WHAT IS... a Stationary Measure?

Alex Furman

The concept of a stationary measure appears in probability, dynamics of group actions, and foliations of manifolds. But it can also be related to such real-life experiences as card shuffling and searching the Internet.

Markov Chains and Markov Operators

Let $P = (p_{ij})$ be a $k \times k$ stochastic matrix, i.e., a matrix with nonnegative entries satisfying $p_{i1} + \cdots + p_{ik} = 1$ for each $1 \le i \le k$. We view p_{ij} as the probability of moving from state i to state j. Given P and some distribution ν_0 on $\{1, \ldots, k\}$, the corresponding *Markov chain* is a sequence $\{X_n\}_{n\ge0}$ of random variables with values in $\{1, \ldots, k\}$, where X_0 is chosen according to ν_0, and for each $n \ge 0$ the value of X_{n+1} given $X_0, \ldots, X_n$ depends only on the value of X_n, and $X_{n+1} = j$ with probability $p_{i,j}$ provided $X_n = i$. Then the distribution ν_n of X_n is $\nu_n = P^*\nu_{n-1} = \cdots = (P^*)^n\nu_0$, where $P^*_{ij} = p_{ji}$. A *P-stationary measure* is a solution to the equation: $\nu = P^*\nu$.

Observe that the distribution ν_n of the nth step X_n of the Markov chain defined by a stationary measure ν remains stable: $\nu_n = \nu$. Any finite Markov chain is guaranteed to have a stationary measure. Indeed, any stochastic P satisfies $P\mathbf{1} = \mathbf{1}$. Thus 1 is an eigenvalue for P and therefore also for P^*. Writing a P^* invariant ν as $\nu = \nu^+ - \nu^-$ with $\nu^+, \nu^- \in (\mathbb{R}_+)^k$, we obtain $P^*\nu^\pm = \nu^\pm$ because P^* preserves the positive cone; if $\nu^+ \ne 0$ take $\nu = (\sum \nu_i^+)^{-1} \cdot \nu^+$, otherwise normalize ν^-. Another argument for the existence of stationary measures is by Brouwer's fixed point theorem applied to the P^*-invariant simplex of probability measures $\Delta = \{q_i \ge 0, \sum q_i = 1\}$.

Often (e.g., if $P^{n_0}_{ij} > 0$ for some n_0 and all i, j) there is only one stationary measure ν, and given any initial distribution ν_0 the distributions $\nu_n = (P^*)^n\nu_0$ converge to ν exponentially fast (Perron-Frobenius theorem).

Apparently Google uses this phenomenon in its page-ranking algorithm, where thousands of sites containing the searched-for words have to be ranked by their "relevance". Imagine a graph whose vertices are the sites found in a particular search and interconnected by links between them. One assumes that sites that are better connected (linked to or from) within this graph are most relevant and that the stationary measure for this graph can be used to determine the ranking of the search results. Since the convergence to the stationary measure is very fast, it can be approximated by the nth step of the random walk, rather than by a calculation of the eigenfunctions of the potentially huge matrix.

Card shuffling is another, much older, example. Performing a number of shuffles, say cutting the deck at random places, is implicitly assumed to produce permutations of the deck with an approximately uniform distribution. From a mathematical standpoint, a finite group (S_{52}) acts transitively on a set (fifty-two cards), and some probability measure μ is given on a generating set (cut-shuffles) of the group. This defines a Markov chain $p_{ij} = \mu\{g : gi = j\}$, and one can show that the uniform measure is the unique stationary one on this set. This example appears already in Poincaré's *Calcul des probabilités* (1912).

Markov chains are generalized by *Markov operators* on, say, a compact space X. Let $\{\mu_x\}_{x\in X}$ be a family of Borel probability measures on X, so that $x \mapsto \mu_x$ is continuous in the weak-* topology. This defines a (positive, normalized) operator on $C(X)$ by $Pf(x) = \int_X f(y)\,d\mu_x(y)$. The dual operator P^* acting on $C(X)^* = \mathcal{M}(X)$ preserves the convex weak-* compact subset $\mathrm{Prob}(X)$ of probability measures. In this setting, stationary measures are solutions in $\mathrm{Prob}(X)$ to $\nu = P^*\nu$. Existence of stationary measures is now guaranteed by the Markov-Kakutani fixed point theorem.

μ-Stationary Measures

Next consider a continuous action $G \curvearrowright X$ of some locally compact group G on a compact space X. We say that X is a compact G-space. Fix a probability measure μ on G and define a Markov operator $\{\mu_x\}$ on X by pushing μ forward via $g \mapsto g.x$.

Alex Furman is professor of mathematics at the University of Illinois at Chicago. His email address is `furman@math.uic.edu`.

This Markov operator on $\mathrm{Prob}(X)$ is given by the convolution $\mu * \nu = \int g_* \nu \, d\mu(g)$. Stationary (or rather *μ-stationary*) measures are solutions to

$$\nu = \mu * \nu. \tag{1}$$

Note that any G-invariant measure on X is μ-stationary for any μ on G; however, many important actions of large groups (precisely the nonamenable ones) leave no probability measure invariant. Yet any compact G-space X is entitled to a μ-stationary measure ν.

Here is a beautiful application of this fact (observed by Deroin-Kleptsyn-Navas): *any countable subgroup $G < \mathrm{Homeo}(S^1)$ of the group of homeomorphisms of the circle is conjugate to the subgroup of bi-Lipschitz homeomorphisms.* Indeed, fix a measure μ of full support on G and let ν be a μ-stationary measure on the circle. Then it follows from (1) that $\frac{dg_*\nu}{d\nu} \le \mu(g)^{-1}$, and the required conjugation can be realized by any circle homeomorphism mapping ν to the Lebesgue measure.

Poisson-Furstenberg Boundaries

Now let us very briefly discuss the notions of μ-stationary measures, μ-harmonic functions, and Poisson (or rather Poisson-Furstenberg) boundaries that were developed by Furstenberg in his study of random walks on groups [2] and that continue to play an important role in rigidity, dynamics, and geometry.

Let ν be a μ-stationary measure on some G-space X. If μ is sufficiently nice, say absolutely continuous with respect to the Haar measure on G and not supported on a proper closed subgroup, then ν is G-quasi-invariant. Consider the transform $\Pi : L^\infty(X, \nu) \to L^\infty(G)$ given by

$$\Pi f(g) = \int_X f(x)\, dg_*\nu(x) = \int_X \frac{dg_*\nu}{d\nu}(x) f(x)\, d\nu(x). \tag{2}$$

Equation (1) implies that $u = \Pi f$ satisfies a μ-mean value property

$$u(g) = \int_G u(gh)\, d\mu(h).$$

Such functions are called *bounded μ-harmonic.* In the case of $G = \mathrm{PSL}_2(\mathbb{R})$ acting on the circle $X = \mathbb{R} \cup \{\infty\}$ and μ being a bi-$\mathrm{SO}(2)$-invariant measure, bounded μ-harmonic functions u on G are lifts of the classical harmonic functions on the Poincaré disc $\mathrm{PSL}_2(\mathbb{R})/\mathrm{SO}(2)$, and (2) is the classical Poisson transform.

This construction has an analogue in the completely general setting of an arbitrary locally compact group G and a nice probability measure μ on it. Namely, there exists a G-space X with a μ-stationary measure ν, so that transform (2) is an isometric isomorphism between $L^\infty(X, \nu)$ and the subspace of bounded μ-harmonic functions in $L^\infty(G)$. Such an (X, ν) is defined uniquely as a measurable G-space; it is called the *Poisson* or the *Poisson-Furstenberg* boundary $\mathrm{PF}_\mu(G)$ of (G, μ).

Such boundaries have been identified explicitly in many examples, including semisimple Lie groups and their lattices, where PF_μ is a compact homogeneous space of the Lie group ([2]); Gromov hyperbolic groups Γ, where PF_μ is realized on the Gromov boundary $\partial\Gamma$; and many other situations (see [3]).

The G-action on PF_μ has important dynamical properties—the action is *amenable* and *ergodic* in a very strong sense. The dynamics of $G \curvearrowright \mathrm{PF}_\mu$ and, in particular, the structure of their equivariant quotients play a crucial role in proofs of many remarkable rigidity results for representations of lattices, group actions on manifolds, computations of bounded cohomology, and more.

Stiff Actions

Studying dynamics of a single transformation $T : X \to X$, one is interested in closed invariant sets, invariant measures, and limiting distributions for orbits of individual points. Famously, Ratner's theorems provide a complete classification of these objects for unipotent flows on homogeneous spaces.

Recently, significant progress has been made on analogous problems for algebraic actions of large groups Γ, such as $\Gamma < \mathrm{SL}_d(\mathbb{Z})$ acting on the torus $X = \mathbb{R}^d/\mathbb{Z}^d$ ([1]), or $\Gamma < \mathrm{SL}_d(\mathbb{R})$ acting on $X = \mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$ ([4]). In this context, any Γ not contained in a proper algebraic subgroup $L \lneq \mathrm{SL}_d(\mathbb{R})$ is "large" enough. The acting group Γ is nonamenable; thus the role played by invariant probability measures in Ratner's theorems is taken here by stationary ones. For example, classification of closed invariant sets is deduced from the classification of stationary measures. These turn out to be invariant (such actions are called *stiff*): either atomic on a finite Γ-orbit, or uniform, or convex combinations of these. Hence using a few matrices from $\mathrm{SL}_d(\mathbb{Z})$ at random, one can very effectively shuffle the deck of torus points to get a uniform distribution.

References

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