocycle $\alpha(g, -)$ is essentially bounded, due to the uniform bound on $\|Dp_x(0)\|$ and on the derivative of g over the compact manifold M .

Since each Γ_i has property (T), the ergodicity of the action $\Gamma_i \curvearrowright M$, which is equivalent to the absence of Γ_i -invariant vectors in $L_0^2(M)$, yields a spectral gap in this representation. Hence we are in position to apply Cocycle Superrigidity Theorem D, with G_i being Γ_i (endowed with the discrete topology) and $X = M$.

Suppose first that all factors H_i appearing in the statement of Theorem D are non-trivial (and thus non-compact). Then we have finite index subgroups Γ_i^* with Zariski-dense representations $\rho_i^* : \Gamma_i^* \rightarrow H_i$ and a virtual semi-conjugacy of the derivative cocycle to the product representation ϱ . This is precisely the case of the linearity phenomena (Lin) in Theorem A; we just need to justify $\dim(M) \geq 3n$. We first observe that each H_i has at least one simple factor with property (T). Indeed, we recall that the image of a group with property (T) in a semi-simple Lie group without (T) is always bounded (and thus not Zariski-dense) since such Lie groups have the Haagerup property [9]. Now it suffices to observe that non-compact simple Lie groups with property (T) have no unbounded real representation of dimension less than 3. (This follows immediately, for instance, from the Haagerup property of $\mathrm{SL}_2(\mathbf{R})$ together with the fact that Kazhdan groups have compact Abelianization [29].)

It remains to show that the only alternative is the case described in (Isom₁)–(Isom₃). Hence, assume that at least one of the factors H_i in Theorem D is compact. In this case, at least one of the factors of Γ , say Γ_1 , has the property that the restriction $\alpha|_{\Gamma_1}$ is cohomologous to a cocycle ranging in an amenable group (namely in a finite extension of the amenable radical of L^0).

It is well-known [55, 9.1.1] that any measurable cocycle of an ergodic action of a property (T) group to an amenable group is cohomologous to a cocycle ranging in some compact subgroup. Hence the restriction $\alpha|_{\Gamma_1}$ is cohomologous to a cocycle ranging in a compact subgroup of $\mathrm{GL}_d^1(\mathbf{R})$; upon conjugating further, we may assume it ranges in $\mathrm{O}_d(\mathbf{R})$. This is equivalent to saying that Γ_1 preserves a measurable Riemannian structure on M .

This is precisely a situation analysed by Zimmer in [59], where he proves (Theorem 1.7) that in such case the action $\Gamma_1 \curvearrowright M$ has discrete spectrum: that is, $L^2(M)$ splits as a direct sum of finite dimensional Γ_1 -invariant subspaces. By Mackey's measure-theoretical converse to the Peter–Weyl theorem [31], such an action is measure-theoretically isomorphic to an isometric action (the case of a single transformation was previously established by Halmos and von Neumann [52, 26]). This means that there exist a compact group K , a homomorphism $\tau_1 : \Gamma_1 \rightarrow K$ a closed subgroup $L < K$, and a measure space isomorphism

$$T : M \xrightarrow{\cong} K/L \quad \text{with} \quad T(g_1 x) = \tau_1(g_1)T(x)$$

for a.e. $x \in M$ and all $g_1 \in \Gamma_1$. We can assume that L does not contain non-trivial closed subgroups that are normal in K upon dividing by the kernel of the K -action on K/L .

Note that the group $N_K(L)/L$ acting on K/L faithfully from the right commutes with the Γ_1 -action by left translations. Denote by $\mathrm{Aut}(K/L, m_{K/L})$ the group of all measure space automorphisms, where everything is understood modulo null sets. We recall the following easy

Lemma 3.1 (see [20, 7.2]). *The centraliser of $\tau_1(\Gamma_1)$ in $\mathrm{Aut}(K/L, m_{K/L})$ is precisely $N_K(L)/L$.* \square

Denote $\Gamma'_1 = \Gamma_2 \times \cdots \times \Gamma_n$, so that $\Gamma = \Gamma_1 \times \Gamma'_1$. By the above Lemma, the Γ'_1 -action, which commutes with the Γ_1 -action on $M \cong K/L$, defines a homomorphism $\tau : \Gamma'_1 \rightarrow N_K(L)/L$. Ergodicity of the Γ'_1 -action implies that $N_K(L)/L$ also acts ergodically on K/L . But $N_K(L)$ and $N_K(L)/L$ are compact groups, so ergodicity means that the action is transitive, i.e., $N_K(L) = K$ and L is normal in K . By our convention this means that L is trivial, i.e. $M \cong K$. In particular, the representation τ of Γ'_1 ranges into K itself.

The ergodicity assumption of the action of each Γ_i on $M \cong K$ means that the images $\tau(\Gamma_i)$ are dense in K . We claim that $n = 2$, i.e. $\Gamma'_1 = \Gamma_2$. Indeed if $n \geq 3$ then K contains two commuting subgroups $\tau(\Gamma_2)$ and $\tau(\Gamma_3)$, each being dense in K . This forces K to be commutative. Property (T) of, say Γ_1 , implies that $\tau_1(\Gamma_1)$ is finite, hence so is K . But this contradicts the measure-theoretic isomorphism of K with M since the volume has no atoms. This completes the description of $\Gamma \curvearrowright M$ in this case, and thus the proof of Theorem A. \square

For the variation on Theorem A, we recall from Remark 1.2 that cocycle superrigidity also holds if the action of one factor lacks the spectral gap property. Therefore, we can follow the above proof by grouping all factors without property (T) into one factor and reason with

$$\Gamma_1 \times \cdots \times \Gamma_k \times (\Gamma_{k+1} \times \cdots \times \Gamma_n).$$

Thus, the argument above can be repeated *verbatim* with the only difference that the distinction of the two cases in the alternative hinges upon whether at least one factor H_i is compact for $i \leq k$. We emphasize that the conclusion (Ism₁) is still $n = 2$ (rather than only $k = 2$).

3.B. Corollaries. First we recall how to deduce the entropy formula (1.i). We refer to [22] or [55, 9.4] for more precisions on the following exposition. For any (finite-dimensional) linear operator A , denote by $\{\lambda_k\}$ the set of *Lyapunov (characteristic) exponents*, that is, the set of logarithms $\log |a_j|$ of all eigenvalues a_j of A with $|a_j| > 1$. Thus all λ_k are distinct and positive; the total multiplicity m_k is the sum of the multiplicities of all a_j with $\log |a_j| = \lambda_k$. One has the relation

$$\sum_k m_k \lambda_k = \max_p \lim_{n \rightarrow \infty} \frac{1}{n} \log \| \wedge^p A^n \|,$$

wherein the symbol \wedge denotes the exterior p -power. Returning to Remark 1.1 and taking $A = \varrho(g)$, we observe that the quantity

$$\max_p \lim_{n \rightarrow \infty} \frac{1}{n} \log \| \wedge^p \varrho(g^n) \|$$

is well-defined and is not affected by the (virtual) semi-conjugacy that Theorem A produces. On the other hand, Pesin's formula [42] gives

$$h(g, M) = \max_p \lim_{n \rightarrow \infty} \frac{1}{n} \log \| \wedge^p Dg^n|_x \| \quad (\text{a.e. } x \in M)$$

and hence (1.i) follows as claimed. For more on convergence of the above limit and characteristic exponents, see Section 6.

Before continuing, we recall the elementary fact that if a group has a finite index subgroup that admits a linear representation with infinite (or unbounded) image, then so does the initial group.

Proof of Corollary 1.3. The statement follows immediately from Theorem A if one recalls that mixing implies ergodic irreducibility (since the factors are assumed infinite) and precludes discrete spectrum. \square

Corollary 1.4 hinges on the well-known fact that a finitely generated group is residually finite if and only if it has an injective morphism to a compact group, see the proof:

Proof of Corollary 1.4. It suffices to show that both conclusions offered by Theorem A are incompatible with our assumptions. We can suppose that Γ_1 has only finitely many finite quotients. Recall that property (T) implies that Γ_1 is finitely generated [29]. Since finitely generated linear groups are residually finite [32], every linear image of Γ_1 is finite. This already rules out case (Lin).

We consider now a homomorphism $\tau_1 : \Gamma_1 \rightarrow K$ as in case (Ism) and seek a contradiction. By the Peter–Weyl theorem, K embeds into a product $\prod_{n \in \mathbb{N}} U_n$ of (finite-dimensional) unitary groups U_n . By the above discussion, the image of $\tau_1(\Gamma_1)$ in each U_n is finite, and hence by density the same statement holds for K . Thus K is profinite; this implies that the image of Γ_1 in K is finite. Thus K is a finite group, contradicting the fact that M has positive dimension and is measurably isomorphic to K . \square

Finally, we briefly recall that there are many known infinite groups with Kazhdan’s property (T) that do not admit unbounded linear representations over \mathbf{R} . These include: (a) Quotients (by infinite kernels) of lattices in $\mathrm{Sp}_{n,1}(\mathbf{R})$, in view of Corlette’s superrigidity [10]; (b) Lattices in semi-simple groups of higher rank over non-Archimedean fields, by Margulis’ superrigidity [37]; (c) Suitable Kac–Moody groups [46, 47, 8, 14]; (d) Gromov’s constructions of simple groups or torsion groups as quotients of arbitrary non-elementary hyperbolic groups [24], which are Kazhdan as soon as the corresponding hyperbolic group is so; (e) Gromov’s random groups [25]. Some of the groups listed under (c) and (d) above have no non-trivial finite quotients.

Of course it is expected that there are many more property (T) groups outside the linear realm. In fact, several of the above examples have much stronger properties than those needed for Corollary 1.3 and are therefore not really an illustration of our results. It has been observed on several occasions (we learned it from Sh. Matsumoto) that the random groups of [25] do not have mixing smooth volume-preserving actions on compact manifolds. Indeed, they have the fixed point property for isometric actions on “induced” spaces of the form

$$\int_M \mathrm{GL}(T_x M)/\mathrm{O}(g_x) \, dx,$$

where g is the Riemannian metric; see *e.g.* [38, Ex. 47] and compare Section 9.A below.* Groups as in (a) also have strong restrictions on their cocycles, see Fisher–Hitchman [15].

4. PRELIMINARIES FOR THE COCYCLE SUPERRIGIDITY THEOREM

We recall that a group action on a standard Borel space is *tame* (or “smooth” in [55]) if the quotient Borel structure is countably separated. For continuous actions of lcsc groups on separable metrisable spaces, this is equivalent to having locally closed orbits by the Effros–Glimm theorem [55, 2.1.14]. We shall repeatedly use the following fundamental facts.

Theorem 4.1. *Let $G < \mathrm{GL}_N(\mathbf{R})$ be a real algebraic group.*

- (i) *If $H, L < G$ are algebraic subgroups, then the H -action on G/L is tame.*
- (ii) *The G -action on $\mathrm{Prob}(\mathbf{P}^{N-1})$ is tame.*

On the proof. The first statement is apparently an unpublished result of Chevalley from the 1950s, see the introduction of [12]; for the proof, see [13] and [55, 3.1.3]. The second statement is due to Zimmer (see [55, 3.2.12]) and uses a result of Furstenberg [21]. (A statement for measurable maps was given by Margulis in [36], see also [54, §7].) \square

The following fact adapted from [39] is very general; it holds even for cocycles over not necessarily measure-preserving actions.

Proposition 4.2 (Cocycle Splitting). *Let $G = G_1 \times G_2$ and H be lcsc groups, $G \curvearrowright (X, \mu)$ a measurable measure class-preserving action on a standard probability space and $\alpha : G \times X \rightarrow H$ a cocycle. Assume:*

- *The restriction $\alpha|_{G_1}$ ranges in a closed subgroup $H_1 < H$, and is not cohomologous to a cocycle ranging in any proper subgroup of H of the form $h^{-1}H_1h \cap H_1$ for some $h \in H$;*
- *The action of H_1 on H/H_1 is tame;*
- *$G_1 \curvearrowright (X, \mu)$ is ergodic.*

Then α ranges in the normaliser $N_H(H_1)$ of H_1 and, passing to the quotient, the cocycle $G \times X \rightarrow N_H(H_1)/H_1$ is a homomorphism $G \rightarrow N_H(H_1)/H_1$ factoring through G_2 .

Proof. The arguments given on page 413 of [39] apply word for word (with G_1 and $G_2 = G'_1$ exchanged; the more specific assumptions in [39] are not used for that proposition). \square

The following is probably well-known.

*D. Fisher has informed us that upon reading the preprint version of this paper he made the following observation with L. Silberman [17]: if a group fixes a point in the above Hilbert manifold and has no finite quotient, then the underlying action on the manifold is trivial. (Indeed, one combines as above Zimmer’s result [59] with Peter–Weyl and Mal’cev.) The authors of [17] then give many very interesting illustrations of this statement.

Lemma 4.3. *Let G be a lcsc group, X an ergodic probability G -space and $\alpha : G \times X \rightarrow H$ a cocycle ranging in an algebraic subgroup $H < L$ of an algebraic group L .*

Then the algebraic hull of α viewed as a cocycle in H is the same as in L .

Proof. Without loss of generality, α is Zariski-dense in H . Let $H' < L$ be its hull in L ; we have to show that H, H' are conjugated and it suffices to show that H can be conjugated into H' .

Let $\varphi : X \rightarrow L$ be a measurable map with α^φ ranging in H' . The condition $\alpha^\varphi(g, x) \in H'$ can be written as

$$\varphi(gx)H' = \alpha(g, x)\varphi(x)H'.$$

Since the left H -action on L/H' is tame by Theorem 4.1, it follows by the ergodicity that almost all $\varphi(x)H'$ belong to a single H -orbit $H\ell_0H'$. Hence for some measurable $\psi : X \rightarrow H$ we have

$$\varphi(x)H' = \psi(x)\ell_0H',$$

which means that α^ψ ranges in $\ell_0H'\ell_0^{-1}$, and thus in $\ell_0H'\ell_0 \cap H$. By Zariski-density of α in H , that range is H and thus indeed $\ell_0^{-1}H\ell_0 \subseteq H'$. \square

We shall prove the following statement, which is a variant of the ideas used by Zimmer in [57] for the case where L is compact.

Proposition 4.4 (Controlled Conjugation). *Let G, H be lcsc groups, $L < H$ a closed subgroup, (X, μ) a probability G -space and $\alpha : G \times X \rightarrow H$ a cocycle. Fix some measurable norm on H and assume that for some measurable $f : X \rightarrow H$ the cocycle α^f ranges in L . Assume:*

- (SG) *The G -action on X has a spectral gap;*
- (L^∞) *$\|\alpha(g, -)\| \in L^\infty(X)$ for every $g \in G$.*

Then there exists a measurable map $F : X \rightarrow H$ with $\|F(-)\| \in L^\delta(X)$ for some $\delta > 0$ and such that α^F also ranges in L . In particular,

$$(L^1) \quad \int_X \log \|\alpha^F(g, x)\| d\mu(x) < \infty \quad (\forall g \in G).$$

Lemma 4.5 (Zimmer [57]). *Let G be a lcsc group and (X, μ) an ergodic probability G -space with spectral gap. Then there is a compact subset $K \subseteq G$ and $0 < \lambda < 1$ such that for every measurable $A_0 \subset X$ with $\mu(A_0) \geq 1/2$, there exists a sequence $\{g_n\}$ in K such that the sequence of sets defined by $A_{n+1} = A_n \cup g_n A_n$ satisfies $\mu(A_n) \geq 1 - \lambda^n$.*

Proof. Recall from (2.i) that there is K compact and $\epsilon > 0$ such that

$$\forall \varphi \in L_0^2(X) \exists g \in K : \quad \|\varphi \circ g - \varphi\| \geq \epsilon \cdot \|\varphi\|.$$

Define $\lambda = 1 - \epsilon^2/4$. For a measurable set $A \subset X$, the function $p_A(x) = 1_A(x) - \mu(A)$ is in $L_0^2(X)$ and has $\|p_A\|^2 = \mu(A) \cdot (1 - \mu(A))$. If $A, B \subset X$ are of the same

measure, then $\|p_A - p_B\|^2 = \mu(A \triangle B)$. It now follows that for any $A \subset X$ with $\mu(A) > 1/2$ there exists $g \in K$ such that

$$\mu(gA \cap (X \setminus A)) = \frac{1}{2}\mu(gA \triangle A) \geq \frac{\epsilon^2}{2}\mu(A)(1 - \mu(A)) \geq \frac{\epsilon^2}{4}\mu(X \setminus A)$$

and therefore $1 - \mu(A \cup gA) \geq \lambda(1 - \mu(A))$. Applying this argument inductively starting from $A = A_0$, the claimed estimates follow. \square

Proof of Proposition 4.4. As α^f ranges in L , for $g \in G$ a.e. $f(gx)L = \alpha(g, x)f(x)L$. We shall construct a measurable $F : X \rightarrow H$ with $\|F\|^\delta$ integrable and such that a.e. $F(x)L = f(x)L$. Let K , ϵ and λ be as before; we claim that the expression

$$C = \sup_{g \in K} \text{ess sup}_{x \in X} \|\alpha(g, x)\|$$

is finite. Indeed, the map $G \rightarrow L^\infty(X)$ defined by $g \mapsto \|\alpha(g, -)\|$ is weak-* continuous, and thus the image of K is weak-*bounded. By the Banach–Steinhaus principle, it is norm bounded, whence the claim.

Choose now $\delta > 0$ small enough to ensure $\lambda C^\delta < 1$ and $M < \infty$ large enough so that the set

$$A_0 = \{x \in X : \|f(x)\| \leq M\}$$

satisfies $\mu(A_0) \geq 1/2$. Construct a sequence $\{A_n\}$ as in Lemma 4.5. We define F on the conull set $\bigcup_{n=0}^{\infty} A_n \subset X$ by induction: Let $F(x) = f(x)$ for $x \in A_0$, and for $x \in A_{n+1} \setminus A_n$

$$F(x) = \alpha(g_n, y)F(y)$$

where $x = g_n y$ with $y \in A_n$. Thus $\|F(x)\| \leq C\|F(y)\|$ because $g_n \in K$. This gives the estimate

$$\|F(x)\| \leq M \cdot 1_{A_0}(x) + \sum_{n=1}^{\infty} M \cdot C^{n+1} \cdot 1_{A_{n+1} \setminus A_n}(x).$$

Since $\mu(A_{n+1} \setminus A_n) \leq \lambda^n$, the choice of δ yields integrability of $\|F\|^\delta$. We claim that $F(x)L = f(x)L$ holds on the conull set $\bigcup_{n=0}^{\infty} A_n$. Indeed, for $x \in A_0$ one has $F(x) = f(x)$; for $x \in A_{n+1} \setminus A_n$, writing $y = g_n^{-1}x \in A_n$, we have

$$F(x)L = \alpha(g_n, y)F(y)L = \alpha(g_n, y)f(y)L = f(x)L$$

using $F(y)L = f(y)L$ in the induction assumption. \square

Let $\alpha : G \times X \rightarrow H$ be any cocycle satisfying the (L^1) condition, where G, H are lcsc groups and H has some measurable norm. Consider the (finite) expression

$$(4.i) \quad \ell(g) = \int_X \log \|\alpha(g, x)\| d\mu(x).$$

We observe that ℓ is subadditive since

$$\begin{aligned} \ell(g_1g_2) &\leq \int_X \log \|\alpha(g_1, g_2x)\| d\mu(x) + \int_X \log \|\alpha(g_2, x)\| d\mu(x) \\ &= \ell(g_1) + \ell(g_2). \end{aligned}$$

In other words, $\exp(\ell)$ is a measurable norm on G ; moreover, ℓ is bounded on compact sets (Lemma 2.1).

Remark 4.6. Subadditivity implies also that if G is a finitely generated group with some word-length ℓ_G , then ℓ admits a linear bound in terms of ℓ_G .

5. COCYCLES, UNITARY REPRESENTATIONS AND INVARIANT MEASURES

This section contains some general considerations relating quasi-regular representations and existence of invariant measures. These are used in the proof of Theorem 6.1, but seem to be of independent interest.

Let G be lcsc group and (X, μ) an ergodic probability G -space. Denote by π the unitary representation on $L^2(X)$ and by π_0 its restriction to $L_0^2(X)$. Let B be a compact metrisable space with some given Borel-regular probability measure ν of full support, and let $H < \text{Homeo}(B)$ be a (lcsc) group of homeomorphisms which leaves the measure class of ν invariant. Let $\alpha : G \times X \rightarrow H$ be a measurable cocycle. We denote by $\tilde{\pi}$ the quasi-regular unitary G -representation on $L^2(X \ltimes B, \mu \times \nu)$. Therefore, writing everything explicitly,

$$(\tilde{\pi}(g^{-1})F)(x, b) = \left(\frac{d\alpha(g, x)_*^{-1}\nu}{d\nu}(b) \right)^{1/2} F(gx, \alpha(g, x)b).$$

Let $\text{Prob}_\mu(X \times B)$ denote the space of all probability measures on $X \times B$ that project to μ . The G -action on $X \ltimes B$ defines a G -action on $\text{Prob}_\mu(X \times B)$ since μ is G -invariant. By disintegration with respect to μ , any measure $\eta \in \text{Prob}_\mu(X \times B)$ can be written as

$$\eta = \int_X (\delta_x \times \eta_x) d\mu(x)$$

where $x \mapsto \eta_x \in \text{Prob}(B)$ is a measurable map. Such a measure η is G -invariant if and only if $\eta_{gx} = \alpha(g, x)_*\eta_x$ holds for all $g \in G$ and μ -a.e. $x \in X$.

It is straightforward to verify that $\tilde{\pi}$ contains $\mathbf{1}_G$ if and only if G preserves a measure in $\text{Prob}_\mu(X \times B)$ that is absolutely continuous with respect to $\mu \times \nu$. We shall however need a more refined statement relating the existence of arbitrary G -fixed measures in $\text{Prob}_\mu(X \times B)$ and the spectral properties of $\tilde{\pi}$, as follows.

Proposition 5.1. *If $\pi_0 \not\simeq \mathbf{1}_G$ but $\tilde{\pi} \simeq \mathbf{1}_G$, then G preserves some measure $\eta \in \text{Prob}_\mu(X \times B)$.*

Proof. Suppose that F_n is a sequence of unit vectors in $L^2(X \times B)$ with

$$d_n(g) = \|\tilde{\pi}(g)F_n - F_n\| \longrightarrow 0$$

uniformly on compact sets. Upon replacing F_n by $|F_n|$ the value d_n will only decrease, so we can assume $F_n \geq 0$. Consider the sequence of unit vectors $\{f_n\} \in L^2(X)$ given by

$$f_n(x) = \left(\int_B F_n(x, b)^2 d\nu(b) \right)^{1/2}.$$

We claim that f_n is a sequence of almost invariant unit vectors for π , and thus the assumption $\pi_0 \neq \mathbf{1}_G$ gives

$$(5.i) \quad \|f_n - \mathbf{1}\| \longrightarrow 0.$$

To verify almost invariance of $\{f_n\}$, note that in view of the elementary inequality $|a - b|^2 \leq |a^2 - b^2|$ we have

$$\begin{aligned} \|f_n - \pi(g)f_n\|^2 &\leq \int_X |f_n^2 - (\pi(g)f_n)^2| \, d\mu \\ &\leq \int_X \int_B |F_n^2 - (\tilde{\pi}(g)F_n)^2| \, d\mu \, d\nu \\ &= \int_X \int_B |F_n + \tilde{\pi}(g)F_n| \cdot |F_n - \tilde{\pi}(g)F_n| \, d\mu \, d\nu \\ &\leq \|F_n + \tilde{\pi}(g)F_n\| \cdot \|F_n - \tilde{\pi}(g)F_n\| \\ &\leq 2 \cdot d_n(g) \longrightarrow 0 \end{aligned}$$

uniformly on compact sets in G ; hence (5.i) follows. We shall now define a probability measure η on $X \times B$ as a functional on $L^1(X, \mu) \otimes C(B)$. Let $\psi_i : X \rightarrow [0, 1]$ be a sequence of measurable functions spanning a dense subspace in $L^1(X, \mu)$, and $\varphi_j : B \rightarrow [0, 1]$ be a sequence of continuous functions spanning a dense subspace in $C(B)$. Assume that $\psi_0 = \mathbf{1}_X$ and $\varphi_0 = \mathbf{1}_B$ constant one functions. For each i, j the following sequence in n is non-negative and satisfies

$$\begin{aligned} \langle \psi_i \otimes \varphi_j, F_n^2 \rangle &= \int_X \int_B \psi_i(x) \varphi_j(b) F_n(x, b)^2 \, d\nu(b) \, d\mu(x) \\ &\leq \int_X f_n^2 \, d\mu \longrightarrow 1. \end{aligned}$$

Applying the diagonal process, one finds a subsequence $\{n_k\}$ along which the LHS above converges for all i, j . We can now define η by

$$\int_{X \times B} \psi_i(x) \varphi_j(b) \, d\eta(x, b) = \lim_{k \rightarrow \infty} \langle \psi_i \otimes \varphi_j, F_{n_k}^2 \rangle.$$

More precisely, extending η linearly to the span of $\psi_i \otimes \varphi_j$ we note that it is a positive, normalised functional satisfying

$$\langle \psi_i \otimes \mathbf{1}_B, \eta \rangle = \lim_{k \rightarrow \infty} \int_X \psi_i f_{n_k}^2 \, d\mu = \int_X \psi_i \, d\mu$$

Thus it corresponds to a measure η on $X \times B$ projecting onto μ . This measure is G -invariant; indeed for fixed ψ_i, φ_j we have

$$\langle \psi_i \otimes \varphi_j, g\eta - \eta \rangle = \lim_{n \rightarrow \infty} \langle \psi_i \otimes \varphi_j, (\tilde{\pi}(g)F_{n_k})^2 - F_{n_k}^2 \rangle$$

whilst

$$\begin{aligned} & |\langle \psi_i \otimes \varphi_j, (\tilde{\pi}(g)F_{n_k})^2 - F_{n_k}^2 \rangle| \\ & \leq \int_X \int_B \psi_i \varphi_j |\tilde{\pi}(g)F_{n_k} + F_{n_k}| \cdot |\tilde{\pi}(g)F_{n_k} - F_{n_k}| \, d\mu \, d\nu \\ & \leq \|\tilde{\pi}(g)F_{n_k} + F_{n_k}\| \cdot \|\tilde{\pi}(g)F_{n_k} - F_{n_k}\| \leq 2d_{n_k} \longrightarrow 0. \end{aligned}$$

□

We now specialise to the case where: $H = \mathrm{SL}_d(\mathbf{R})$, $B = \mathbf{P}^{d-1}$ is the projective space and ν is the unique $\mathrm{SO}_d(\mathbf{R})$ -invariant probability measure on B .

Lemma 5.2 (Zimmer's Cocycle Reduction). *Let G be a lcsc group, (X, μ) an ergodic probability G -space and $\alpha : G \times X \rightarrow \mathrm{SL}_d(\mathbf{R})$ a cocycle. If the corresponding G -action on $X \ltimes \mathbf{P}^{d-1}$ preserves a probability measure η projecting to μ , then either*

- (1) α is cohomologous to a cocycle $\alpha' : G \times X \rightarrow \mathrm{SO}_d(\mathbf{R})$, or
- (2) α is cohomologous to a cocycle $\alpha' : G \times X \rightarrow L$ where L has a finite index subgroup that is reducible on \mathbf{R}^d .

If $\eta \prec \mu \times \nu$, then case (1) holds.

Proof. A G -invariant measure has the form $\eta = \int_X (\delta_x \times \eta_x) \, d\mu(x)$ with

$$\eta_{gx} = \alpha(g, x)_* \eta_x \quad \mu\text{-a.e. on } X.$$

The action of $H = \mathrm{SL}_d(\mathbf{R})$ on the space $\mathrm{Prob}(\mathbf{P}^{d-1})$ of probability measures is tame by Theorem 4.1. In view of the ergodicity of G -action on (X, μ) this implies that μ -almost all η_x lie on a single H -orbit:

$$\eta_x = \varphi(x) \eta_0$$

for some $\eta_0 \in \mathrm{Prob}(\mathbf{P}^{d-1})$ and a measurable map $\varphi : X \rightarrow H$. Denoting by $H_0 = \{h \in \mathrm{SL}_d(\mathbf{R}) : h_* \eta_0 = \eta_0\}$ the stabiliser of this measure, we get that the cocycle

$$\alpha'(g, x) = \varphi(gx)^{-1} \alpha(g, x) \varphi(x)$$

ranges in H_0 . Furstenberg's Lemma, which can be found *e.g.* as Corollary 3.2.2 in [55, Cor 3.2.2], implies that H_0 is either compact or virtually reducible on \mathbf{R}^d , and that moreover the former case holds when $\eta_0 \prec \nu$. It remains only to observe that $\eta \prec \mu \times \nu$ implies $\eta_0 \prec \nu$ and to recall that any compact subgroup of $\mathrm{SL}_d(\mathbf{R})$ can be conjugated into $\mathrm{SO}_d(\mathbf{R})$. □

6. RANDOM WALKS AND A FURSTENBERG CONDITION

This section investigates the growth of matrix-valued cocycles along random walks. The main result is a cocycle analogue of the famous Furstenberg condition for positivity of the top Lyapunov exponent. We recall the definition of the following integrability condition:

$$(L^1) \quad \int_X \log \|\alpha(g, x)\| \, d\mu(x) < \infty \quad (\forall g \in G).$$

Our presentation is based on [19].

Theorem 6.1 (Cocycle Version of Furstenberg's Theorem). *Let G be a lcsc group, (X, μ) an ergodic probability G -space with spectral gap and $\alpha : G \times X \rightarrow \mathrm{SL}_d(\mathbf{R})$ a cocycle satisfying (L^1) . Suppose that α is not equivalent to a cocycle ranging into a compact or virtually reducible subgroup of $\mathrm{SL}_d(\mathbf{R})$.*

Then, for any absolutely continuous generating measure σ on G :

$$\liminf_{n \rightarrow \infty} \int_G \int_X \frac{1}{n} \log \|\alpha(g, x)\| d\mu(x) d\sigma^{*n}(g) > 0.$$

This result will be applied when σ satisfies

$$\int_G \int_X \log \|\alpha(g, x)\| d\mu(x) d\sigma(g) < \infty,$$

in which case the limit in the theorem is finite and will be denoted by $\lambda_1 = \lambda_1(\alpha, \sigma)$. We begin with some preparations and recall the subadditive function ℓ from (4.i).

Lemma 6.2. *There exist symmetric absolutely continuous probability measures σ of full support on G such that $\ell \in L^1(G, \sigma)$, i.e.:*

$$(6.i) \quad \int_G \int_X \log \|\alpha(g, x)\| d\mu(x) d\sigma(g) < \infty.$$

(In the case where G is discrete, this is obvious.)

Proof. Let U be a compact neighbourhood of the identity in G ; in particular, $m_G(gU) = m_G(U) < \infty$ for all g . By Lemma 2.1, ℓ is bounded on U . Choose a countable set $\{g_n\}$ so that $G = \bigcup g_i U$ and let $\sigma' = \sum 2^{-n} \ell(g_i)^{-1} \cdot m_{g_n U}$ where m_A is the restriction of m_G to a Borel subset $A \subset G$. Then σ' is a finite positive measure, equivalent to m_G , and $\ell \in L^1(G, \sigma')$. We may now take σ to be the normalised symmetrised measure

$$\sigma(E) = (\sigma'(E) + \sigma'(E^{-1})) / 2\sigma'(G) \quad (E \subset G).$$

□

Proof of Theorem 6.1. We consider the G -space $X \ltimes \mathbf{P}^{d-1}$ as in Lemma 5.2 and recall that we chose for ν the (unique) $\mathrm{SO}_d(\mathbf{R})$ -invariant probability measure on \mathbf{P}^{d-1} . Consider the quasi-regular G -representation $\tilde{\pi}$ on $L^2(X \ltimes \mathbf{P}^{d-1}, \mu \times \nu)$ defined in Section 5. Applying Proposition 5.1 and Lemma 5.2, we deduce that $\tilde{\pi}$ has a spectral gap. By the Kesten-type characterization (Lemma 2.2), it follows

$$(6.ii) \quad \|\tilde{\pi}(\sigma)\|_{\mathrm{sp}} < 1$$

for an arbitrary absolutely continuous generating probability measure σ on G . This gap will allow us to estimate the growth of the cocycle using the following lemma; for shorter notation, we denote by

$$\varrho(h, \xi) = \frac{dh_*^{-1}\nu}{d\nu}(\xi)$$

the Radon–Nikodým derivative for $h \in \mathrm{SL}_d(\mathbf{R})$, $\xi \in \mathbf{P}^{d-1}$.

Lemma 6.3.

$$\|h\| \geq \left(\int_{\mathbf{P}^{d-1}} \sqrt{\varrho(h, \xi)} d\nu(\xi) \right)^{-d/2}$$

Proof. Let $\varrho_{max}(h) = \max_{\xi \in \mathbf{P}^{d-1}} \varrho(h, \xi)$. Since $\int_{\mathbf{P}^{d-1}} \varrho(h, \xi) d\nu(\xi) = 1$, we have

$$1 \leq \sqrt{\varrho_{max}(h)} \cdot \int_{\mathbf{P}^{d-1}} \sqrt{\varrho(h, \xi)} d\nu(\xi).$$

It now suffices to show that $\varrho_{max}(h) = \|h\|^d$. Using the Cartan (polar) decomposition and since ν is $\mathrm{SO}_d(\mathbf{R})$ -invariant, it is enough to consider for h a diagonal matrix $h = \mathrm{diag}[a_1, \dots, a_d]$ with $a_1 \geq a_2 \geq \dots \geq a_d > 0$ and $a_1 \cdots a_d = 1$, where

$$\varrho_{max}(h) = \varrho(h, \mathbf{R}e_1) = \frac{a_1}{a_2} \times \dots \times \frac{a_1}{a_d} = a_1^d.$$

On the other hand, $\|h\| = a_1$. \square

Using the above estimate we deduce:

$$\begin{aligned} & \int_G \int_X \log \|\alpha(g, x)\| d\mu(x) d\sigma(g) \\ & \geq \int_G \int_X -\frac{d}{2} \cdot \log \left(\int_{\mathbf{P}^{d-1}} \sqrt{\varrho(\alpha(g, x), \xi)} d\nu(\xi) \right) d\mu(x) d\sigma(g) \\ & \geq -\frac{d}{2} \cdot \log \left(\int_G \int_{X \times \mathbf{P}^{d-1}} \sqrt{\varrho(\alpha(g, x), \xi)} d\sigma(g) d(\mu \times \nu)(x, \xi) \right) \\ & = -\frac{d}{2} \cdot \log \langle \tilde{\pi}(\sigma) \mathbf{1}, \mathbf{1} \rangle \geq -\frac{d}{2} \cdot \log \|\tilde{\pi}(\sigma)\|. \end{aligned}$$

Replacing σ by σ^{*n} , where σ is generating, we get

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \frac{1}{n} \int_G \int_X \log \|\alpha(g, x)\| d\mu(x) d\sigma^{*n}(g) \\ & \geq \lim_{n \rightarrow \infty} -\frac{d}{2} \cdot \log \|\tilde{\pi}(\sigma^{*n})\|^{1/n} = -\frac{d}{2} \cdot \log \|\tilde{\pi}(\sigma)\|_{\mathrm{sp}}. \end{aligned}$$

This last term is strictly positive by (6.ii), concluding the proof of Theorem 6.1. \square

We shall now recall Oseledets' multiplicative ergodic theorem and establish some additional information that becomes available when $\lambda_1 > 0$ thanks to the above theorem.

Let σ be a probability measure as in Lemma 6.2. Consider the one-sided Bernoulli shift ϑ acting on $\Omega = G^{\mathbf{N}}$ equipped with the product measure $\sigma^{\mathbf{N}}$ by $(\vartheta\omega)_i = \omega_{i+1}$. Using the G -action on X , one defines a transformation T on $Z = \Omega \times X$, preserving the measure $\sigma^{\mathbf{N}} \times \mu$, by

$$T(\omega, x) = (\vartheta\omega, \omega_1 x).$$

In fact, by Kakutani's random ergodic theorem [28], the assumption that σ has full support on G together with ergodicity of the G -action on X implies that T is ergodic. The cocycle $\alpha : G \times X \rightarrow \mathrm{SL}_d(\mathbf{R})$ gives rise to a function $A : Z \rightarrow \mathrm{SL}_d(\mathbf{R})$

$$A(\omega, x) = \alpha(\omega_1, x)$$

for which $\log \|A(-)\|$ is in $L^1(Z, \sigma^{\mathbf{N}} \times \mu)$ by (6.i). The associated \mathbf{N} -cocycle $\mathbf{N} \times Z \rightarrow \mathrm{SL}_d(\mathbf{R})$ takes the following form

$$A_n(\omega, x) = (A \circ T^{n-1}) \cdots (A \circ T) A(\omega, x) = \alpha(\omega_n \cdots \omega_1, x).$$

To such a function one associates the non-negative quantity

$$\begin{aligned} \lambda_1 = \lambda_1(\alpha, \sigma) &= \liminf_{n \rightarrow \infty} \int_X \frac{1}{n} \log \|A_n(z)\| dz \\ &= \liminf_{n \rightarrow \infty} \int_X \int_G \frac{1}{n} \log \|\alpha(g, x)\| d\mu(x) d\sigma^{*n}(g). \end{aligned}$$

It follows from Kingman's subadditive ergodic theorem that the above \liminf is actually a limit (converging to the infimum); moreover, the convergence to the constant function λ_1 holds not only for the integral, but also almost-everywhere and in L^1 .

When $\lambda_1 > 0$, Oseledets' theorem ([41]; see also [45, 49]) gives further structure, namely there exist:

- An integer $1 < k \leq d$, integers $d = d_1 > \cdots > d_k > d_{k+1} = 0$ and reals $\lambda_1 > \cdots > \lambda_k$;
- A measurable family $\{E_{\omega,j}(x)\}$ of (d_1, d_2, \dots, d_k) -flags

$$\mathbf{R}^d = E_{\omega,1}(x) \supset E_{\omega,2}(x) \supset \cdots \supset E_{\omega,k}(x)$$

with $\dim E_{\omega,j} = d_j$ and such that for a.e. $(\omega, x) \in \Omega \times X$

$$E_{\omega,j}(x) = \left\{ v \in \mathbf{R}^d : \limsup_{n \rightarrow \infty} \frac{1}{n} \log \|\alpha(\omega_n \cdots \omega_1, x)v\| \leq \lambda_j \right\}.$$

Proposition 6.4. *Let $G = G_1 \times G'_1$ be a lcsc group, (X, μ) a probability G -space on which G_1 is ergodic and $\alpha : G \times X \rightarrow \mathrm{SL}_d(\mathbf{R})$ a cocycle satisfying (L¹).*

Then G'_1 leaves invariant the characteristic filtrations associated to random walks on G_1 . More precisely, if σ is a probability measure on G_1 satisfying (6.i) and $\lambda_1(\alpha|_{G_1}, \sigma) > 0$, then

$$\alpha(h, x) E_{\omega,j}(x) = E_{\omega,j}(hx) \quad (\forall h \in G'_1, \text{ a.e. } x \in X, \omega \in \Omega).$$

Proof. Fix some $h \in G'_1$. Given $g \in G_1$, $x \in X$ and $v \in \mathbf{R}^d$, we write $y = hx$ and $w = \alpha(h, x)^{-1}v$. Since g and h commute, we have

$$\begin{aligned} \alpha(g, x)w &= \alpha(h^{-1}gh, x)w \\ &= \alpha(h^{-1}g, hx) \alpha(h, x)w = \alpha(h^{-1}g, y)v \\ &= \alpha(h^{-1}, gy) \alpha(g, y)v \end{aligned}$$

which gives the estimate

$$\left| \log \|\alpha(g, x)w\| - \log \|\alpha(g, y)v\| \right| \leq f(gy)$$

with $f(-) = \log \|\alpha(h^{-1}, -)\| \in L^1(X)$. Recall that for any L^1 -function φ on an ergodic system (Z, T) , Birkhoff's ergodic theorem implies

$$\frac{1}{n}\varphi(T^n z) = \frac{1}{n}\varphi(z) + \frac{1}{n} \sum_{k=0}^{n-1} (\varphi \circ T - \varphi)(z) \longrightarrow 0$$

for a.e. z . Viewing f as an L^1 -function on $Z = \Omega \times X$, we deduce

$$\left| \frac{1}{n} \log \|\alpha(g_n, x)w\| - \frac{1}{n} \log \|\alpha(g_n, hx)v\| \right| \leq \frac{1}{n} f(g_n y) \longrightarrow 0$$

for a.e. $x \in X$ and a.e. $\omega \in \Omega$. This shows that $v \in E_{\omega,j}(hx)$ is equivalent to $w \in E_{\omega,j}(x)$, finishing the proof. \square

7. THE COCYCLE SUPERRIGIDITY THEOREMS

7.A. Semi-Simple Hulls. We will construct the homomorphism $\varrho : G \rightarrow H$ as a product of homomorphisms $\varrho_i : G_i \rightarrow H_i$, where $H = H_1 \times \cdots \times H_n$ will be a splitting of the given connected centre-free group H into (possibly trivial) connected semi-simple Lie groups.

We first consider the case of H simple (thus the above splitting will have only one non-trivial factor). We recall that it was assumed in Theorem C that H has no compact factors; in particular, H is non-compact (compare with Section 9.B).

Proof of Theorem C for H simple. The assertion to prove is that there exists a single factor G_{i_1} and a representation $\varrho_{i_1} : G_{i_1} \rightarrow H$ with Zariski-dense image such that

$$\alpha(g, x) = f(gx)^{-1} \varrho_{i_1}(g_{i_1}) f(x)$$

for some $f : X \rightarrow H$.

Write $G = G_1 \times G'_1$ where $G'_1 = G_2 \times \cdots \times G_n$, and let H_1 and H'_1 be the algebraic hulls in H of the restrictions $\alpha|_{G_1}$ and $\alpha|_{G'_1}$, respectively. Thus there are measurable maps $f, f' : X \rightarrow H$ such that α^f ranges in H_1 on $G_1 \times X$ and $\alpha^{f'}$ ranges in H'_1 on $G'_1 \times X$.

Since $H_1 < H$ is an inclusion of algebraic groups, the action $H_1 \curvearrowright H/H_1$ is tame by Theorem 4.1. Hence, applying Proposition 4.2, we deduce that the cocycle α^f ranges in $N_H(H_1)$. Since α was assumed to be Zariski-dense in H , the same holds for α^f . But $N_H(H_1)$ being an algebraic subgroup of H , it follows that $N_H(H_1) = H$. As H is simple, we have either $H_1 = H$ or $H_1 = \{e\}$.

Case $H_1 = \{e\}$. It follows from Proposition 4.2 that the restricted cocycle $\alpha^f|_{G'_1}$ is a homomorphism $\varrho' : G'_1 \rightarrow H$. Note that the image $\varrho'(G'_1)$ is Zariski-dense in H . Observe that for each $2 \leq i \leq n$ the image $\varrho'(G_i)$ is normalised by $\varrho'(G'_1)$, and therefore by all of H , hence each $\varrho'(G_i)$ is either trivial, or is Zariski-dense in H . However, for all but one $2 \leq i \leq n$ the image $\varrho'(G_i)$ is trivial, for otherwise the simple group H would contain two commuting Zariski-dense subgroups $\varrho'(G_i)$ and $\varrho'(G_j)$, which is impossible. Hence ϱ' factors through a Zariski-dense homomorphism $\varrho_{i_1} : G_{i_1} \rightarrow H$ of a single factor.

We can thus assume $H_1 = H$ for the remainder of this proof. Applying Proposition 4.2 to $\alpha^{f'}|_{G'_1}$, we deduce that $H'_1 \triangleleft H$ and therefore either $H'_1 = \{e\}$ or $H'_1 = H$. If $H'_1 = \{e\}$, then Proposition 4.2 shows that the cocycle $\alpha^{f'}$, being trivial on $G'_1 \times X$, is a homomorphism $\varrho_1 : G_1 \rightarrow H$ when restricted to $G_1 \times X$. Therefore, the main point is to prove the following key proposition, for which we shall also indicate an alternative approach in Section 9.A.

Proposition 7.1. *One cannot have $H'_1 = H$.*

Towards a contradiction, let us assume $H'_1 = H$. Choose an irreducible faithful representation $\pi : H \rightarrow \mathrm{SL}_r(\mathbf{R})$ and consider the resulting linear cocycle

$$\beta = \pi \circ \alpha : G \times X \xrightarrow{\alpha} H \xrightarrow{\pi} \mathrm{SL}_r(\mathbf{R}).$$

Lemma 7.2. *The restrictions $\beta|_{G_1}$ and $\beta|_{G'_1}$ are both not cohomologous to cocycles ranging in a compact subgroup, nor in a subgroup leaving invariant a finite family of proper subspaces in \mathbf{R}^r .*

Proof of the lemma. Recall that compact groups, or groups leaving invariant a finite union of subspaces in \mathbf{R}^r , are both examples of \mathbf{R} -algebraic groups in $\mathrm{SL}_r(\mathbf{R})$. Lemma 4.3 implies that the algebraic hull of $\beta|_{G_1}$ is $\pi(H)$. Had $\beta|_{G_1}$ been cohomologous to a cocycle ranging in an algebraic subgroup $L < \mathrm{SL}_r(\mathbf{R})$, it would follow that $\pi(H) < L$ up to replacing L by its conjugate. However $\pi(H)$ is not a subgroup of a compact group and does not leave invariant a finite union of proper subspaces, because $\pi(H)$ is connected, irreducible and non-compact.

The same argument can be applied to $\beta|_{G'_1}$ since we are assuming $H'_1 = H$. \square

The pointwise integrability condition (L^1) on α is inherited by β , and passed on to $\beta|_{G_1}$. Choose a probability measure σ on G_1 as in Lemma 6.2 for $\beta|_{G_1}$. That is, σ is absolutely continuous, symmetric and satisfies the average integrability condition

$$\int_{G_1} \int_X \log \|\beta|_{G_1}(g, x)\| d\mu(x) d\sigma(g) < \infty.$$

One may now look at the behavior of $\beta|_{G_1}$ along a path of a σ -random walk as is described in Section 6. In particular, Lemma 7.2 allows us to apply Theorem 6.1 which ensures strict positivity of the top Lyapunov exponent

$$\lambda(\beta|_{G_1}, \sigma) = \lim_{n \rightarrow \infty} \frac{1}{n} \int_{G_1} \int_X \log \|\beta|_{G_1}(g, x)\| d\mu(x) d\sigma^{*n}(g) > 0.$$

This fact yields the non-triviality of the Lyapunov filtrations (see Section 6). In particular, for some proper intermediate dimension $0 < d_2 < r$ there is a measurable family $\{E_{\omega,2}(x)\}_{\omega \in \Omega}$ of d_2 -dimensional vector subspaces $E_{\omega,2}(x) < \mathbf{R}^r$ which are intrinsically defined by $\beta|_{G_1} : G_1 \times X \rightarrow \mathrm{SL}_r(\mathbf{R})$ (describing the exceptional space of “slow” vectors under the ω -path of the random walk).

Being naturally associated to G_1 -action each of these sections $E_{\omega,2} : X \rightarrow \mathrm{Gr}(r, d_2)$ is invariant under the action of the commuting group G'_1 . Namely by Proposition 6.4, for almost every $\omega \in \Omega$ the section $E_{\omega,2}$ satisfies

$$\beta|_{G'_1}(g, x) E_{\omega,2}(x) = E_{\omega,2}(gx) \quad (g \in G'_1).$$

Fix a d_2 -dimensional subspace $E_0 < \mathbf{R}^r$, and choose a measurable map $\varphi : X \rightarrow \mathrm{SO}_r(\mathbf{R})$ such that $E_{\omega,2}(x) = \varphi(x)E_0$. Then the φ -conjugate of $\beta|_{G'_1} : G'_1 \times X \rightarrow \mathrm{SL}_r(\mathbf{R})$ ranges in the group $\{g \in \mathrm{SL}_d(\mathbf{R}) : gE_0 = E_0\}$, contradicting Lemma 7.2. This contradiction proves Proposition 7.1 and therefore completes the proof of Theorem C in the case of H being simple. \square

Completion of the proof of Theorem C in general. Now consider a general connected centre-free semi-simple Lie group H without compact factors. Then H can be written as a direct product of simple factors $H = S_1 \times \cdots \times S_N$. (Indeed, the connectedness implies that H is the product of its simple factors, and the triviality of the centre implies that the latter product is direct.) Let $\pi_t : H \rightarrow S_t$ denote the projections and apply the case of a simple target to the cocycles

$$\pi_t \circ \alpha : G \times X \longrightarrow H \longrightarrow S_t \quad (t = 1, \dots, N).$$

We get homomorphisms $\tau_t : G_{i(t)} \rightarrow S_t$, where $i(t) \in \{1, \dots, n\}$, and corresponding conjugating maps $f_t : X \rightarrow S_t$. For $j = 1, \dots, n$ we define

$$H_j = \prod_{i(t)=j} S_t, \quad \varrho_j = \prod_{i(t)=j} \tau_t : G_j \rightarrow H_j$$

being understood that the product over an empty set is the trivial group or morphism. We have an identification $H = H_1 \times \cdots \times H_n$. Arranging the maps f_t accordingly into a single map $f : X \rightarrow H$, we have

$$\alpha(g, x) = f(gx)^{-1} (\varrho_1(g_1), \dots, \varrho_n(g_n)) f(x).$$

Note that $\varrho_j(G_j)$ is Zariski-dense in H_j because α is Zariski-dense in H ; define $\varrho = \prod_i \varrho_i$. This concludes the proof. \square

7.B. Unrestricted Hulls. We turn now to the proof of Theorem D, where the cocycle is no longer assumed to be Zariski-dense in a connected semi-simple Lie group. There is a number of issues to address before it is possible to reduce the argument to the proof of the semi-simple case, notably:

- Controlled conjugation into the algebraic hull;
- Non-connectedness of the algebraic hull;
- The amenable radical.

Proof of Theorem D. Keep the notation of the theorem. Let

$$\beta(g, x) = f(gx)^{-1} \alpha(g, x) f(x)$$

be a conjugate of α that ranges in L . Conditions (SG) and (L^∞) allow us to apply Proposition 4.4 in order to choose f such that $\|f\|^\delta$ is integrable for some $\delta > 0$. Recall that $L^0 \triangleleft L$ is the connected component of the identity. Consider the finite extension $Y = X \ltimes (L/L^0)$ of the G -action on X , which is the skew product by β endowed with the invariant measure $\nu = \mu \times m_{L/L^0}$. It is shown in [55, 9.2.6] that this action is ergodic (otherwise, one could conjugate β to a cocycle ranging in an intermediate subgroup $L^0 < L_1 < L$, which is incompatible with Zariski-density).

The point of the finite extension $\pi : Y \rightarrow X$ is that the lift of the cocycle β to $G \times Y \rightarrow L$ becomes cohomologous to a cocycle ranging in L^0 . Indeed, if

$\xi : L/L^0 \rightarrow L$ is a choice of coset representatives and $\varphi(x, \ell L^0) = \xi(\ell L^0)$, then the cocycle $G \times Y \rightarrow L$ defined by $\varphi(gy)^{-1} \beta(g, \pi(y)) \varphi(y)$ ranges in L^0 (in fact, L^0 is its algebraic hull [55, 9.2.6]).

We now have a finite ergodic G -equivariant extension $\pi : Y \rightarrow X$ such that the lift of α to $G \times Y \rightarrow \mathrm{GL}_d^1(\mathbf{R})$ is cohomologous to a cocycle ranging in L^0 :

$$\vartheta : G \times Y \longrightarrow L^0, \quad \vartheta(g, y) = \varphi(gy)^{-1} f(\pi(y))^{-1} \alpha(g, \pi(y)) f(\pi(y)) \varphi(y).$$

Our restriction on f and the fact that φ has finite range imply that ϑ satisfies the integrability condition

$$(L^1) \quad \int_Y \log \|\vartheta(g, y)\| d\nu(y) < \infty \quad (\forall g \in G).$$

Let $p : L^0 \rightarrow H = L^0/\mathrm{Ramen}(L^0)$ denote the quotient by the amenable radical. Then the cocycle

$$(7.i) \quad p \circ \vartheta : G \times Y \longrightarrow L^0 \longrightarrow H$$

is a Zariski-dense in a connected centre-free semi-simple Lie group H without compact factors. This cocycle satisfies the (L^1) condition as well, and is almost ready for an application of Theorem C.

The only issue here is that the action of $G = G_1 \times \dots \times G_n$ on (Y, ν) might fail to be ergodically irreducible: although $G \curvearrowright (Y, \nu)$ is ergodic, this may not be true for the actions of the individual factors G_i . However, since each G_i acts ergodically on (X, μ) , the G_i -action on the finite extension (Y, ν) has a finite number of ergodic components (at most $[L : L^0]$). For $i = 1, \dots, n$ denote by

$$P^{(i)} = (Y_1^{(i)}, \dots, Y_{k_i}^{(i)})$$

the partition of (Y, ν) into the G_i -ergodic components.

Lemma 7.3.

- (1) *The partitions $P^{(i)}$ are independent and the elements of the joint partition $P = P^{(1)} \vee \dots \vee P^{(n)}$ are transitively permuted by G ;*
- (2) *If G^* denotes the stabiliser of an element, say $Z = \bigcap_{i=1}^n Y_1^{(i)}$, of the joint partition P , then the map*

$$q : Y \longrightarrow G/G^*, \quad q|_{gZ} \equiv gG^*$$

is G -equivariant;

- (3) *The group G^* is a direct product $G^* = G_1^* \times \dots \times G_n^*$ of finite index subgroups $G_i^* < G_i$, and the action of each of G_i^* on Z is ergodic.*

Proof. We describe the case $n = 2$, the general case following by induction. Since G acts ergodically on (Y, ν) , the group $G_2 \cong G/G_1$ permutes transitively the collection $P^{(1)}$ of G_1 -ergodic components. Hence, for each $1 \leq i \leq k_2$, the G_2 -action transitively permutes the k_1 sets $Y_j^{(1)} \cap Y_i^{(2)}$, $1 \leq j \leq k_1$. In particular, each of these sets has size $\mu(Y_j^{(1)} \cap Y_i^{(2)}) = \mu(Y_i^{(2)})/k_1$. Similarly, G_1 transitively

permutes the collection $\{Y_j^{(1)} \cap Y_i^{(2)} : 1 \leq i \leq k_2\}$ and we obtain $\mu(Y_j^{(1)} \cap Y_i^{(2)}) = \mu(Y_j^{(1)})/k_2$. Thus $P^{(1)} \perp P^{(2)}$ and we proved claim (1).

It further follows that the G -action on the elements of the joint partition $P^{(1)} \vee P^{(2)}$ is through G_2 in the $P^{(1)}$ -coordinate, and through G_1 in the $P^{(2)}$ -coordinate. Thus, the stabiliser of $Z = Y_1^{(1)} \cap Y_1^{(2)}$ has the form $G^* = G_1^* \times G_2^*$ with $[G_1 : G_1^*] = k_2$ and $[G_2 : G_2^*] = k_1$. Therefore, the G -action on the elements of $P = P^{(1)} \vee P^{(2)}$ is through G/G^* as in (2).

Finally, to see the ergodicity of the G^* -action on Z , consider a positive measure subset $A \subset Z$ and a generic point $z \in Z$. Viewing z and A in Y , one can use the ergodicity of $G \curvearrowright (Y, \nu)$ to deduce that $gz \in A$ for some $g \in G$. But since both z and gz are in $Z \subset Y_1^{(1)}$, the G_2 -component of $g \in G = G_1 \times G_2$ is in G_2^* . Similarly, $z, gz \in Z \subset Y_1^{(2)}$ implies that the G_1 -component of g is in G_1^* . Hence we proved that for a generic $z \in Z$ and a positive measure $A \subset Z$ there is an element $g \in G^* = G_1^* \times G_2^*$ with $gz \in A$. This proves the ergodicity claim in (3). \square

We now return to the cocycle $p \circ \vartheta$ in (7.i). Let $Z \subset Y$ and G^* be as in Lemma 7.3. Then the restriction $\delta : G^* \times Z \longrightarrow H$ of $p \circ \vartheta$ to $G^* \times Z$ satisfies the (L^1) condition, and the action $G^* \curvearrowright Z$ is ergodically irreducible. The G_i^* -representation on $L_0^2(Z)$ is easily seen to inherit the spectral gap property from the G_i -representation on $L_0^2(X)$. Hence we may apply Theorem C to deduce that δ is cohomologous to the homomorphism $\varrho^* : G^* \rightarrow H$ obtained in Section 7.A. Using a measurable cross-section $H \rightarrow L^0 < \mathrm{GL}_d(\mathbf{R})$ we may re-adjust the initial conjugation map $f : Y \rightarrow \mathrm{GL}_d(\mathbf{R})$ to achieve a situation where the cocycle $\tilde{\beta}$ lifted from β

$$\tilde{\beta} : G \times Y \longrightarrow L^0 \quad \tilde{\beta}(g, y) = f(gy)^{-1} \alpha(g, \pi(y)) f(y)$$

has the property that its restriction to $G^* \times Z$ projected via $L^0 \xrightarrow{p} H$ is the homomorphism $\varrho^* : G^* \rightarrow H$

$$f(gz)^{-1} \alpha(g, \pi(z)) f(z) = \varrho^*(g) \quad (g \in G^* < G, z \in Z \subset Y).$$

Finally, the fact that the $G \curvearrowright Y$ has $G \curvearrowright G/G^*$ as a factor with Z being the preimage of $eG^* \in G/G^*$ means that the whole cocycle $p \circ \beta : G \times Y \rightarrow H$ is induced from the homomorphism $\varrho^* : G^* \rightarrow H$ via $Y \rightarrow G/G^*$ as claimed. \square

8. LATTICES IN PRODUCTS

In this section, we address Theorem B. As a general fact for a lattice Γ in a product group G , recall that it is equivalent to assume property (T) for Γ or for G or for all factors of G , see *e.g.* [11].

Completion of the proof of Theorem B. We have a cocycle $\alpha : \Gamma \times M \rightarrow \mathrm{GL}_d^1(\mathbf{R})$ verifying the (L^∞) condition just as in the proof of Theorem A. Since Γ is finitely generated by property (T) [29], we may choose a word-length ℓ_Γ , noting that what

follows will not depend on this choice. The integrability assumption on Γ means that there is a cocycle $c : G \times G/\Gamma \rightarrow \Gamma$ in the canonical class such that

$$(8.i) \quad \int_{G/\Gamma} \ell_\Gamma(c(g, q)) dm_{G/\Gamma}(q) < \infty \quad (\forall g \in G).$$

This integrability condition is known to hold for classical lattices (see § 2 in [50]) and for Kac–Moody groups viewed as lattices (see [48]).

Let L be the algebraic hull of α , L^0 its neutral component and $H = L^0/\text{Ramen}(L^0)$. We record the following.

Lemma 8.1. *The group H is non-compact.*

Proof. Otherwise, L would be amenable since L^0 has finite index in L . Since Γ has property (T), we recall that this would imply that the hull L is compact [55, 9.1.1]. In other words, α could be conjugated into $O_d(\mathbf{R})$, which means that it preserves a measurable Riemannian structure on M . Thus, $\Gamma \curvearrowright M$ would have discrete spectrum by Zimmer’s result recalled in Section 3.A, contradicting the mixing assumption. \square

We consider now the induced G -space $X = G/\Gamma \ltimes M$.

Lemma 8.2. *Each G_i acts ergodically and with spectral gap on X .*

Proof. The ergodicity of $G_i \curvearrowright G/\Gamma \ltimes M$ is equivalent to the ergodicity of Γ on $G/G_i \times M$ (Gel’fand–Fomin duality principle). Since Γ is irreducible in G , it acts ergodically on G/G_i . Thus the ergodicity on $G/G_i \times M$ is a well-known consequence of the fact that $\Gamma \curvearrowright M$ is mixing (see e.g. [51, 3.7]). The spectral gap follows from ergodicity by property (T) of G_i . \square

By property (T) of Γ , any ergodic Γ -action has the spectral gap; using Proposition 4.4, we can assume that some conjugate α^f of α ranges in L and at the same time satisfies the (L^1) condition. Consider the induced cocycle

$$\beta : G \times X \longrightarrow L, \quad \beta(g, (q, x)) = \alpha^f(c(g, q), x).$$

It is a general fact that the operation of inducing cocycles does not change the algebraic hull, see Lemma 3.1 in [51]. Therefore:

Lemma 8.3. *The cocycle β is Zariski-dense in L .* \square

Further, we claim that induction using an integrable cocycle c preserves integrability:

Lemma 8.4. *The cocycle $\beta : G \times X \rightarrow L$ satisfies the (L^1) condition.*

Proof. Let $g \in G$ and decompose $G/\Gamma = \bigsqcup_{\gamma \in \Gamma} A_\gamma$ along the fibres A_γ of $c(g, -)$ over Γ . Thus (8.i) can be written

$$\sum_{\gamma \in \Gamma} m_{G/\Gamma}(A_\gamma) \cdot \ell_\Gamma(\gamma) < \infty.$$

Let μ denote the probability measure on M defined using the volume form. Our choice of the conjugate α^f ensures

$$\ell(\gamma) := \int_X \log \|\alpha^f(\gamma, x)\| d\mu(x) < \infty \quad (\forall \gamma \in \Gamma).$$

Now the lemma follows from the identity

$$\begin{aligned} \int_{G/\Gamma \times M} \log \|\beta(g, (q, x))\| d(m_{G/\Gamma} \times \mu)(q, x) \\ = \sum_{\gamma \in \Gamma} \int_{A_\gamma} \int_X \log \|\alpha^f(\gamma, x)\| d\mu(x) dm_{G/\Gamma} = \sum_{\gamma \in \Gamma} m_{G/\Gamma}(A_\gamma) \cdot \ell(\gamma) \end{aligned}$$

together with the fact that ℓ is linearly controlled by ℓ_Γ , see Remark 4.6. \square

We now argue as in the proof of Theorem D and consider the G -equivariant finite extension $\pi : Y \rightarrow X$ given by $Y = X \ltimes (L/L^0)$, which we recall is G -ergodic. As we have seen in that proof, this provides us with a cocycle

$$\eta : G \times Y \longrightarrow L^0 \longrightarrow H$$

which still retains the integrability and Zariski-density conditions established above for β . We can therefore continue as in Section 7.B. Thus we have Zariski-dense representations of finite index subgroups $G_i^* < G$ to connected groups H_i such that $H = \prod_i H_i$ and such that the product representation of $G^* = \prod_i G_i^*$ is conjugate to $\eta|_{G^*}$. Upon possibly passing to finite extensions, this yields a representation of G which is unbounded by Lemma 8.1. \square

9. ADDITIONAL CONSIDERATIONS

9.A. Geometric Approach to Cocycle Superrigidity. Let (M, g) be a compact Riemannian manifold. For each $x \in M$, consider the symmetric space $\mathrm{GL}(T_x M)/\mathrm{O}(g_x)$. The *Pythagorean integral* (or *induced space*)

$$\int_X \mathrm{GL}(T_x M)/\mathrm{O}(g_x) dx$$

is by definition the space of L^2 -sections of this bundle ([38, Ex. 47]) and is a complete CAT(0) space, indeed a Hilbert manifold of non-positive sectional curvature. Volume-preserving diffeomorphisms of M yield isometries of this space, and thus our original motivation for investigating actions of product groups upon M was the possibility to apply to this situation the splitting theorem for general CAT(0) spaces given in [38]. Whilst we have preferred in this article to take advantage of the projective dynamics available in the much more specific case of symmetric spaces, we shall nevertheless give below a very brief outline of this alternative approach.

The assumption of the splitting theorem is that the action on the CAT(0) space is not *evanescent*, which means that there should not be an unbounded set on which each group element has bounded displacement. In order to reduce to that situation, one first replaces the above bundle by a smaller sub-bundle. More

specifically, we shall now sketch how to prove Theorem C following this geometric approach. In view of the definition of the Pythagorean integral, we shall however replace the L^1 -condition by the slightly stronger L^2 -condition (which holds in the situation arising from actions on compact manifolds).

As we have seen in Section 7.A, we may assume that our cocycle $\alpha : G \times X \rightarrow H$, with $G = G_1 \times G'_1$, is Zariski-dense when restricted to G_1 and we need to prove that its restriction to G'_1 cannot be Zariski-dense. In other words, we shall explain how geometric splitting implies the key Proposition 7.1 for a simple group H . In view of the cocycle reduction (Lemma 5.2), it essentially suffices to derive a contradiction from the assumption that neither G_1 nor G'_1 admits an equivariant measurable map from X to probability measures on the geometric boundary of the symmetric space Y of H . Consider the induced G -space $S = \int_X Y$; the L^2 -condition ensures that the obvious isometric G -action is well-defined. The proof of Proposition 5.1 can be modified to yield the following statement for any proper CAT(0) space Y :

Let $L < G$ be a subgroup whose action on X has the spectral gap. If the L -action on S is evanescent, then there is an $\alpha|_L$ -equivariant measurable map from X to probabilities on ∂Y .

In particular, the G -action is non-evanescent and therefore the splitting theorem (Theorem 9 in [38]) provides a canonical G -invariant subspace $Z \subseteq S$ with an isometric equivariant splitting $Z = Z_1 \times Z'_1$ into G_1 - and G'_1 -spaces.

In fact, we shall use only a weaker statement which is a preliminary step in this splitting theorem, namely the fact that S contains a minimal (non-empty) G'_1 -invariant CAT(0) subspace Z'_1 , compare Proposition 35 in [38].

We now have the following dichotomy. Either Z'_1 is bounded, in which case it is a point by minimality (and the circumcentre lemma). Then G'_1 fixes a point in S , which means that $\alpha|_{G'_1}$ is conjugated into a compact subgroup, contrary to our assumption. (Notice that this argument would not be possible if the simple Lie group H were allowed to be compact, compare Section 9.B.) If on the other hand Z'_1 is unbounded, then it witnesses the evanescence of the G_1 -action on S . Indeed, minimality and convexity of the metric forces the displacement lengths of elements of G_1 to be constant on Z'_1 . Applying the above statement to $L = G_1$, we have also a contradiction.

We observe that in the above outline of argumentation, just like in our main random walk argument, one needs only to assume the spectral gap property for all but one factor (Remark 1.2).

9.B. Compact Targets. Finally, we explain why it is necessary in Theorem C to assume that the semi-simple target group H has no compact factors. More specifically, the standard arithmetic construction below shows that the conclusions of that theorem fail if the target group is a simple compact Lie group (compare with Proposition 7.1).

Fix some $n \geq 2$. Let $F = \mathbf{Q}(\xi)$ be a totally real separable extension of \mathbf{Q} of degree $n+1$; we denote by $Gal(F/\mathbf{Q})$ the corresponding Galois group and realize it as $Gal(F/\mathbf{Q}) = \{\sigma_0, \dots, \sigma_n\}$ where $\sigma_i : F \rightarrow \mathbf{R}$ are Galois embeddings. Upon

replacing ξ by a suitable rational translate, one can assume that $\sigma_0(\xi) < 0 < \sigma_1(\xi), \dots, \sigma_n(\xi)$. Let D denote the diagonal matrix $D = \text{diag}[1, 1, 1, -\xi, -\xi]$ and consider the algebraic group $\mathbf{G} = \{A \in \text{SL}_5 \mid A^T D A = D\}$ defined over F . Under the Galois embeddings σ_i , the quadratic form defined by D has signature $(3, 2)$ for $1 \leq i \leq n$, and is positive definite for $i = 0$. Denoting by k_i the Archimedean completions coming from $\sigma_i : F \rightarrow \mathbf{R}$, we get $G_i = \mathbf{G}(k_i) \cong \text{SO}_{3,2}(\mathbf{R})$ for $1 \leq i \leq n$, and $K = \mathbf{G}(k_0) \cong \text{SO}_5(\mathbf{R})$. Let \mathcal{O}_F denote the ring of integers of F . The group $\tilde{\Gamma} = \mathbf{G}(\mathcal{O}_F)$ embeds as a (uniform) lattice in the locally compact group

$$\tilde{G} = \prod_{i=0}^n \mathbf{G}(k_i) = K \times G_1 \times \cdots \times G_n$$

having dense injective projections on every sub-factor of the product [5]. In particular, the projection $\tau : \tilde{G} \rightarrow G = G_1 \times \cdots \times G_n$ maps $\tilde{\Gamma}$ to a lattice $\Gamma < G$, while $\pi : \tilde{\Gamma} \rightarrow K$ is a dense embedding. Starting from a cocycle $c : G \times G/\Gamma \rightarrow \Gamma$ in the canonical class, construct the cocycle

$$(9.i) \quad \alpha = \pi \circ \tau^{-1} \circ c : G \times G/\Gamma \rightarrow K.$$

We claim that its restriction $\alpha|_{G_i}$ is Zariski dense in $K \cong \text{SO}_5(\mathbf{R})$ for each $1 \leq i \leq n$. Since all the groups G_i have Kazhdan's property (T) which ensures the spectral gap assumption, this claim indeed shows that Theorem C cannot hold for compact targets.

The proof of the claim relies on a well-known change of viewpoint (which we have already used in an earlier section); namely, it is equivalent to the ergodicity of the G_i -action on the skew product $G/\Gamma \ltimes K$ associated to α . This latter action is isomorphic to

$$G_i \curvearrowright \tilde{G}/\tilde{\Gamma} = (K \times G)/\tilde{\Gamma}.$$

We now recall that the following conditions are equivalent (Gel'fand–Fomin duality principle):

- (1) $G_i \curvearrowright \tilde{G}/\tilde{\Gamma}$ is ergodic;
- (2) $\tilde{\Gamma} \curvearrowright \tilde{G}/G_i = K \times G'_i$ is ergodic, where $G'_i = \prod_{j \neq i} G_j$;
- (3) $\tilde{\Gamma}$ has a dense projection to $K \times G'_i$;
- (4) Γ has a dense projection to G'_i .

These conditions are satisfied by construction, proving the claim.

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