RIGIDITY OF GROUP ACTIONS

I. Introduction to Super-Rigidity

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- G is a "higher rank" lcsc group
- $\Gamma < G$ – an (irreducible) lattice
- $\rho : \Gamma \longrightarrow H$ with $\rho(\Gamma)$ "non-elemntary" in H.

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Example (Geometric)

$\Gamma = \pi_1(M)$ for M – loc. symmetric, compact (or $\mathrm{vol}(M) < \infty$)
is a lattice in $G = \mathrm{Isom}(\widetilde{M})$.

Margulis' Higher rank Super-rigidity

Theorem (Superrigidity, Margulis 1970s)

Assume

$G = \prod G_i$ – *semi-simple Lie group with* $\underline{rk(G) \geq 2}$

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In higher rank all irreducible lattices are arithmetic !

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Observation

$$\{\rho : \Gamma \longrightarrow H\}/\mathrm{conj} \quad \cong \quad \{c : G \times G/\Gamma \to H\}/\sim .$$

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- Proofs combine Algebraic groups with Ergodic Theory

  G-boundary $(B, \nu) = (G/P, \mathrm{Haar})$ plays a key role

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  $\rightsquigarrow \quad \alpha_T : \Gamma \times X \to \Lambda$ by $\quad T(\gamma.x) = \alpha(g, x).T(x)$

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Margulis (1974) Zimmer Ann.Math. (1981)

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More Margulis-Zimmer like Super-rigidity results

Targets H	$\mathbf{H}(k)$	$\mathrm{CAT}(-1)$	δ-Hyp	$\curvearrowright S^1$	Mod_g	$\mathrm{Isom}(\mathcal{H})$
$\Gamma < \mathbf{G}$ alg alg $\mathbf{G} \times X$	Margulis Zimmer	Bu-Mzs Adams	Furst Adams	Ghys Wi-Zi	K-M	(T) (T)
$\Gamma < \Lambda < G$	Margulis	Bu-Mzs				Shalom
$\Gamma < \prod G_i$ $\prod G_i \times X$	Monod (F-Md)	Md-Sh and H-K	M-M-S and H-K			Sh Md (Shalom)

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Furstenberg Bull.AMS (1967)
Ghys Inven. (1999)
Witte-Zimmer Geom.Ded.(2001)
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$\Gamma < \prod G_i$ $\prod G_i \times X$	Monod (F-Md)	Md-Sh and H-K	M-M-S and H-K	B-F-S B-F-S		Sh Md (Shalom)
$\Gamma \curvearrowright \tilde{A}_2$ $\Gamma \times X$				B-F-S B-F-S		(T) (T)

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Bader-Furman (2007)

Boundary and the Weyl group

Definition (G-Boundaries, after Burger-Monod)

G – a general lcsc grp.
A G-boundary is a msbl G-space $(B, [\nu])$ so that

- $G \curvearrowright (B, [\nu])$ is amenable
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- G amenable, can take trivial B and W
- $G = \prod^n G_i$ with non-amenble factors, $(\mathbf{Z}/2\mathbf{Z})^n < W_{G,\prod B_i}$.