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Weak Notions of Normality and Vanishing up to Rank in L^2-Cohomology

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We study vanishing results for L^2-cohomology of countable groups under the presence of subgroups that satisfy some weak normality condition. As a consequence, we show that the L^2-Betti numbers of $\mathrm{SL}_n(R)$ for any infinite integral domain R vanish below degree $n-1$. Another application is the vanishing of all L^2-Betti numbers for Thompson's groups F and T.

1 Introduction

An important application of the algebraic theory of L^2-Betti numbers [10] (see Farber [8] for an alternative approach) is that the L^2-Betti numbers $\beta_i^{(2)}(\Gamma)$ of a group Γ vanish if it has a normal subgroup whose L^2-Betti numbers vanish. With regard to the *first* L^2-Betti number, one can significantly relax the normality condition to obtain similar vanishing results [14]. Peterson and Thom prove in [14] that the first L^2-Betti number of a group vanishes if it has a s-normal subgroup (defined below) with vanishing first L^2-Betti number.

The aim of this article is to extend such vanishing results to arbitrary degrees and to present some applications. Next, we describe the main notions and results in greater detail.

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We denote the γ-conjugate $\gamma^{-1}\Lambda\gamma$ of a subgroup $\Lambda<\Gamma$ by Λ^{γ}. Unless stated otherwise, all groups are discrete and countable, and all modules are left modules.

Definition 1.1. A subgroup Λ of a group Γ is called

(1) *n-step s-normal* if for every $(n+1)$-tuple $\omega=(\gamma_0,\dots,\gamma_n)\in\Gamma^{n+1}$ the intersection

$$\Lambda^{\omega}:=\Lambda^{\gamma_0}\cap\cdots\cap\Lambda^{\gamma_n}$$

is infinite.

(2) *s-normal* if it is one-step s-normal. □

Example 1.2. The subgroup of upper triangular matrices

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} < \mathrm{SL}_2(\mathbb{Z}[1/p])$$

inside $\mathrm{SL}_2(\mathbb{Z}[1/p])$ is one-step s-normal but not two-step s-normal. The fact that it is s-normal can be verified directly or is a special case of the more general results in Section 4.1. The fact that it is not two-step s-normal can again be verified directly; it is also a consequence of Corollary 1.5 below and the nonvanishing

$$\beta_2^{(2)}(\mathrm{SL}_2(\mathbb{Z}[1/p]))\neq 0 \tag{1.1}$$

of the second L^2-Betti number of $\mathrm{SL}_2(\mathbb{Z}[1/p])$. The group $\mathrm{SL}_2(\mathbb{Z}[1/p])$ is an irreducible lattice in $\mathrm{SL}_2(\mathbb{R})\times\mathrm{SL}_2(\mathbb{Q}_p)$. The latter locally compact group contains a product of nonabelian free groups as a (reducible) lattice. Hence, $\mathrm{SL}_2(\mathbb{Z}[1/p])$ is measure equivalent to a product of nonabelian free groups. By an important theorem of Gaboriau [9], the nonvanishing of $\beta_n^{(2)}$ is an invariant under measure equivalence, and the second L^2-Betti number of a product of nonabelian free groups is nonzero by the Kuenneth formula for L^2-cohomology. □

The following is our main result. Recall that the zeroth L^2-Betti number of a group is zero if and only if the group is infinite.

Theorem 1.3. Let $\Lambda<\Gamma$ be a subgroup. Assume that

$$\beta_i^{(2)}(\Lambda^{\omega})=0,$$

for all integers $i, k \geq 0$ with $i + k \leq n$ and every $\omega \in \Gamma^{k+1}$. In particular, Λ is an n-step s-normal subgroup of Γ. Then

$$\beta_i^{(2)}(\Gamma) = 0 \quad \text{for every } i \in \{0, \ldots, n\}. \qquad \square$$

Recall that $\Lambda < \Gamma$ is called *commensurated* if $\Lambda \cap \Lambda^\gamma$ is of a finite index in Λ and Λ^γ for every $\gamma \in \Gamma$. The corollary follows from the preceding theorem and the fact that one has the relation $\beta_i^{(2)}(\Gamma') = [\Gamma : \Gamma'] \cdot \beta_i^{(2)}(\Gamma)$ for a subgroup $\Gamma' < \Gamma$ of a finite index.

Corollary 1.4. Let $\Lambda < \Gamma$ be a commensurated subgroup. If $\beta_i^{(2)}(\Lambda) = 0$ for every $i \in \{0, \ldots, n\}$, then also $\beta_i^{(2)}(\Gamma) = 0$ for every $i \in \{0, \ldots, n\}$. $\square$

The theorem above implies together with the vanishing of L^2-Betti numbers of infinite amenable groups [5] the following.

Corollary 1.5. Let $\Lambda < \Gamma$ be an n-step s-normal and amenable subgroup. Then the L^2-Betti numbers of Γ vanish up to degree n, that is, $\beta_i^{(2)}(\Gamma) = 0$ for every $i \in \{0, \ldots, n\}$. $\square$

By taking a suitable subgroup Λ inside the special linear group $\Gamma = \mathrm{SL}_n(R)$ over a ring R, Theorem 1.3 yields the following application (proved in Section 4.1).

Theorem 1.6. Let R be an infinite integral domain. Let $n \geq 2$. Then

$$\beta_i^{(2)}(\mathrm{SL}_n(R)) = 0 \quad \text{for every } i \in \{0, \ldots, n-2\}. \qquad \square$$

In addition, we have a statement about degree $n - 1$.

Theorem 1.7. Assume that a ring R satisfies at least one of the following properties:

(1) R is an infinite field.
(2) R is a subring of the field $F(t)$ of rational functions over a finite field F and R contains an invertible element α that is not a root of unity.
(3) R is a subring of the field $\bar{\mathbb{Q}}$ of algebraic numbers, and R contains an invertible element α that is not a root of unity.

Then one has

$$\beta_{n-1}^{(2)}(\mathrm{SL}_n(R)) = 0. \qquad \square$$

If $\mathrm{SL}_n(R)$ is a lattice in a semisimple Lie group, for example, in the case $R=\mathbb{Z}$ or, more generally, R being a subring of algebraic *integers*, then much more is known than in the preceding theorems. It follows from results of Borel, which rely on global analysis on the associated symmetric space, that the L^2-Betti numbers vanish except possibly in the middle dimension of the symmetric space [3, 13]. However, the interesting and new case of the preceding theorems is the one where $\mathrm{SL}_n(R)$ is *not* a lattice in a semisimple Lie group; take, for example, $R=\mathbb{Z}[x_1, x_2, \ldots, x_d]$. According to results of Shalom [17] and Vaserstein the so-called *universal lattice* $\mathrm{SL}_n(\mathbb{Z}[x_1, \ldots, x_d])$ has property (T) provided that $n \geq 3$; Mimura [12] showed that for $n \geq 4$ the universal lattice has even property F_{L^p}, $p \in (1, \infty)$, as defined by Bader–Furman–Gelander–Monod. Ershov and Jaikin-Zapirain showed property (T) for the groups $\mathrm{EL}_n(\mathbb{Z}\langle x_1, \ldots, x_d\rangle)$, $n \geq 3$, of *noncommutative universal lattices* [7].

Of course, property (T) implies the vanishing of the first L^2-Betti number, but nothing was known before about the L^2-Betti numbers of universal lattices in higher degrees.

The following application of Theorem 1.3 (proved in Section 4.2) was kindly pointed to us by Monod remarking on an earlier draft of this paper.

Theorem 1.8. All L^2-Betti numbers of Thompson's groups F and T vanish. □

The groups F and T were invented by Thompson in 1965. In unpublished work Thompson proved that the group T is a finitely presented, infinite, and simple group. The vanishing of L^2-Betti numbers for Thompson's group F was proved before in a different way by Lück [11, Theorem 7.10., p. 298].

Remark 1.9. In a forthcoming paper [1], we show that, if a locally compact group G has a noncompact amenable radical, then every lattice of G has an infinite amenable commensurated subgroup. In particular, every lattice of G has vanishing L^2-Betti numbers by a theorem of Cheeger and Gromov [5] and Corollary 1.4. □

Example 1.10. The subgroup $\mathbb{Z} \cong \langle x\rangle$ of the Baumslag–Solitar group

$$\mathrm{BS}(p, q) = \langle x, t \mid tx^p t^{-1} = x^q\rangle$$

is commensurated but not normal. Corollary 1.4 yields that the L^2-Betti numbers of $\mathrm{BS}(p, q)$ vanish. This result is part of earlier work of Dicks and Linnell [6] about L^2-Betti numbers of one-relator groups. □

2 L^2-Cohomology

Our background reference for L^2-Betti numbers is Lück [11]. L^2-Betti numbers have various definitions with different levels of generality. A modern and algebraic description that applies to arbitrary groups was given by Lück [10]. He introduced a dimension function for arbitrary modules over the group von Neumann algebra $L(\Gamma)$ and showed that the ith L^2-Betti number $\beta_i^{(2)}(\Gamma)$ in the sense of Cheeger and Gromov [5] can be expressed as follows:

$$\beta_i^{(2)}(\Gamma) = \dim_{L(\Gamma)} H_i(\Gamma, L(\Gamma)).$$

The dimension function $\dim_{L(\Gamma)}$ extends to a dimension function $\dim_{\mathcal{U}(\Gamma)}$ for modules over the algebra $\mathcal{U}(\Gamma)$ of densely defined, closed operators affiliated to $L(\Gamma)$ in the sense that

$$\dim_{L(\Gamma)}(M) = \dim_{\mathcal{U}(\Gamma)}(\mathcal{U}(\Gamma) \otimes_{L(\Gamma)} M),$$

for every $L(\Gamma)$-module M [15, Proposition 3.8]. One has [15, Proposition 5.1]

$$\beta_i^{(2)}(\Gamma) = \dim_{\mathcal{U}(\Gamma)} H_i(\Gamma, \mathcal{U}(\Gamma)). \tag{2.1}$$

We refer to [11, chapter 8] for more information about this way of defining L^2-Betti numbers. The algebra $\mathcal{U}(\Gamma)$ of affiliated operators is a *self-injective* ring, that is, the functor $M \mapsto \hom_{\mathcal{U}(\Gamma)}(M, \mathcal{U}(\Gamma))$ is exact [2]. Thom firstly exploited this property for the computation of L^2-invariants [18]. Later we need the following lemma.

Lemma 2.1. Let $\Lambda < \Gamma$ be a subgroup. If $\beta_i^{(2)}(\Lambda) = 0$, then

$$H^i(\Lambda, \mathcal{U}(\Gamma)) = 0. \qquad \square$$

Proof. The ring $\mathcal{U}(\Lambda)$ is von Neumann regular [11, Theorem 8.22, p. 327]. Thus, $\mathcal{U}(\Gamma)$ is a flat $\mathcal{U}(\Lambda)$-module [11, Lemma 8.18, p. 326]. So we have

$$H_i(\Lambda, \mathcal{U}(\Gamma)) \cong \mathcal{U}(\Gamma) \otimes_{\mathcal{U}(\Lambda)} H_i(\Lambda, \mathcal{U}(\Lambda)).$$

The uniqueness of $\dim_{\mathcal{U}(\Lambda)}$-dimension [15, Theorem 3.11] and the flatness of the functor $\mathcal{U}(\Gamma)\otimes_{\mathcal{U}(\Lambda)}$ yield that for any $\mathcal{U}(\Lambda)$-module M we have

$$\dim_{\mathcal{U}(\Gamma)}(\mathcal{U}(\Gamma) \otimes_{\mathcal{U}(\Lambda)} M) = \dim_{\mathcal{U}(\Lambda)}(M).$$

In particular, it follows that

$$\dim_{\mathcal{U}(\Gamma)}(H_i(\Lambda,\mathcal{U}(\Gamma)))=\dim_{\mathcal{U}(\Lambda)}(H_i(\Lambda,\mathcal{U}(\Lambda)))\overset{(2.1)}{=}\beta_i^{(2)}(\Lambda)=0.$$

By Thom [18, Corollary 3.3], this yields that

$$\hom_{\mathcal{U}(\Gamma)}(H_i(\Lambda,\mathcal{U}(\Gamma)),\mathcal{U}(\Gamma))=0.$$

Since $\mathcal{U}(\Gamma)$ is self-injective, as mentioned above, the latter module is isomorphic to $H^i(\Lambda,\mathcal{U}(\Gamma))$. ■

3 Proof of Theorem 1.3

For a $\mathbb{C}\Gamma$-module M, we use the notation

$$M^\Gamma=\{m\in M\mid \gamma m=m \quad \text{for every } \gamma\in\Gamma\}.$$

For a subgroup $\Lambda<\Gamma$ and a $\mathbb{C}\Lambda$-module M, the $\mathbb{C}\Gamma$-module

$$\operatorname{coind}_\Lambda^\Gamma(M):=\hom_{\mathbb{C}\Lambda}(\mathbb{C}\Gamma,M)$$

given by the Γ-action

$$(\gamma_0 f)(x)=f(x\gamma_0)\quad \text{for } f\in\operatorname{coind}_\Lambda^\Gamma(M) \text{ and } \gamma\in\Gamma$$

is called the *co-induced* $\mathbb{C}\Gamma$-module [4, III.5]. For a $\mathbb{C}\Gamma$-module N, we denote the *restriction* of N to a $\mathbb{C}\Lambda$-module by $\operatorname{res}_\Lambda^\Gamma(N)$. We use the notation for the restriction only for emphasis; we often drop the $\operatorname{res}_\Lambda^\Gamma$-notation.

3.1 A sequence of modules for dimension shifting

In the sequel let Γ be a group, $\Lambda<\Gamma$ a subgroup, and let

$$M_0=\mathcal{U}(\Gamma)$$

regarded as a $\mathbb{C}\Gamma$-module. Starting with M_0, consider the following inductively defined sequence of $\mathbb{C}\Gamma$-modules, whose study is motivated by the use of dimension shifting in the proof of Theorem 1.3.

$$M_{i+1}:=\operatorname{coker}(M_i\to\operatorname{coind}_\Lambda^\Gamma(\operatorname{res}_\Lambda^\Gamma(M_i))). \tag{3.1}$$

The homomorphism $M_i \to \hom_{\mathbb{C}\Lambda}(\mathbb{C}\Gamma, M_i)$ for the cokernel is $m \mapsto (\gamma \mapsto \gamma m)$; it is $\mathbb{C}\Gamma$-equivariant. So this declares inductively the $\mathbb{C}\Gamma$-module structure on M_i.

Lemma 3.1. Assume that for all integers $j, k \geq 0$ with $j + k \leq n$ and for every $\omega \in \Gamma^{k+1}$ one has

$$\beta_j^{(2)}(\Lambda^\omega) = 0.$$

Then for all integers $i, j, k \geq 0$ with $i + j + k \leq n$ and for every $\omega \in \Gamma^{k+1}$ one has

$$H^j(\Lambda^\omega, \operatorname{res}^\Gamma_{\Lambda^\omega}(M_i)) = 0, \tag{3.2}$$

$$H^j(\Lambda^\omega, \operatorname{res}^\Gamma_{\Lambda^\omega}(\operatorname{coind}^\Gamma_\Lambda(M_{i-1}))) = 0 \quad \text{if } i \geq 1. \tag{3.3}$$

□

Proof. We run an induction over $i \geq 0$. By Lemma 2.1, the basis $i = 0$ is equivalent to our assumption. Assume the statement is true for a fixed $i \geq 0$ and all $j, k \geq 0$ with $i + j + k \leq n$. We show that the assertion holds for $i + 1$ and all $j, k \geq 0$ with $i + 1 + j + k \leq n$.

The short exact sequence of $\mathbb{C}\Lambda^\omega$-modules

$$0 \to \operatorname{res}^\Gamma_{\Lambda^\omega}(M_i) \to \operatorname{res}^\Gamma_{\Lambda^\omega}(\operatorname{coind}^\Gamma_\Lambda(M_i)) \to \operatorname{res}^\Gamma_{\Lambda^\omega}(M_{i+1}) \to 0$$

induces a long exact sequence in cohomology for which we consider the following part:

$$\cdots \to H^j(\Lambda^\omega, \operatorname{res}^\Gamma_{\Lambda^\omega}(\operatorname{coind}^\Gamma_\Lambda(M_i))) \to H^j(\Lambda^\omega, \operatorname{res}^\Gamma_{\Lambda^\omega}(M_{i+1})) \to H^{j+1}(\Lambda^\omega, \operatorname{res}^\Gamma_{\Lambda^\omega}(M_i)) \to \cdots$$

The homology group on the right vanishes by induction hypothesis. It remains to show that the homology group on the left vanishes. Mackey's double coset formula [4, III.5] says that after a choice of a set E of representatives of the double coset space $\Lambda^\omega \backslash \Gamma / \Lambda$, we obtain an isomorphism of $\mathbb{C}\Lambda^\omega$-modules as follows:

$$\operatorname{res}^\Gamma_{\Lambda^\omega}(\operatorname{coind}^\Gamma_\Lambda(M_i)) \cong \prod_{\gamma \in E} \operatorname{coind}^{\Lambda^\omega}_{\Lambda^\omega \cap \Lambda^{\gamma^{-1}}}(\operatorname{res}^{\Lambda^{\gamma^{-1}}}_{\Lambda^\omega \cap \Lambda^{\gamma^{-1}}}(\gamma M_i)).$$

Applying the Shapiro lemma and the induction hypothesis yields

$$\begin{aligned} H^j(\Lambda^\omega, \operatorname{res}^\Gamma_{\Lambda^\omega}(\operatorname{coind}^\Gamma_\Lambda(M_i))) &= \prod_{\gamma \in E} H^j(\Lambda^\omega, \operatorname{coind}^{\Lambda^\omega}_{\Lambda^\omega \cap \Lambda^{\gamma^{-1}}}(\operatorname{res}^{\Lambda^{\gamma^{-1}}}_{\Lambda^\omega \cap \Lambda^{\gamma^{-1}}}(\gamma M_i))) \\ &= \prod_{\gamma \in E} H^j(\Lambda^\omega \cap \Lambda^{\gamma^{-1}}, \operatorname{res}^{\Lambda^{\gamma^{-1}}}_{\Lambda^\omega \cap \Lambda^{\gamma^{-1}}}(\gamma M_i)) \end{aligned}$$

$$= \prod_{\gamma \in E} H^j(\gamma^{-1}\Lambda^\omega\gamma \cap \Lambda, \mathrm{res}^{\Lambda}_{\gamma^{-1}\Lambda^\omega\gamma\cap\Lambda}(M_i))$$

$$= 0. \qquad \blacksquare$$

3.2 Conclusion of proof of Theorem 1.3

Retain the setting of Theorem 1.3. It suffices to verify that the restriction homomorphism

$$\mathrm{res}\colon H^i(\Gamma, M_0) \to H^i(\Lambda, M_0)$$

is injective for every $i \in \{1, \ldots, n\}$. We employ the technique of dimension shifting [4, III.7]:

For $i, j \geq 0$ with $i + j \leq n$, the Shapiro lemma and (3.2) yield that

$$H^j(\Gamma, \mathrm{coind}^{\Gamma}_{\Lambda}(M_i)) \cong H^j(\Lambda, M_i) = 0.$$

From the long exact sequence

$$\cdots \to H^j(\Gamma, \mathrm{coind}^{\Gamma}_{\Lambda}(M_i)) \to H^j(\Gamma, M_{i+1}) \xrightarrow{\partial} H^{j+1}(\Gamma, M_i) \to H^{j+1}(\Gamma, \mathrm{coind}^{\Gamma}_{\Lambda}(M_i)) \to \cdots$$

one obtains, for any $i \in \{0, \ldots, n\}$, natural isomorphisms

$$H^i(\Gamma, M_0) \xleftarrow{\cong} H^{i-1}(\Gamma, M_1) \xleftarrow{\cong} \cdots \xleftarrow{\cong} H^1(\Gamma, M_{i-1}) \xleftarrow{\cong} H^0(\Gamma, M_i).$$

Using 3.3, we argue similarly to see that there is a sequence of injective homomorphisms

$$H^i(\Lambda, M_0) \hookleftarrow H^{i-1}(\Lambda, M_1) \hookleftarrow \cdots \hookleftarrow H^1(\Lambda, M_{i-1}) \hookleftarrow H^0(\Lambda, M_i),$$

for any $i \in \{0, \ldots, n\}$. In particular, we obtain, for $i \in \{0, \ldots, n\}$, the following commutative square with an upper horizontal isomorphism and a lower horizontal monomorphism:

$$\begin{tikzcd}
H^0(\Gamma, M_i) \arrow[r, "\cong"] \arrow[d, "\mathrm{res}"] & H^i(\Gamma, M_0) \arrow[d, "\mathrm{res}"] \\
H^0(\Lambda, M_i) \arrow[r, hook] & H^i(\Lambda, M_0)
\end{tikzcd}$$

So it is enough to show that the left restriction map is injective. Since it is given by the inclusion $M_i^{\Gamma} \hookrightarrow M_i^{\Lambda}$, this is obvious.

4 Applications

4.1 The groups SL_n and EL_n over general rings

The subgroup of $\mathrm{GL}_n(R)$ that is generated by elementary matrices is denoted by $\mathrm{EL}_n(R)$.

Theorem 4.1. Let R be an infinite integral domain and K be its field of fractions. For some $n \geq 2$, let $\Gamma < \mathrm{GL}_n(K)$ be a countable group which contains a finite index subgroup of $\mathrm{EL}_n(R)$.

Then there exists a subgroup $\Lambda < \Gamma$ such that for every $k < n$ and every $\omega \in \Gamma^k$, Λ^{ω} contains an infinite amenable normal subgroup.

Assume in addition that Γ contains a finite index subgroup of $\mathrm{SL}_n(R)$ and for every ideal $\{0\} \neq I \lhd R$, there exist infinitely many invertible elements $x \in R$ such that $x^n - 1 \in I$. Then also for every $\omega \in \Gamma^n$, Λ^{ω} contains an infinite amenable normal subgroup. □

Proof. We let $V = K^n$, and $e_1, \ldots, e_n$ be the standard basis. We denote by $Q < \mathrm{GL}_n(V)$ the stabilizer of the line $V_1 = \mathrm{span}\{e_1\} \in \mathbb{P}(V)$ and $S \lhd Q$ be the kernel of the obvious homomorphism $Q \to \mathrm{PGL}(V/V_1)$. Clearly, S is two-step solvable, thus amenable. We let $V_2 = \mathrm{span}_K\{e_2, \ldots, e_n\}$, thus $V = V_1 \oplus V_2$.

Let $\Lambda = \Gamma \cap Q$. For given k and $\omega = (\gamma_0, \ldots, \gamma_{k-1}) \in \Gamma^k$, we consider the group Λ^{ω}. Examining whether it contains an infinite amenable normal subgroup, we may and will assume that $\gamma_0 = e$. For $i \in \{1, \ldots, k-1\}$, we let $t_i \in K$ and $u_i \in V_2$ be defined by

$$\gamma_i^{-1} e_1 = u_i + t_i e_1.$$

We set $U = \mathrm{span}\{u_1, \ldots, u_{k-1}\} < V_2$.

Assume $U \lneq V_2$. Then there exists a nontrivial functional $\phi \in V_2^*$ which vanishes on U. Multiplying ϕ by the common denominator of $\phi(e_2), \ldots, \phi(e_n) \in K$, we may assume that $\{\phi(e_2), \ldots, \phi(e_n)\} \subset R$. For $r \in R$, we define $T_r : V \to V$ by

$$T_r(v) = v + r\phi \circ p_2(v) \cdot e_1,$$

where $p_2 : V \to V_2$ is the projection. Observe that $r \mapsto T_r$ is an injection of the additive group of R into $\mathrm{EL}_n(R)$, whose image (up to a finite index) is in $\Lambda^{\omega} \cap S$. We deduce that $\Lambda^{\omega} \cap S$ is infinite. This is an infinite amenable normal subgroup of Λ^{ω}, as required.

If $k < n$, looking at the dimensions yields that $U \lneq V_2$, thus proving the first part of the theorem.

We now consider the case $k = n$. We assume further that Γ contains $\mathrm{SL}_n(R)$ up to finite index and that for every ideal $\{0\} \neq I \lhd R$, there exist infinitely many invertible elements $x \in R$ such that $x^n - 1 \in I$. Again we will show that the amenable normal subgroup $\Lambda^\omega \cap S$ is infinite.

By the argument above, it remains to deal with the case $U = V_2$. Hence, we will assume that $U = V_2$, thus $\{u_1, \ldots, u_{n-1}\}$ forms a basis of V_2. We let $\psi \in V_2^*$ be the functional defined by $\psi(u_i) = t_i$. We let $r \in R \backslash \{0\}$ be such that $\{r\psi(e_2), \ldots, r\psi(e_n)\} \subset R$, and we set $I = (r)$ to be the ideal generated by r. Fixing an invertible element $x \in R$ such that $x^n - 1 \in I$, and letting $q_x \in R$ be an element satisfying $x^{-(n-1)} - x = q_x r$, we define $S_x : V \to V$ by setting for $t \in K$ and $u \in V_2$

$$S_x(te_1 + u) = (x^{-(n-1)}t - q_x r\psi(u))e_1 + xu.$$

It is clear that, for every such x, S_x is in $\mathrm{SL}_n(R) \cap S$ and stabilizes $\gamma_i V_1$ for every $i = 0, \ldots n-1$, thus $\Lambda^\omega \cap S$ is infinite. ■

Our next goal will be to show that some integral domains satisfy the condition appearing in the previous theorem.

Proposition 4.2. Assume that a ring R satisfies at least one of the following properties:

(1) R is an infinite field.
(2) R is a subring of the field $F(t)$ of rational functions over a finite field F, and R contains an invertible element α that is not a root of unity.
(3) R is a subring of the field $\bar{\mathbb{Q}}$ of algebraic numbers, and R contains an invertible element α that is not a root of unity.

Then for every $n \in \mathbb{N}$ and for every ideal $\{0\} \neq I \lhd R$, there exist infinitely many invertible elements $x \in R$ such that $x^n - 1 \in I$. □

The proof of the proposition in case R is a ring of algebraic numbers will depend on the following elementary lemma.

Lemma 4.3. Given $\alpha \in \bar{\mathbb{Q}}$ and $0 \neq k \in \mathbb{N}$, the ring $\mathbb{Z}[\alpha]/(k)$ is finite. □

Proof. By the general version of the Chinese remainder theorem (for the ring $\mathbb{Z}[\alpha]$), for coprime $k_1, k_2 \in \mathbb{N}$, the two ideals $(k_1 k_2)$ and $(k_1) \cap (k_2)$ coincide and $\mathbb{Z}[\alpha]/(k_1 k_2) \simeq \mathbb{Z}[\alpha]/(k_1) \times \mathbb{Z}[\alpha]/(k_2)$. It follows that we may assume that $k = p^j$ is a prime power. We now prove the statement that $\mathbb{Z}[\alpha]/(p^j)$ is finite by induction on j. For $j = 1$ the statement is clear, as this is a finite dimensional vector space over $\mathbb{Z}/(p)$. For the induction step, observe that the statement is equivalent to the statement that in $\mathbb{Z}[\alpha]$, for some i, $\alpha^i - 1$ is in the ideal (p^j). For this statement induction applies easily; $\alpha^i = 1 + p^j r$ implies $\alpha^{ip} = (1 + p^j r)^p = 1 + p^{j+1} r'$. ■

Proof of Proposition. The case that R is an infinite field is trivial.

Assume $R < F(t)$ and that $\alpha \in R$ is an invertible element which is not a root of unity. We assume (as we may upon replacing F by $F \cap R$) that R is an F-algebra, thus $F[\alpha, \alpha^{-1}] < R$. Let $\{0\} \neq I \lhd R$ be given. We claim that the image of α in $(R/I)^\times$ is torsion. We first observe that $I \cap F[\alpha, \alpha^{-1}] \neq \{0\}$. Indeed, $F(t)$ is a finite field extension of $F(\alpha)$ (it is finitely generated and of transcendental degree 0), so if $\sum a_i \beta^i$ is a minimal polynomial over $F(\alpha)$ for some nonzero function $\beta \in I$ with $a_i \in F[\alpha]$ then $a_0 \in I$. The claim follows from the obvious fact that $F[\alpha, \alpha^{-1}]/(I \cap F[\alpha, \alpha^{-1}])$ is a finite extension of F, hence, finite. Now, if $\alpha^m - 1 \in I$, then the set $\{\alpha^{jm} \mid j \in \mathbb{Z}\}$ contains, for every n, infinitely many invertible elements x with $x^n - 1 \in I$.

Assume now that $R < \bar{\mathbb{Q}}$. Again, we claim that the image of α in $(R/I)^\times$ is torsion, for any given $\{0\} \neq I \lhd R$. We first observe that $I \cap \mathbb{Z} \neq \{0\}$. Indeed, if $\sum a_i \beta^i$ is a minimal polynomial for some nonzero algebraic number $\beta \in I$ with $a_i \in \mathbb{Z}$ then $a_0 \in I$. Thus, in order to prove the claim it is enough to show that the image of α is torsion in $(R/(k))^\times$ for every $k \in \mathbb{N}$. This follows from Lemma 4.3. As before, if $\alpha^m - 1 \in I$, then the set $\{\alpha^{jm} \mid j \in \mathbb{Z}\}$ contains, for every n, infinitely many invertible elements x with $x^n - 1 \in I$. ■

Proofs of Theorems 1.6 and 1.7. By a theorem of Cheeger and Gromov [5] all L^2-Betti numbers of a group vanish if the group has an infinite normal amenable subgroup. Hence, Theorem 1.3 and the first part of Theorem 4.1 yield Theorem 1.6. Similarly and using Proposition 4.2 in addition, one obtains Theorem 1.7. ■

4.2 Thompson's groups

Thompson's group T is defined as the group of piecewise linear homeomorphisms of the circle $\mathbb{R}/\mathbb{Z}$ that are differentiable except at finitely many dyadic rational numbers, that is, points in $\mathbb{Z}[\frac{1}{2}]/\mathbb{Z}$, and such that the slopes on intervals of differentiability are powers of 2 with respect to the obvious flat structure on $\mathbb{R}/\mathbb{Z}$. Thompson's groups F is defined to be the stabilizer of $0 \in \mathbb{R}/\mathbb{Z}$ in T.

Proof of Theorem 1.8. Let $n \geq 1$. Let $\Lambda \subset F$ be the stabilizer subgroup inside F of a finite set of $(n+1)$ many dyadic rational points. For any $\omega \in F^m$ with $m \geq 1$, the subgroup $\Lambda^\omega \subset F$ is the stabilizer subgroup of a finite set of d dyadic rational points with some $d \in \{n+1, \ldots, m(n+1)\}$. By the description above, it is evident that $\Lambda^\omega \cong F^d$. The L^2-Betti numbers of any d-fold product of infinite groups, thus of Λ^ω, vanish up to degree $d-1 \geq n$ by repeated application of the Kuenneth formula in L^2-cohomology [11, Theorem 6.54., p. 265]. Now Theorem 1.3 implies that the L^2-Betti numbers of F vanish up to degree n, and since n was arbitrary, Theorem 1.8 for the group F is proved. For the group T, we run the almost the same argument, taking Λ to be the stabilizer inside T and considering $\omega \in T^m$. We again obtain that $\Lambda^\omega \cong F^d$ and finish the argument as above. ■

4.3 Permutation group theoretic criterion

Theorem 4.4. Let Γ be a countable group, $\Lambda < \Gamma$ be an amenable subgroup such that the closure of the image of Γ in the Polish group $\mathrm{Sym}(\Gamma/\Lambda)$ is not discrete. Then $\beta_n^{(2)}(\Gamma) = 0$ for every $n \geq 0$. □

Proof. We apply Corollary 1.5. We will be done by showing that for any n and any $\omega \in \Gamma^n$, Λ^ω is infinite. Assume otherwise that for some n and some $\omega \in \Gamma^n$, Λ^ω is finite. Then for some n' and $\omega' \in \Gamma^{n'}$, $\Lambda^{\omega'}$ is the core of Λ, $\bigcap_{\gamma \in \Gamma} \Lambda^\gamma$. That is, the identity element of the image of Γ in $\mathrm{Sym}(\Gamma/\Lambda)$ could be expressed as the intersection of finitely many open subgroups, contradicting the nondiscreteness of this image. ■

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References

[1] Bader, U., A. Furman, and R. Sauer. "Lattice envelopes." (2012) (in preparation).

[2] Berberian, S. K. "The maximal ring of quotients of a finite von Neumann algebra." *The Rocky Mountain Journal of Mathematics.* 12, no. 1 (1982): 149–64.

[3] Borel, A. "The L^2-cohomology of negatively curved Riemannian symmetric spaces." *Annales Academi Scientiarium Fennic Mathematica* 10 (1985): 95–105.

[4] Brown, K. S. "Cohomology of Groups." Graduate Texts in Mathematics 87. New York: Springer, 1982.

[5] Cheeger, J. and M. Gromov. "L_2-cohomology and group cohomology." *Topology* 25, no. 2 (1986): 189–215.

[6] Dicks, W. and P. A. Linnell. "L^2-Betti numbers of one-relator groups." *Mathematische Annalen* 337, no. 4 (2007): 855–74.

[7] Ershov, M. and A. Jaikin-Zapirain, "Property (T) for noncommutative universal lattices." *Inventiones Mathematicae* 179, no. 2 (2010): 303–47.

[8] Farber, M. "von Neumann categories and extended L^2-cohomology." *K-Theory* 15, no. 4 (1998): 347–405.

[9] Gaboriau, D. "Invariants l^2 de relations d'équivalence et de groupes." *Publications Mathématiques. Institut de Hautes Etudes Scientifiques* 95 (2002): 93–150 (French).

[10] Lück, W. "Dimension theory of arbitrary modules over finite von Neumann algebras and L^2-Betti numbers. I. Foundations." *Journal für die Reine und Angewandte Mathematik.* 495 (1998): 135–62.

[11] Lück, W. L^2-invariants: theory and applications to geometry and K-theory. *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]* 44, xvi+595. Berlin: Springer, 2002.

[12] Mimura, M. "Fixed point properties and second bounded cohomology of universal lattices on Banach spaces." *Journal für die Reine und Angewandte Mathematik* 653 (2011): 115–34.

[13] Olbrich, M. "L^2-invariants of locally symmetric spaces." *Documenta Mathematica* 7 (2002): 219–37 (electronic).

[14] Peterson, J. and A. Thom. "Group cocycles and the ring of affiliated operators." *Inventiones Mathematicae* 185, no. 3 (2011): 561–92.

[15] Reich, H. "On the K- and L-theory of the algebra of operators affiliated to a finite von Neumann algebra." *K-Theory* 24, no. 4 (2001): 303–26.

[16] Sauer, R. and A. Thom. "A spectral sequence to compute L^2-Betti numbers of groups and groupoids." *Journal of the London Mathematical Society. Second Series*, 81, no. 3 (2010): 747–73.

[17] Shalom, Y. "The algebraization of Kazhdan's property (T)." *International Congress of Mathematicians II*. 1283–310. Zürich: European Mathematical Society, 2006.

[18] Thom, A. "L^2-cohomology for von Neumann algebras." *Geometric and Functional Analysis* 18, no. 1 (2008): 251–70.