RIGIDITY OF GROUP ACTIONS

1. Introduction to Super-Rigidity

Alex Furman
(University of Illinois at Chicago)

February 28, 2007
For some $\Gamma < G$ representations $\rho : \Gamma \longrightarrow H$ extend to $G$:

$$
\begin{array}{c}
\Gamma \\
\downarrow \rho \\
H \\
\downarrow \bar{\rho} \\
G
\end{array}
$$
The Super-rigidity Phenomenon

For some $\Gamma < G$ representations $\rho : \Gamma \rightarrow H$ extend to $G$:

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G & \xrightarrow{\bar{\rho}} & \end{array}
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provided

$G$ is a “higher rank” lcsc group
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- $G$ is a “higher rank” lcsc group
- $\Gamma < G$ – an (irreducible) lattice
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\bar{\rho}
\end{array}
\begin{array}{c}
\rightarrow \\
\downarrow \\
\rho
\end{array}
\]

provided

- $G$ is a “higher rank” lcsc group
- $\Gamma < G$ – an (irreducible) lattice
- $\rho : \Gamma \rightarrow H$ with $\rho(\Gamma)$ “non-elementary” in $H$. 
Lattices

Definition
Γ < G is a lattice if Γ is discrete and Haar(G/Γ) < ∞.
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$\Gamma \lhd G$ is a lattice if $\Gamma$ is discrete and $\text{Haar}(G/\Gamma) < \infty$.
$\Gamma \lhd G = \prod_{i=1}^{n} G_i$ is irreducible if $\text{pr}_i(\Gamma)$ dense in $G_i$. 

Examples (Arithmetic)
$\Gamma = \mathbb{Z}^n$ in $G = \mathbb{R}^n$
$\Gamma = \text{SL}_n(\mathbb{Z})$ in $G = \text{SL}_n(\mathbb{R})$
$\Gamma = \mathbb{Z}(\sqrt{2})$ in $G = \mathbb{R}^2$ with $(a + b\sqrt{2}, a - b\sqrt{2})$

Example (Geometric)
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- “similar” construction of \( \Gamma \) in \( G = \text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R}) \)
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Γ < G is a lattice if Γ is discrete and Haar(G/Γ) < ∞.

Γ < G = ∏^n_{i=1} G_i is irreducible if pr_i(Γ) dense in G_i.

Examples (Arithmetic)

- Γ = Z^n in G = R^n
- Γ = SL_n(Z) in G = SL_n(R)
- Γ = Z(√2) in G = R^2 with (a + b√2, a − b√2)
- “similar” construction of Γ in G = SL_2(R) × SL_2(R)

Example (Geometric)

Γ = π_1(M) for M – loc. symmetric, compact (or vol(M) < ∞) is a lattice in G = Isom(˜M).
Margulis’ Higher rank Super-rigidity

Theorem (Superrigidity, Margulis 1970s)
Assume

\[ G = \prod G_i \] is semi-simple Lie group with \( \text{rk}(G) \geq 2 \)

\( H \) is simple and center free

\( \Gamma \triangleleft G \) is an irreducible lattice

\( \rho : \Gamma \rightarrow H \) with \( \rho(\Gamma) \) Zariski dense in \( H \).

Theorem (Arithmeticity, Margulis 1970s)
In higher rank all irreducible lattices are arithmetic!
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Measurable Cocycles

$G, H$ – lcsc groups, $G \curvearrowright (X, \mu)$ – prob. m.p. action
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Cocycles: measurable maps $c : G \times X \rightarrow H$ s.t.

$$\forall g_1, g_2 \in G : \quad c(g_1 g_2, x) = c(g_1, g_2.x) \cdot c(g_2, x)$$
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Cohomologous cocycles: $c \sim c'$ if $\exists f : X \to H$ s.t.

$$c'(g, x) = f(g.x)c(g, x)f(x)^{-1}$$
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Observation

$$\{\rho : \Gamma \to H\}/\text{conj} \cong \{c : G \times G/\Gamma \to H\}/\sim.$$
Zimmer’s Cocycle Super-rigidity

Theorem (Cocycle Super-rigidity, Zimmer 1981)

Let $G = \prod G_i$ be a semi-simple Lie group with $\text{rk}(G) \geq 2$. $G \curvearrowright (X, \mu)$ a prob. m.p. action with each $G_i$ ergodic.

Remarks
Margulis' super-rigidity corresponds to $X = G / \Gamma$. Proofs combine Algebraic groups with Ergodic Theory. $G$-boundary $(B, \nu) = (G / \mathcal{P}, \text{Haar})$ plays a key role.
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- Margulis’ super-rigidity corresponds to \( X = G/\Gamma \)
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Cocycles: where from and what for?

- Volume preserving Actions on Manifolds
  \[ \rho : \Gamma \rightarrow \text{Diff}(M, \text{vol}) \]
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Volume preserving Actions on Manifolds

\[ \rho : \Gamma \to \text{Diff}(M, \text{vol}) \]

\[ \Gamma \curvearrowright TM \cong \mathbb{R}^d \times M \text{ where } d = \dim M \]

\[ \sim \quad \alpha : \Gamma \times M \to \text{GL}_d(\mathbb{R}) \text{ or } \alpha : \Gamma \times M \to \text{SL}_d(\mathbb{R}). \]
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- **Orbit Equivalence in Ergodic Theory**
  \( \Gamma \curvearrowleft (X, \mu) \) and \( \Lambda \curvearrowleft (Y, \nu) \) free erg. actions
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  \( \sim \) \[ \alpha_T : \Gamma \times X \rightarrow \Lambda \text{ by } T(\gamma.x) = \alpha(g, x).T(x) \]
Theorem (S. Popa 2006)

Let \( \Gamma \) have \((T)\) and \( \Gamma \acts X = (X_0, \mu_0)^\Gamma \) be a Bernoulli action.
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Theorem (S. Popa 2006)

Let $\Gamma$ have (T) and $\Gamma \curvearrowright X = (X_0, \mu_0)^\Gamma$ be a Bernoulli action. Then for any discrete or compact $\Lambda$ every cocycle $\alpha : \Gamma \times X \to \Lambda$ is cohomologous to a homomorphism $\rho : \Gamma \to \Lambda$. 

Remark ▶ $\Lambda$ arbitrary discrete or compact or in $U_{\text{fin}}$! ▶ No assumptions on $\alpha$! All cocycles $\sim$ to homs in $\Lambda$! ▶ "deformation-rigidity": malleability - spectral assumption (T) ▶ The assumption on the action $\Gamma \curvearrowright X$ rather than on $G$ or $\Gamma$ ▶ leads to "von Neumann rigidity"
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<tr>
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<th>$H(k)$</th>
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Boundary and the Weyl group

Definition (G-Boundaries, after Burger-Monod)

$G$ – a general lcsc grp.
A $G$-boundary is a msbl $G$-space $(B, [\nu])$ so that

- $G \act (B, [\nu])$ is amenable
- $G \act (B \times B, [\nu \times \nu])$ erg with Unitary Coefficients

Definition (Weyl Group, (Bader-F, Bader-F-Shaker))

Given a $G$-boundary $(B, \nu)$ let $W_{G, B} = \text{Aut}(B \times B, [\nu \times \nu])$.

Examples

- $G$-ss alg, $B = G/P$ then $W_{G, B}$ – the classical Weyl (e.g. $G = \text{SL}_n \rightarrow W = \text{S}_n$)
- $G$ hyperbolic-like
- $G$ amenable, can take trivial $B$ and $W$
- $G = \prod_n G_i$ with non-amenble factors, $(Z/2Z)^n < W_{G, B_i}$.
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